TWO INDEPENDENT VARIABLE GENERALIZATIONS
OF OPIAL’S INEQUALITY

1. INTRODUCTION

In 1960, Z. Opial [29] proved the following integral inequality, in case $p = 1$.

If $y(t)$ is absolutely continuous with $y(0) = 0$, then for any $p > 0$,

$$
\int_0^a |y'(x)y^p(x)| \, dx \leq \frac{a^p}{p^{p+1}} \int_0^a |y'(x)|^{p+1} \, dx.
$$

Equality holds only if $y(t) = ct$ for some constant $c$.

After the discovery of the integral inequality resulting from Opial, a large number of papers have been appeared in the literature dealing with the integral inequalities of the Opial type and its various generalizations, of which we mention those of Beesack [8], Beesack and Das [9], Das [18], Yang [59], Boyd and Wong [11], Shum [51] and the references given therein. Recently, Das and Rachpate [19] have established some integral inequalities which are the n-dimensional analogues of the well-known integral inequalities of Z. Opial and others.
However, the constant obtained in the $n$ independent variable generalization of Opial's inequality in [19] is not the best possible. In this chapter we establish some interesting integral inequalities which are the two independent variable analogues of the Opial's inequality and some of its generalizations. The constants obtained in all the theorems are the best possible constants.

2. MAIN RESULTS

In this section we establish some interesting two independent variable generalizations of the integral inequalities of Z. Opial [29] and C. S. Yang [59].

The following theorem is a two independent variable generalization of the well-known Opial's inequality given in Lemma 10.

**Theorem 1.** Let $f(x,y)$ and its partial derivatives $f_x(x,y), f_y(x,y), f_{xy}(x,y)$ be absolutely continuous functions on $[0,a] \times [0,b]$, $f(x,0) = f(a,y) = 0$. Then

\[
(6.1) \quad \int_0^a \int_0^b |f(x,y)f_{xy}(x,y)| \, dx \, dy \leq \frac{ab}{4} \int_0^a \int_0^b |f_{xy}(x,y)|^2 \, dx \, dy.
\]
where the constant \( \frac{ab}{4} \) is the best possible. Equality holds in (6.1) if and only if \( f(x,y) = cxy \) for some constant \( c \).

**Proof.** For \((x,y) \in [0,a] \times [0,b]\), define

\[
(6.2) \quad s(x,y) = \int_0^x \int_0^y |f_{st}(s,t)| \, ds \, dt, \quad s(x,0) = s(0,y) = 0.
\]

Then

\[
(6.3) \quad s_{xy}(x,y) = |f_{xy}(x,y)|.
\]

We note that

\[
(6.4) \quad |f(x,y)| = |\int_0^x \int_0^y f_{st}(s,t) \, ds \, dt| \leq \int_0^x \int_0^y |f_{st}(s,t)| \, ds \, dt = s(x,y).
\]

From (6.3) and (6.4) we have

\[
(6.5) \quad \int_0^a \int_0^b |f(x,y)f_{xy}(x,y)| \, dxdy \leq \int_0^a \int_0^b s(x,y)s_{xy}(x,y) \, dxdy
\]

\[
= \int_0^a \int_0^b \left\{ s(x,y)s_{xy}(x,y) + s(x,y)s_y(x,y) \right\} \, dxdy
\]

\[
- \int_0^a \int_0^b s_x(x,y)s_y(x,y) \, dxdy
\]

\[
= \int_0^a \int_0^b \{ s(x,y)s_z(x,y) \} \, dxdy
\]

\[
- \int_0^a \int_0^b s_x(x,y)s_y(x,y) \, dxdy
\]

\[
= I_1 - I_2.
\]
Now

\[ I_1 = \int_0^a \int_0^b \left\{ s(x,y) s_x(x,y) \right\} \, dx \, dy \]

\[ = \int_0^a s(x,b) s_x(x,b) \, dx \]

\[ = \frac{s^2(a,b)}{2} . \]

We observe that \( s \) is an increasing function of \( x \) and \( y \), \( s_x \) is an increasing function of \( y \) and \( s_y \) is an increasing function of \( x \). For fixed \( s \) in \([0, a]\) and fixed \( t \) in \([0, b]\), and \(0 \leq s \leq x \leq a\), \(0 \leq t \leq y \leq b\), we observe that

\[ 1 \geq \frac{z(x,y)}{s(a,b)} , \]

and

\[ I_2 = \int_0^a \int_0^b \frac{s(x,y)}{s(a,b)} s_x(x,y) s_y(x,y) \, dx \, dy \]

\[ \geq \int_0^a \int_0^b \frac{s(x,y)}{s(a,b)} s_x(x,y) s_y(x,y) \, dx \, dy \]

\[ \geq \int_0^a \int_0^b \frac{s(x,t)}{s(a,b)} s_x(x,t) s_y(x,y) \, dx \, dy \]

\[ = \left( \int_0^a \frac{s(x,t) s_x(x,t)}{s(a,b)} \, dx \right) \left( \int_0^b \frac{s(s,y) s_y(s,y)}{s(a,b)} \, dy \right) \]

\[ = \frac{s^2(a,t)}{2 s(a,b)} \frac{s^2(a,b)}{2 s(a,b)} . \]
Since $s$ and $t$ are arbitrary in $[0, a]$ and $[0, b]$ respectively and the maximum of the right-hand side in the above inequality is attained at $s = a$ and $t = b$, we have

$$I_2 = \frac{s^2(a, b)}{4}$$

Using (6.6) and (6.7) in (6.6), we have

$$\int_0^a \int_0^b |f(x, y)| f(x, y) \, dx \, dy \leq \frac{s^2(a, b)}{4}.$$ 

Now from (6.2) we have

$$s(a, b) = \int_0^a \int_0^b |f_{xy}(x, y)| \, dx \, dy,$$

and by Schwartz's inequality

$$s^2(a, b) \leq ab \left( \int_0^a \int_0^b |f_{xy}(x, y)|^2 \, dx \, dy \right).$$

Using this bound in (6.3) we obtain the desired inequality in (6.1).

A slightly general version of Theorem 1 is embodied in the following theorem.

**Theorem 2.** Let $f(x, y)$ and its partial derivative $f_x(x, y)$, $f_y(x, y)$, $f_{xy}(x, y)$ be absolutely continuous functions on $[0, a] \times [0, b]$, $f(x, 0) = f(x, b) = 0$ and $p > 0$. Then
\[(6.9) \int_{0}^{a} \int_{0}^{b} |f(x,y)|^p |f_{xy}(x,y)| \, dx \, dy \leq \frac{a^2 b^p}{(p+1)^2} \int_{0}^{a} \int_{0}^{b} |f_{xy}(x,y)|^{p+1} \, dx \, dy.\]

Equality holds in (6.9) if and only if \( f(x,y) = cxy \) for some constant \( c \).

**Proof.** Let \( f(x,y) \) be the same function as in (6.2). Then from (6.3) and (6.4) we have

\[(6.10) \int_{0}^{a} \int_{0}^{b} |f(x,y)|^p |f_{xy}(x,y)| \, dx \, dy \leq \int_{0}^{a} \int_{0}^{b} s^p(x,y)z_{xy}(x,y) \, dx \, dy \]

\[= \int_{0}^{a} \int_{0}^{b} \left\{ s^p(x,y)z_{xy}(x,y) + ps^{-1}(x,y)z_x(x,y)a_y(x,y) \right\} \, dx \, dy \]

\[-\int_{0}^{a} \int_{0}^{b} ps^{-1}(x,y)z_x(x,y)a_y(x,y) \, dx \, dy \]

\[= \int_{0}^{a} \int_{0}^{b} \left\{ s^p(x,y)z_x(x,y) \right\} \, dx \, dy \]

\[-\int_{0}^{a} \int_{0}^{b} ps^{-1}(x,y)z_x(x,y)a_y(x,y) \, dx \, dy \]

\[= I_3 - I_4.\]

Now, as in Theorem 1 we obtain

\[(6.11) \quad I_3 \leq \int_{0}^{a} \int_{0}^{b} \frac{a}{\delta y} \left\{ s^p(x,y)z_x(x,y) \right\} \, dx \, dy \leq \frac{s^p(a,b)}{(p+1)}.\]

By monotonic behavior of \( s, z_x \) and \( a_y \) and for \( s \) and \( t \) as in Theorem 1, we observe that
\[ s^{p-1}(x, y) \geq \frac{s^p(x, y)}{s(a, b)} \cdot 1 \geq \frac{s^p(x, y)}{s^p(a, b)} \]

and

\[ I_4 = \int_0^a \int_0^b \frac{s^p(x, y)}{s^p(a, b)} s_x(x, y) s_y(x, y) \, dx \, dy \]

\[ \geq p \int_0^a \int_0^b \frac{s^p(x, y)}{s^p(a, b)} s_x(x, y) \frac{s^p(x, y)}{s^p(a, b)} s_y(x, y) \, dx \, dy \]

\[ \geq p \int_0^a \int_0^b \frac{s^p(x, t)}{s^p(a, b)} s_x(x, t) \frac{s^p(s, y)}{s^p(a, b)} s_y(s, y) \, dx \, dy \]

\[ = p \left( \int_0^a \frac{s^p(x, t) s_x(x, t)}{s^p(a, b)} \, dx \right) \left( \int_0^b \frac{s^p(s, y) s_y(s, y)}{s^p(a, b)} \, dy \right) \]

\[ = \frac{p}{(p+1)^2} \frac{s^{p+1}(a, b)}{s^p(a, b)} \frac{s^{p+1}(s, b)}{s^p(a, b)} . \]

Since \( s \) and \( t \) are arbitrary in \([0, a]\) and \([0, b]\) respectively and the maximum of the right-hand side in the above inequality is attained at \( s = a \) and \( t = b \), we have

\[ (6.12) \quad I_4 = \frac{p}{(p+1)^2} s^{p+1}(a, b). \]

Using (6.11) and (6.12) in (6.10), we have

\[ (6.13) \quad \int_0^a \int_0^b |f(x, y)|^p |f_{xy}(x, y)| \, dx \, dy \leq \frac{s^{p+1}(a, b)}{(p+1)^2} . \]
From (6.2), we have

\[ s(a, b) = \int \int_{o}^{a, b} |f_{xy}(x, y)| \, dx \, dy , \]

and by Holder’s inequality with indices \((p+1)/p\) and \(p+1\),

\[ s^{p+1}(a, b) \leq a^{p} b^{p} \int \int_{o}^{a, b} |f_{xy}(x, y)|^{p+1} \, dx \, dy . \]

Using this bound in (6.13) we obtain the desired inequality in (6.9).

We note that in the case \(p = 1\) our Theorem 2 reduces to Theorem 1.

We next establish a two independent variable generalization
of the integral inequality given by G.S. Yang [59] which in
turn is a generalization of Opial’s inequality.

**Theorem 7.** Let \( f(x, y) \) and its partial derivatives
\( f_{x}(x, y), f_{y}(x, y), f_{xy}(x, y) \) be absolutely continuous functions
on \([0, a] \times [0, b]\), \( f(x, 0) = f(0, y) = 0 \) and if \(p, q \geq 1\), then

\[ \int \int_{o}^{a, b} |f(x, y)|^{p} |f_{xy}(x, y)|^{q} \, dx \, dy \leq \frac{2^{q/(p+q)}}{(p+q)^{2}} \int \int_{o}^{a, b} |f_{xy}(x, y)|^{p+q} \, dx \, dy . \]

Equality holds in (7.14) if and only if \( f(x, y) = c \) or else \( q = 1 \)
and \( f(x, y) = cxy \) for some constant \( c \).
PROOF. For \((x, y) \in [0, a] \times [0, b]\), define

\[
(6.15) \quad s(x, y) = \int_0^x \int_0^y |f_{st}(s, t)|^{p+q} \, ds \, dt, \quad s(x, 0) = s(0, y) = 0.
\]

Then

\[
(6.16) \quad |f_{xy}(x, y)|^q = s_{xy}^{q/(p+q)}(x, y).
\]

We note that

\[
|f(x, y)| \leq \int_0^x \int_0^y |f_{st}(s, t)| \, ds \, dt.
\]

Now applying Hölder's inequality with indices \((p+q)/(p+q-1)\) and \((p+q)\) we obtain

\[
(6.17) \quad |f(x, y)| \leq \left( \int_0^x \int_0^y |f_{st}(s, t)| \, ds \, dt \right)^{(p+q)/(p+q-1)} \left( \int_0^x \int_0^y |f_{st}(s, t)|^{p+q} \, ds \, dt \right)^{1/(p+q)} = s(x, y)^{(p+q-1)/(p+q)} \, s(x, y)^{1/(p+q)}.
\]

From (6.16) and (6.17) we have

\[
\int_0^a \int_0^b |f(x, y)|^p |f_{xy}(x, y)|^q \, dx \, dy \leq \int_0^a \int_0^b |s(x, y)|^{p(p+q-1)/(p+q)} \, s(x, y)^{p/(p+q)} \, dx \, dy \leq \int_0^a \int_0^b \{ s(x, y) \, s_{xy}^{q/(p+q)}(x, y) \} \, dx \, dy.
\]

Again, applying Hölder's inequality with indices \(\frac{p+q}{p}\) and \(\frac{p+q}{q}\)

we have
\[ (6.18) \ \int \int_0^b \left| f(x, y) \right|^p \left| f_{xy}(x, y) \right|^q \, dx \, dy \]

\[ \leq \frac{a^p b^p}{(p+q)^{2p/(p+q)}} \left( \int \int_0^a s(x, y) s_{xy}(x, y) \, dx \, dy \right)^{(p+q)/q} \cdot \]

Let

\[ I_5 = \int \int_0^b \int_0^a s(x, y) s_{xy}(x, y) \, dx \, dy. \]

Setting \( p/q = a \), we have

\[ I_5 = \int \int_0^b \int_0^a s(x, y) s_{xy}(x, y) \, dx \, dy \]

Now following the same steps as in the proof of Theorem 2 we obtain the estimate on \( I_5 \) such that

\[ I_5 = \frac{a^{a+1}}{s(a^a b)} - a \frac{a^{a+1}}{s(a^a b) (a+1)^2} \]

\[ = \frac{a^2}{(p+q)^2} s(a^a b) \]

Substituting this value of \( I_5 \) in (6.18) and observing \( s(a^a b) \) from (6.15) we obtain the desired inequality (6.14).

We note that in the case \( p=q=1 \), (6.14) reduces to the result (6.1) of Theorem 1.
In this section we establish two independent variable
generalizations of some of the inequalities given by Beesack and
Das [9] which in turn are the further generalizations of Opial's
inequality.

**Theorem 4.** Let \( p, q \) be real numbers such that \( pq > 0 \), and
either \( p+q > 1 \), or \( p+q < 0 \), and let \( x = r(x, y) \), \( y = s(x, y) \) be
nonnegative, measurable functions on \((a, b) \times (c, d)\) such that
\[
\int_a^b \int_c^d \frac{x^y}{(p+q-1)} \, dx \, dy < \infty,
\]
and
\[
(6.19) \quad \chi(x, y, p, q)
\]
\[
= \left( \frac{a^x}{p+q} \right) \left( \frac{b^y}{p+q} \right) \left[ \frac{x^y (p+q)}{a^x b^y} - \frac{(q/p)}{a^x b^y} \right] \left( \frac{1}{(p+q-1)} \right) \int_a^b \int_c^d \frac{1}{x^y} \, dx \, dy
\]
if finite, where \(-\infty < a < b < \infty, -\infty < c < d < \infty\). If \( f(x, y) \) and
its partial derivatives \( f_x(x, y), f_y(x, y), f_{xy}(x, y) \) be absolutely
continuous functions on \([a, b] \times [c, d]\), \( f(x, b) = f(a, y) = 0 \),
and \( f_x(x, y), f_y(x, y), f_{xy}(x, y) \) do not change signs on \((a, b) \times (c, d)\),
then
\[
(6.20) \quad \int_a^b \int_c^d |f(x, y)| \, dx \, dy \leq \chi(x, y, p, q) \int_a^b \int_c^d |f_{xy}(x, y)| \, dx \, dy
\]
Equality holds in \((6.20)\) if and only if either \( q > 0 \) and
\( f(x, y) = 0 \), or
\begin{equation}
\tag{6.21}
\mathcal{L} = a_1 \int_a^b \frac{(q-1)/(p+q-1)}{x^{q-1} \left(\int_a^b \frac{1}{x^{p-1}} \right)^{1/(p+q-1)}} \exp\left\{\frac{p(1-q)}{q} \right\} \, \text{d}x
\end{equation}

and

\begin{equation}
\tag{6.22}
f(x,y) = a_2 \int_a^b \frac{x^y - 1/(p+q-1)}{x^{1/(p+q)}} \, \text{d}x
\end{equation}

for some constants \( a_1 \geq 0, \ a_2 \) real.

\textbf{Proof.} Since \( f_x(x,y), f_y(x,y) \) and \( f_{xy}(x,y) \) do not change signs on \((a,x) \times (b,y)\) we have

\begin{equation}
\tag{6.23}
|f(x,y)| = \int_a^b \left| f_{st}(a,t) \right| \, \text{d}t
\end{equation}

Then it follows from Hölder's inequality with indices \( (p+q)/(p+q-1) \) and \( (p+q) \) that

\begin{equation}
\tag{6.24}
\int_a^b \left| f_{st}(a,t) \right| \, \text{d}t \leq \left( \int_a^b \left| f_{st}(a,t) \right|^\frac{p+q}{p} \, \text{d}t \right)^{1/(p+q)}
\end{equation}

if \( p+q > 1 \), while

\begin{equation}
\tag{6.25}
\int_a^b \left| f_{st}(a,t) \right| \, \text{d}t \geq \left( \int_a^b \left| f_{st}(a,t) \right|^\frac{1}{p+q} \, \text{d}t \right)^{p+q}/(p+q)
\end{equation}
if either \( p+q < 0 \) or \( 0 < p+q < 1 \). Taking the case \( p+q > 1 \), we suppose first that \( p > 0, q > 0 \). Then,

\[
(6.25) \quad |f(x,y)|^p \leq \left( \int_a^b x \left[ 1/(p+q-1) \right] \frac{p(p+q-1)}{(p+q)} ds \right)^{\frac{p}{p+q}} \int_a^b f_{xy}(x,t)^{p+q} \frac{p}{p+q} \quad \text{for} \quad a \leq x \leq X, \quad b \leq y \leq Y.
\]

Now let

\[
s(x,y) = \int_a^b x (s,t)^{p+q} \quad \text{so}
\]

\[
x_{xy}(x,y) = x(x,y) |f_{xy}(x,y)|^{p+q},
\]

and

\[
|f_{xy}(x,y)|^q = z = \left\{ \frac{q}{(p+q)} \right\} (x_{xy}(x,y))^{q/(p+q)}.
\]

As \( \ell (x,y) \) is nonnegative measurable function on \((a,X) \times (b,Y)\), we have

\[
(6.27) \quad \int_a^X \int_b^Y |f(x,y)|^p |f_{xy}(x,y)|^q dx dy \leq \int_a^X \int_b^Y \left[ \left( \frac{q}{p+q} \right) \right] \left( \frac{1}{p+q-1} \right) \frac{p(p+q)}{(p+q)} \left( \frac{p}{p+q} \right) s(x,y) n_{xy}(x,y) dx dy
\]

\[
\leq \int_a^X \int_b^Y \left( \frac{q}{p+q} \right) \left( \frac{1}{p+q-1} \right) \frac{p(p+q)}{(p+q)} \left( \frac{p}{p+q} \right) s(x,y) n_{xy}(x,y) dx dy.
\]
Now applying Hölder's inequality, with indices \( (p+q)/p \) and \( (p+q)/q \), we obtain

\[
(6.20) \quad \int_a^b \int_a^b |f(x,y)| \mathcal{P}_{xy}(x,y) \, dxdy \leq \left[ \int_a^b \int_a^b \frac{f(x,y)}{p} \, dx \right]^{(p/q)} \left[ \int_a^b \int_a^b \frac{\mathcal{P}_{xy}(x,y)}{q} \, dxdy \right]^{(q/p)}
\]

By following the same steps as in the proof of Theorem 3 we obtain

\[
\int_a^b \int_a^b \frac{f(x,y)}{p} \mathcal{P}_{xy}(x,y) \, dxdy = \frac{2}{(p+q)^2} \mathcal{Q}^{(p+q)/q} (x,y) .
\]

Now substituting this value in (6.20) we obtain the desired inequality (6.20).

Similarly, if \( p < 0 \) and \( q < 0 \), then (6.20) again follows from (6.25) and (6.25). As above, since \( \frac{p+q}{p} > 1 \) and \( \frac{p+q}{q} > 1 \) again, we obtain inequality (6.20).

It only remains to prove the assertions concerning (6.29) and (6.22). Now, equality holds in (6.20) only if it holds in (6.24) or (6.25) and (6.27) in which Hölder's inequality is applied leading to (6.20); that is, only if both
\[ f(x, y) = a_2 \int_a^b \frac{x}{p+q-1} \, ds \, dt, \]

since \( f(a, y) = f(x, b) = y \). Thus we have established the equation of \( s(x, y) \), the second of the above conditions reduces to

\[ f(x, y) = a_2 \int_a^b \frac{x}{p+q-1} \, ds \, dt. \]

That is,

\[ \frac{B+q}{A} = \frac{(p+q) / p - (q/p) - 1/(p+q-1)}{(p+q) / p - (q/p) - 1/(p+q-1)}, \]

so

\[ \frac{B}{A} = \frac{(p+q) / p - (p+q) / p (1-q) / p (p+q-1)}{p(1-q)} \]

which is the equation in (6.22). Finally, if \( \lambda \) is given by (6.21), we have
\[(6.29) \quad I_q(x, y, p; q)\]

\[
= \left(\frac{2q}{p+q}\right) \alpha_q \left[ \int_a^b \int_a^b dxdy - \left(\frac{1}{p+q-1}\right) \int_a^b \int_a^b \frac{1}{x} \frac{1}{y} dxdy \right]
\]

If we let,

\[
R(z, y) = \int_a^b \frac{1}{x} dx \quad dxdy, \quad R(a, y) = R(x, b) = 0,
\]

then

\[
R_{xy}(x, y) = \left\{ \frac{1}{x+y-1} \right\}.
\]

We observe that \( R \) is an increasing function of \( x \) and \( y \); \( R_x(x, y) \) is an increasing function of \( y \) and \( R_y(x, y) \) is an increasing function of \( x \), so that

\[
\int_a^b \int_a^b \frac{1}{x+y-1} dx \quad dxdy = \int_a^b \int_a^b \frac{1}{x+y-1} dx \quad dxdy
\]

\[
= \int_a^b \int_a^b R_{xy}(x, y) dx \quad dxdy
\]

\[
= \left(\frac{q}{p+q}\right)^2 \frac{2}{(p+q)/q} \quad \int_a^b \int_a^b \frac{1}{x+y-1} dx \quad dxdy
\]

\[
= \left(\frac{q}{p+q}\right)^2 \left(\frac{2}{(p+q)/q}\right) \quad \int_a^b \int_a^b \frac{1}{x+y-1} dx \quad dxdy
\]
Using this in (6.29), we have

\[ x_1(x, y, p, q) = a_1 \left( \frac{a}{p+q} \right)^2 \left( \int \int r^{x+y-\{1/(p+q-1)\}} \, dx \, dy \right)^{p/q} \]

which is finite. Similarly, choosing \( f(x, y) \) as in (6.22) we have

\[ \int \int r |f(x, y)|^{p+q} \, dx \, dy \]

\[ = \int \int r |a_2|^{p+q} \, dx \, dy \]

\[ = |a_2|^{p+q} \left( \int \int r^{x+y-\{1/(p+q-1)\}} \, dx \, dy \right) < \infty, \]

completing the proof of the theorem.

We only state the next theorem, since its proof is the same as that of Theorem 4, with \([a, c] \) replaced by \([x, c] \) and \([b, y] \) replaced by \([y, d] \) throughout.

**Theorem 5.** Let \( p, q \) be real numbers satisfying the conditions as in Theorem 4, and let \( r = r(x, y), \quad \lambda = \lambda(x, y) \) be nonnegative measurable functions on \((x, c) \times (y, d)\), where

\[-\infty \leq x < c \leq \infty, \quad -\infty < y < d \leq \infty, \quad \text{such that} \quad \int \int r \, dx \, dy < \infty \]

and the constant
(6.30) \[ K_2(x,y,p,q) = \frac{(\frac{\partial}{\partial x})}{p+q-1} \int \int \ell(x,y) \cdot x(x,y) \left( \int \int x(x,y) dxdy \right) \]

is finite. If \( f(x,y) \) and its partial derivatives \( f_x(x,y), f_y(x,y), f_{xy}(x,y) \) be absolutely continuous functions on \([x,a] \times [y,b]\), \( f(x,a) = f(x,b) = 0 \) and \( f_x(x,y), f_y(x,y), f_{xy}(x,y) \) do not change signs on \((x,a) \times (y,b)\), then

\[
(6.31) \quad \int \int |f(x,y)|^p |f_{xy}(x,y)|^q dxdy
\]

\[
\leq K_2(x,y,p,q) \int \int x(x,y)^{p+q} dxdy.
\]

Equality holds in (6.31) if and only if either \( q > 0 \) and \( f(x,y) = 0 \), or

\[
(6.32) \quad \lambda = \alpha_3 \int \int \frac{(\frac{\partial}{\partial x})}{p+q-1} \cdot \left\{ \frac{1}{(p+q-1)} \right\} \cdot x(x,y)^p(1-q)/q \cdot dxdy,
\]

and

\[
(6.33) \quad f(x,y) = \alpha_4 \int \int \frac{\partial}{\partial x} \cdot \left\{ \frac{1}{(p+q-1)} \right\} \cdot dxdy,
\]

for some constants \( \alpha_3 > 0, \alpha_4 \) real.
4. **EXAMPLES**

1. Taking \( \gamma = 1 \), \( \lambda = 1 \) and \( a = b = c \) in (6.20), we obtain

\[
(6.34) \int_0^Y \int_0^X |f(x,y)|^p |f_{xy}(x,y)|^q \, dx \, dy
\]

\[
\leq \frac{2a/(p+q)}{(p+q)^2} Y^p \int_0^X \int_0^Y |f_{xy}(x,y)|^p \, dx \, dy
\]

if \( p \leq q \leq 1 \), which is the bound established in (6.34). In case \( p = q = 1 \), the inequality (6.34) reduces to the inequality (6.1).

2. Taking \( \gamma = 1 \), \( \lambda = \{(x-a)(y-b)\} \) in Theorem 4, we have

\[
X_q(x,y,p+q) = \left(\frac{a}{p+q}\right)^2 \left[ (x-a)(y-b) \right]^{p+q}
\]

and (6.20) reduces to

\[
(6.35) \int_0^Y \int_0^X \{(x-a)(y-b)\}^{p+q} |f(x,y)|^p |f_{xy}(x,y)|^q \, dx \, dy
\]

\[
\leq \left(\frac{a}{p+q}\right)^2 \{(x-a)(y-b)\}^{p+q} \int_0^X \int_0^Y |f_{xy}(x,y)|^p \, dx \, dy
\]

Equality holds in (6.35) if and only if either \( q > 0 \) and \( f(x,y) = 0 \), or \( f(x,y) = \lambda (x-a)(y-b) \). In the latter case, each side in (6.35) is equal to

\[
\lambda^{(p+q)} \left(\frac{a}{p+q}\right)^2 \{(x-a)(y-b)\}^{(p+q)/q}
\]
3. Taking \( r = \{(x-a)(y-b)\} \) for \( l \neq 1 \) in Theorem 4, we have

\[
K_q(x,y,p,q) = \left( \frac{q}{p+q} \right)^{2(1-p)} \{ (x-a)(y-b) \}^{p/(p+q)}
\]

and (6.20) reduces to

\[
(6.36) \int \int_{a \ b} \frac{X \ Y}{2} \left| f(x,y) \right|^p \left| f_{xy}(x,y) \right|^q \ dxdy
\leq \left( \frac{q}{p+q} \right)^{2(1-p)} \{ (x-a)(y-b) \}^{p/(p+q)}
\]

\[
\cdot \int \int_{a \ b} \{ (x-a)(y-b) \} \frac{p(p+q-1)}{(p+q)} \left| f_{xy}(x,y) \right|^{p+q} \ dxdy.
\]

Equality holds in (6.36) if and only if either \( q > 0 \) and \( f(x,y) = 0 \) or
\[
f(x,y) = A \{ (x-a)(y-b) \}^{q/(p+q)}.
\]

In the latter case, each side in (6.36) is equal to

\[
A^{(p+q)} \left( \frac{q}{p+q} \right)^{2q} (x-a)(y-b).
\]

5. INEQUALITIES WITH LOWER BOUNDS

In this section we establish two independent variable generalizations of some of the inequalities obtained by Beesack and Das [9] for lower bounds in case of one independent variable.
THEOREM 6. Let $p, q$ be real numbers such that either $p < 0$ and $p + q > 1$, or $p > 0$ and $p + q < 0$. Let $x = x(x, y)$, $y = y(x, y)$ be nonnegative measurable functions on $(a, x) \times (b, y)$, where $-\infty < a < x < \infty$, $-\infty < b < y < \infty$, and the constant $K(x, y, p, q)$ defined by (6.19) is finite. If $f(x, y)$ and its partial derivatives $f_x(x, y), f_y(x, y), f_{xy}(x, y)$ be absolutely continuous functions on $[a, x] \times [b, y]$, $f(x, b) = f(a, y) = 0$ and $f_x(x, y), f_y(x, y), f_{xy}(x, y)$ do not change signs on $(a, x) \times (b, y)$, then

\[
\int_a^b \int_{f_x(x, y)}^{f_y(x, y)} |f_{xy}(x, y)| \, dx \, dy \geq K(x, y, p, q) \int_a^b \int_{f_x(x, y)}^{f_y(x, y)} |f_{xy}(x, y)| \, dx \, dy.
\]

There is equality in (6.37) if and only if $\mathcal{L}$ and $f(x, y)$ are as defined in (6.21) and (6.22).

PROOF. We consider first the case when $p + q > 1$. If, in addition, $p < 0$, then (6.26) yields

\[
|f(x, y)|^p \geq \left( \int_a^b \int_{f_x(x, y)}^{f_y(x, y)} |f_{xy}(x, y)| \, dx \, dy \right)^{p/(p+q)}.
\]

As $\mathcal{L}(x, y)$ is nonnegative measurable function on $(a, x) \times (b, y)$, we have
\[ l \left| f(x,y) \right|^p \left| f_{xy}(x,y) \right|^q \geq l \int_a^b \left( \int_x^y \left( \int_a^b \right) \right) \frac{1}{p+q-1} \left( \int_a^b \frac{1}{p+q-1} \right) \frac{p(p+q-1)}{(p+q)} \]

\[ \frac{p}{(p+q)} - \frac{q}{(p+q)} , \]

where

\[ s(x,y) = \int_a^b \int_x^y |f_{st}(s,t)|^p \, ds \, dt. \]

Then

\[ \int_a^b \int_x^y |f(x,y)|^p \left| f_{xy}(x,y) \right|^q \, dx \, dy \]

\[ \geq l \int_a^b \int_x^y \left( \int_a^b \left( \int_a^b \right) \right) \frac{1}{p+q-1} \left( \int_a^b \frac{1}{p+q-1} \right) \frac{p(p+q-1)}{(p+q)} \]

\[ \frac{p}{(p+q)} - \frac{q}{(p+q)} , \]

\[ s(x,y) \cdot (s_{xy}(x,y))^{q/(p+q)} \]

Now applying Holder's inequality with indices \((p+q)/p\) and \((p+q)/q\), (note that \((p+q)/q\) lies between 0 and 1) we obtain the desired inequality in (6.37).

Similarly, if \( p > 0 \) and \((p+q) < 0\), then (6.25) yields (6.38). As \( l(x,y) \) is nonnegative measurable function on \((a,x) \times (b,Y)\), again, Holder's inequality with indices \((p+q)/p\) and \((p+q)/q\), (note that \(0 < (p+q)/q < 1\) still holds) leads to (6.37). Equality holds in (6.37) if and only if it holds in (6.24) or (6.25) and in Holder's inequality leading to (6.37); that is, if and only if, \( l \) and \( f(x,y) \) are given by (6.21) and (6.22).
We only state the next theorem, since its proof is the same as that of Theorem 6, with \([a, x]\) replaced by \([x, c]\) and \([b, y]\) replaced by \([y, d]\) throughout Theorem 6.

**Theorem 7.** Let \(p, q\) be real numbers satisfying the same conditions as in Theorem 6, and \(\lambda = \lambda(x, y)\) be nonnegative measurable functions on \((x, c) \times (y, d)\), where \(-\infty \leq x < c \leq -\infty\), \(-\infty \leq y < d \leq -\infty\), such that

\[
\frac{c - d}{\lambda} \leq \frac{1}{(p+q-1)} \quad \text{and} \quad \lambda_2(x, y, p, q)
\]
defined by (6.30) is finite. If \(f(x, y)\) and its partial derivatives \(f_x(x, y), f_y(x, y), f_{xy}(x, y)\) be absolutely continuous on \([x, c] \times [y, d]\), \(f(x, c) = f(c, y) = 0\), and \(f_x(x, y), f_y(x, y)\), \(f_{xy}(x, y)\) do not change sign on \((x, c) \times (y, d)\), then

\[
(6.39) \quad \int_x^c \int_y^d \left| f(x, y) \right|^p \left| f_{xy}(x, y) \right|^q \, dx dy
\]

\[
\geq \lambda_2(x, y, p, q) \int_x^c \int_y^d \left| f_{xy}(x, y) \right|^{p+q} \, dx dy.
\]

Equality holds in (6.39) if and only if

\[
(6.40) \quad \lambda = \alpha \left( \int_x^c \int_y^d \lambda_2(x, y, p, q) \, dx dy \right)^{(q-1)/(p+q-1)}\left( \int_x^c \int_y^d \lambda_2(x, y, p, q) \, dx dy \right)^{(p-1)/(p+q-1)}
\]

\[
\alpha = \left( \int_x^c \int_y^d \lambda_2(x, y, p, q) \, dx dy \right)^{(p-1)/(p+q-1)}\left( \int_x^c \int_y^d \lambda_2(x, y, p, q) \, dx dy \right)^{(q-1)/(p+q-1)}.
\]
\[ f(x, y) = a_4 \int \int_{x y} \frac{e^d}{y^{(y+\alpha-1)}} \, dx \, dy, \]

for some constants \( a_2(\geq 0), a_4 \) real.

In concluding this chapter we note that the results obtained here can be generalized to \( n \) independent variables. In our future work we wish to obtain further generalizations in two or more independent variables of results obtained by D.T. Shum \([51]\) and others.