CHAPTER 0

INTRODUCTION.
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The theory of inequalities began its development from the time when C.F. Gauss (1777-1855), A.L. Cauchy (1789-1857) and P.L. Chebyshev (1821-1894) to mention only the most important, laid the theoretical foundation for approximative methods. Around the end of the 19th and the beginning of the 20th century, numerous inequalities were proved, some of which became classic, while most remained as isolated and unconnected results. The famous monograph "Inequalities" by G.H. Hardy, J.E. Littlewood and G. Pólya, which appeared in 1934, transformed the field of inequalities from a collection of isolated formulas into a systematic discipline. Since then an enormous amount of effort has been devoted to the sharpening and extension of the classical inequalities. The book "Inequalities" by E.F. Beckenbach and R. Bellman, which appeared in 1961, contains an account of some results on inequalities obtained in the period 1934-1960. "Analytic Inequalities" by D.S. Mitrinovic, which appeared in 1970, contains most of the results which are not included in the two monographs mentioned above.

However, a large number of papers have been appeared since 1970, papers partly inspired by the aforementioned monographs, and probably even more so by the challenge of research in various branches of pure and applied mathematics, where inequalities are often the basis of important lemmas for proving various theorems or for approximating various functions.
Most mathematicians regard inequalities as auxiliary in character and would perhaps not think of them as constituting a domain of principal interest apart from applications. In the conference held at the University of California, Los Angeles, during September–October 1954, Richard Bellman [3] has expressed his views about the inequalities as follows. "It has been said that mathematics is the science of tautology, which is to say that mathematicians spend their time proving that equal quantities are equal. This statement is wrong on two counts. In the first place, mathematics is not a science, it is an art; in the second place, it is fundamentally the study of inequalities rather than equalities". Taking into consideration the views expressed by Richard Bellman and the role played by the inequalities in various branches of mathematics, we strongly feel that the theory of inequalities is a fascinating subject in itself.

In recent years inequalities are playing a very active role in many parts of analysis. As examples, let us cite the fields of ordinary and partial differential equations, which are dominated by inequalities and variational principles involving functions and their derivatives. One of the most used integral inequality in the theory of ordinary differential equations is due to T.H. Gronwall [22]. The inequality established by Gronwall is embodied in the following
Lemma 1 (Gronwall [22]). Let the real-valued nonnegative function \( u(t) \) be continuous on the interval \([a, b]\) with

\[
    u(t) \leq \int_a^t \{ K + M u(s) \} \, ds , \quad t \in [a, b],
\]

where \( K, M \) are nonnegative constants. Then

\[
    u(t) \leq (b-a) K \exp \{ M(b-a) \} , \quad t \in [a, b].
\]

In the year 1885, G. Peano [48] explicitly proved the special case of Lemma 1 when \( K = 0 \), and proved some quite general results concerning differential inequalities as well as maximal and minimal solutions of differential equations. After the discovery of the integral inequality resulting from Gronwall, R. Bellman [4] has proved a useful generalization of Lemma 1 also known as "Gronwall-Bellman" lemma which has found many applications in the theory of ordinary differential equations. In a thesis submitted by W.T. Reid [49] published in 1930, stated without proof a result which is essentially the same as that of Bellman, except that continuity assumption was relaxed to Lebesgue integrability.

In the year 1956, I. Bihari [40] has proved a very useful nonlinear generalization of Gronwall-Bellman inequality in the following form
**Lemma 2** (Bihari [10]). (i) Let \( u \) and \( F \) be
nonnegative continuous functions on \([0,a)\), and

(ii) \( v \) a positive nondecreasing continuous function on \((0,\infty)\)
such that

\[
u(t) \leq k\int_0^t F(s)W(u(s))ds, \quad t \in [0,a),
\]

where \( k \) and \( M \) are positive constants. Then

\[
u(t) \leq G^{-1}\left[G(k) + M\int_0^t F(s)ds\right], \quad t \in [0,a_1),
\]

where

\[
g(x) = \int_0^x \frac{ds}{W(s)} \quad , \quad x \geq x_0 > 0,
\]

and \( G^{-1} \) is the inverse function of \( G \) and

\[a_1 = \operatorname{Sup} \{ t \in [0,a) : G(\infty) \geq G(k) + M\int_0^t F(s)ds \} \]

Three years before Bihari, B.N. Babkin [1] proved
a more general inequality than that of Lemma 2, in which
\( F(s)W(v) \) is replaced by \( f(s,v) \), and the constant \( k \) replaced
by a function. Other useful linear and nonlinear generali-
sations of Lemma 1 are also given by D. Willett [57],
V. Lakshmikantham [25], B. Viswanathan [54], D. Willett
and J.S.W. Wong [58], S.C. Chu and P.T. Metcalf [13],
B.G. Pachpatte (1975) and others.
In the year 1973, another interesting and very useful linear generalization of Lemma 1 is established by B. G. Pachpatte in the following form

Lemma 3 (Pachpatte [30]). Let $u(t)$, $f(t)$ and $g(t)$ be real-valued nonnegative continuous functions defined on $I$; for which the inequality

$$u(t) \leq u_0 + \int_0^t f(s)u(s)ds + \int_0^t f(s)(\int_0^s g(\tau)u(\tau)d\tau)ds, \quad t \in I,$$

holds, where $u_0$ is a nonnegative constant. Then

$$u(t) \leq u_0 \left[ 1 + \int_0^t f(s) \exp(\int_0^s (f(\tau)+g(\tau))d\tau)ds \right], \quad t \in I.$$

In a series of papers [31, 34, 40, 41, 42 and 43], Pachpatte has established a number of linear and nonlinear generalizations and applications of Lemma 3 and its variants. In 1974, the same author has obtained a nonlinear generalization of Lemma 3 (for $f=g$) in the following form

Lemma 4 (Pachpatte [33]). Let $u(t)$, $g(t)$ be real-valued nonnegative continuous functions defined on $I$; $H(u)$ be a positive continuous, strictly increasing function for $u>0$, $H(0) = 0$, and suppose further that the inequality

$$u(t) \leq u_0 + \int_0^t g(s)(u(s) + \int_0^s g(\tau)H(u(\tau))d\tau)ds$$

holds.
is satisfied. Then for \( t \in I_q \),

\[
u(t) \leq u_0 + \int_a^t g(s) G^{-1} \left[ G(u_0) + \int_a^s g(\tau) d\tau \right] ds
\]

where

\[
G(x) = \int_{x_0}^{x} \frac{ds}{g(s + g(s))}, \quad x \geq x_0 > 0.
\]

and \( G^{-1} \) is the inverse of \( G \) and

\[
I_q = \{ \ t \in I : G(u) \geq G(u_0) + \int_a^t g(\tau) d\tau \}.
\]

Although the integral inequalities of Gronwall–Bellman and Pachpatte have been widely known and used in the theory of ordinary differential and integral equations, a two independent variable generalization of Gronwall's inequality due to Wendroff given in the book "Inequalities" by Beesonbach and Bellman [2, p. 154] is overlooked by researchers in mathematics. The inequality established by Wendroff is given in the following

**Lemma 5** (Wendroff [2, p. 154]). Let \( \phi(x, y) \) and \( G(x, y) \) be nonnegative continuous functions defined for \( x > 0, \ y > 0 \) for which the inequality

\[
\phi(x, y) \leq a(x) + b(y) + \int_0^x \int_0^y G(s, t) \phi(s, t) ds dt
\]
holds for $x \geq 0$, $y \geq 0$ where $a(x), b(y) > 0$; $a'(x), b'(y) \geq 0$ are continuous functions defined for $x \geq 0$, $y \geq 0$. Then

$$\phi(x,y) \leq \frac{[a(x)+b(y)] [a(x)+b(y)]}{[a(x)+b(y)]} \exp(\int_0^x \int_0^y c(s,t) \, ds \, dt)$$

for all $x \geq 0$, $y \geq 0$.

Various generalizations of Lemma 5 are recently given by Ghoshal and Masood [20], Chandra and Devi [12], Pachpatte [44]-[47], Headley [24] and others.

In the year 1972, D.R. Snow [52] has established a useful two independent variable generalization of Lemma 6 in the following form

**Lemma 6 (Snow [52])**. Suppose $\phi(x,y)$, $a(x,y)$ and $b(x,y)$ are continuous on a domain $D$ with $b > 0$ there. Let $P_0(x_0,y_0)$ and $P(x,y)$ be two points in $D$ such that $(x-x_0)(y-y_0) > 0$ and let $R$ be a rectangular region whose opposite corners are the points $P_0$ and $P$. Let $V(s,t;x,y)$ be the solution of the characteristic initial-value problem

$$L[V] = V_{st} - b(s,t) V = 0, \quad V(s,y) = 0, \quad V(x,t) = 1$$

and let $D^+$ be a connected subdomain in $D$ which contains $P$ and on which $V > 0$. Then, if $R \subset D^+$ and $\phi$ satisfies
\[ \phi(x, y) \leq a(x, y) + \int_X \int_Y b(s, t) \phi(s, t) \, ds \, dt. \]

\[ \phi \] also satisfies

\[ \phi(x, y) \leq a(x, y) + \int_X \int_Y a(s, t) b(s, t) V(s, t; x, y) \, ds \, dt. \]

In the year 1975, E.C. Young [60] has established a useful independent variable generalization of Lemma 6 in the following form

**Lemma 7** (Young[60]). Suppose \( \phi(x) \), \( a(x) \) and \( b(x) \geq 0 \) are continuous functions in \( \Omega \). Let \( V(\xi; x) \) be a solution of the characteristic initial-value problem

\[ (-1)^n V_{\xi_1 \ldots \xi_n}(\xi; x) - b(\xi)V(\xi; x) = 0 \quad \text{in} \quad \Omega, \]

\[ V(\xi; x) = 1 \quad \text{on} \quad \xi_i = x_i, \quad i=1, \ldots, n, \]

and let \( D^+ \) be a connected subdomain of \( \Omega \) containing \( x \) such that \( V > 0 \) for all \( \xi \in D^+ \). If \( D \subset D^+ \) and

\[ \phi(x) \leq a(x) + \int_X b(\xi) \phi(\xi) \, d\xi \]

then

\[ \phi(x) \leq a(x) + \int_X a(\xi) b(\xi) V(\xi; x) \, d\xi \]
(Here $\int_{x_0}^{x_1} \ldots \int_{x_0}^{x_n} d\varepsilon_1 d\varepsilon_2 \ldots d\varepsilon_n$ indicates the n-fold integral)

\[
\int_{x_0}^{x_1} \ldots \int_{x_0}^{x_n} d\varepsilon_1 d\varepsilon_2 \ldots d\varepsilon_n.
\]

In the year 1960, Z. Opial [29] has proved a fundamental integral inequality involving a function and its derivative in the following form

**Lemma 8 (Opial [29]).** If $f$ is an absolutely continuous function on $[0, a]$ and if $f(0) = 0$, the following inequality holds

\[
\int_0^a |f(x)f'(x)| \,dx \leq \frac{a}{2} \int_0^a |f'(x)|^2 \,dx.
\]

where $a/2$ is the best possible constant. Equality holds if and only if $f(x) = cx$, where $c$ is a constant.

After the discovery of the integral inequality resulting from Z. Opial, many authors have written numerous articles which give the simplest proof, various generalizations and discrete analogues of Opial's inequality and its generalizations.

However, a large number of papers have been written on integral inequalities in one independent variable and their applications in
various fields of ordinary differential and integral equations, it appears that the integral inequalities in two or more independent variables which have their origin in the field of partial differential and integral equations have been overlooked from this point of view. It is natural to expect that the integral inequalities involving two or more independent variables will have as many applications for partial differential and integral equations as the classical Gronwall-type inequalities have had for ordinary differential and integral equations. The present work is strongly inspired by the integral inequalities recently established by Rachpate [35], Snow [52], Young [60] and others.

The aim of the present work is to establish a number of integral inequalities in two or more independent variables which can be used as handy tools in the theory of partial differential and integral equations. The areas of applications of these inequalities are boundedness, uniqueness, continuous dependence, stability and other problems in the theory of partial differential and integral equations.

Chapter 1 deals with the two independent variable generalizations of the integral inequalities given in Lemma 1 and 3, which claim their origin in the field of partial differential equations. The nonlinear integral inequalities which may be considered as the further generalizations of lemma 2 and 4 are also established. The inequalities proved here
can be used in the analysis of partial differential and integrodifferential equations in essentially the same capacity as the integral inequalities of the Gronwall, Bihari and Pachpatte type are used in the theory of ordinary differential and integral equations.

In Chapter 2, we establish some useful two independent variable generalizations of the well-known integral inequalities of Gollwitzer [21] Langenhop [27] and Pachpatte [35] which give us the lower bounds on unknown functions. The bounds provided by these inequalities are adequate in many applications in the theory of partial differential and integral equations. Some applications of these inequalities are given so as to illustrate their usefulness.

In Chapter 3, we first establish the further generalizations of some of the integral inequalities established in Chapter 1. Furthermore an elementary method used by Snow [52] to establish a two independent variable generalization of a Gronwall's inequality is used to establish some useful generalizations of the inequalities by Gollwitzer and Pachpatte which involves the Riemann function in place of exponential function. The results obtained can be used in the analysis of various problems in the theory of partial differential and integral equations.
Chapter 4 deals with the $n$ independent variable generalizations of the integral inequalities given in Lemmas 1-5. The inequalities established in this Chapter can be used to study the boundedness, uniqueness, continuous dependence and other problems in the theory of nonlinear partial differential and integrodifferential equations of the hyperbolic type.

In Chapter 5, we use the method of Young [60] to obtain further generalizations of the integral inequalities recently established by Pachpatte [30, 32, 44 and 46] in $n$ independent variables which can be used in some problems in the theory of partial differential equations involving $n$ independent variables.

Chapter 6 deals with the two independent variable generalizations of the well-known integral inequality resulting from Z.Opial [29] and some of its generalizations due to Beesack and Das [9]. Examples are also given to illustrate that the constant obtained in two independent variable generalization of Opial's inequality is the best possible.