CHAPTER I
INTRODUCTION

1.1 SUMMARY

In the present study, seven classes of autoregressive moving average processes (hereafter ARMA processes) with marginals, namely Mittag-Leffler, semi-Mittag-Leffler, geometric stable, geometric semi-stable, generalised geometric stable, discrete Mittag-Leffler and generalised Pareto distributions are introduced and their properties are studied. The concept of specialised class $L$ property is introduced and a characterization of the class of random variables possessing this property is obtained. A new class of discrete distributions, discrete Mittag-Leffler distributions, is introduced and studied. A general stationary Markov process with innovation is introduced and its properties are studied. A new class of continuous distributions, generalised Pareto distributions, is introduced and its role in the study of stationary minification ARMA process is established.

Chapter II deals with autoregressive moving average Mittag-Leffler process (hereafter MLARMA($p,q$) process). The solution of the first order autoregressive (hereafter AR(1)) equation (hereafter we write w.p. for with probability)

$$X_n = \begin{cases} 
\rho X_{n-1} & \text{w.p. } \rho^\alpha \\
\rho X_{n-1} + \varepsilon_n & \text{w.p. } (1-\rho^\alpha)
\end{cases}$$

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\( 0 \leq \rho < 1, 0 < \alpha \leq 1 \) where \( \{\varepsilon_n, n \geq 1\} \) is a sequence of independent and identically distributed positive valued random variables with \( \varepsilon_0 \overset{d}{\sim} \mathcal{E}_1 \) (hereafter \( \overset{d}{\sim} \) stands for is distributed as) is obtained. This gives rise to first order autoregressive semi-Mittag-Leffler process (hereafter SMLAR(1) process).

The innovation distribution of \( p^{th} \) order autoregressive Mittag-Leffler process (hereafter MLAR(p) process) is derived in Chapter III. This gives rise to a class of distributions which is a proper subclass of the class L distributions. Specialised class L distributions are introduced and characterized. A \( p^{th} \) order autoregressive process (hereafter AR(p) process) with innovation is introduced and characterized.

First order autoregressive geometric semi-stable process (hereafter GSSAR(1) process) and first order autoregressive moving average geometric semi-stable process (hereafter GSSARMA(1,1) process) are studied in Chapter IV. The class L property of geometric semi-stable distributions is established. It is shown that geometric semi-stable distributions arise as solution of a stationary AR(1) equation. GSSARMA(1,1) process is introduced and a class of stationary first order autoregressive moving average models (hereafter ARMA(1,1) models) is characterized. The role of generalised geometric stable distributions in the study of stationary ARMA(p,q) process is established.

In chapter V, a new class of discrete distributions is introduced which is named as discrete Mittag-Leffler (hereafter
DML. It is seen that the DML distribution is the discrete analogue of Mittag-Leffler distribution and is a generalisation of the geometric distribution. The mathematical origin of DML distributions is presented and some of its theoretical properties are studied. The role of this distribution in reliability studies is established. Autoregressive - moving average discrete Mittag-Leffler process (hereafter DMLARMA(p,q) process) is developed.

Among class L distributions, geometric semi-stable distribution is characterized through time series property. Discrete Mittag-Leffler distribution is characterized among discrete class L distributions. These results are presented in Chapter VI.

A general stationary Markov process with innovation is introduced and studied in Chapter VII. A characterization of first order autoregressive semi-Pareto process (hereafter SPAR(1) process) is given. A new class of continuous distributions, which we named as generalised Pareto is introduced. This generalised Pareto distribution is a generalisation of the semi-Pareto distribution which itself is a generalisation of the well-known Pareto type III distribution. It is shown that generalised Pareto distribution is the solution of a stationary minification ARMA(ν,ν) equation.

Some simulation studies of both first order autoregressive Mittag-Leffler and first order autoregressive geometric stable processes are done in Chapter VIII. Simulation
study of first order autoregressive Mittag-Leffler (MLAR(1)) process is done and the MLAR(1) process is compared with the first order autoregressive exponential process (hereafter EAR(1) process) of Gaver and Lewis (1980). The role of MLAR(1) process when the EAR(1) model becomes unrealistic is thus stressed.

Simulation study of first order autoregressive geometric stable process (hereafter GSAR(1) process) is also done. Monte Carlo study of first order autoregressive discrete Mittag-Leffler process (hereafter DMLAR(1) process) is done and the DMLAR(1) process is compared with the first order autoregressive geometric process (hereafter GAR(1) process) of McKenzie (1986) through the behaviour of simulated sample paths. Finally, as an illustration the MLAR(1) model is fitted to weekly stream flows of Kallada river of Kerala.

1.2 AUTOREGRESSIVE - MOVING AVERAGE PROCESSES

A stochastic model which can be extremely useful in the representation of certain practically occurring series is the autoregressive model. In this model the current value of the process is expressed as a finite, linear aggregate of previous values of the process and a shock $\varepsilon_n$. Let the values of a process at equally spaced times $n$, $n-1$, $\ldots$ be $X_n$, $X_{n-1}$, $\ldots$ respectively. Then

$$X_n = b_1 X_{n-1} + b_2 X_{n-2} + \ldots + b_p X_{n-p} + \varepsilon_n \quad (1.2.1)$$
is called an autoregressive process of order $p$ (AR($p$)).

Another kind of model, of great practical importance in the representation of observed time series is the finite moving average process. Here $X_n$ is linearly dependent on a finite number $q$ of previous $\varepsilon$'s. Thus

$$X_n = \varepsilon_n - a_1 \varepsilon_{n-1} - a_2 \varepsilon_{n-2} - \ldots - a_q \varepsilon_{n-q}$$ (1.2.2)

is called a moving average process of order $q$ (MA($q$)).

To achieve greater flexibility in fitting of actual time series, it is sometimes advantageous to include both autoregressive and moving average terms in the model. This leads to autoregressive-moving average (ARMA($p,q$)) model

$$X_n = b_1 X_{n-1} + b_2 X_{n-2} + \ldots + b_p X_{n-p} + \varepsilon_n - a_1 \varepsilon_{n-1} - \ldots - a_q \varepsilon_{n-q}$$ (1.2.3)

A detailed discussion of ARMA($p,q$) process is given in Box and Jenkins (1970).

Lawrance and Lewis (1982 a,b) noted that Box-Jenkins or ARMA models, wellknown in time series analysis are mainly suitable for modelling time series with Gaussian marginal distributions, and are inappropriate for positive variables, such as response times at a computer terminal. Since many hydrologic series are discrete and non-Gaussian in nature, non-Gaussian distributions provide a better fit than the usual Gaussian distributions. Lawrance and Kottegoda (1977) stressed the use of ARMA process with non-Gaussian marginal distributions, in
describing the behaviour of hydrologic series, especially river flow time series. The autoregressive-moving average exponential process (hereafter EARMA(p,q) process) was introduced and studied by Lawrance and Lewis (1980). The EARMA(p,q) model is useful in modelling stream flows and other hydrological data (see Hutton (1990)). Sim (1987) fitted a mixed gamma ARMA(1,1) model to monthly stream flows of the Perak river in Malaysia and observed that the simulated data have a close resemblance to the historical data. Lawrance and Lewis (1982a) introduced autoregressive-moving average mixed exponential process, for stochastic modelling in operations analysis. Gibson (1986) discussed the use of the first order autoregressive Laplace process in image source modelling in data compression task.


Cline and Brockwell (1985) discussed the linear prediction of ARMA process with infinite variance. Davis and Resnick (1989) studied the properties of max-ARMA process. After introducing a natural metric between two jointly max-stable random variables, Davis and Resnick (1989) considered the prediction problem of max-ARMA process.

A brief description of certain classes of distributions, which we need subsequently, is given in the following sections.

1.3 CLASS L DISTRIBUTIONS

The infinitely divisible laws are the possible limit

$$\sum_{k=1}^{n_k}$$

laws for $n \to \infty$ of $\sum_{k=1}^{n_k} X_{nk}$

where the random variables $X_{nk}$

1) for each fixed $n$ are independent.

2) satisfy the uniformly asymptotic negligibility (hereafter u.a.n) condition, 

i.e. $\max_{1 \leq k \leq n} P\{|X_{nk}| > \varepsilon\} \to 0$, $n \to \infty$, for every $\varepsilon > 0$.

An interesting subclass of such limit laws consists of those which corresponds to $X_{nk} = \frac{X_k}{b_n}$ where $X_k$, $k=1,2,...$ are independent random variables and $b_n$ are suitable constants such that $b_n \to \infty$, $\frac{b_{n+1}}{b_n} \to 1$, when $n \to \infty$. 

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DEFINITION 1.3.1. Let \( \{X_n, n \geq 1\} \) be a sequence of independent random variables and let \( \{b_n\} \) be a sequence of positive real numbers such that \( X_n^k = \frac{b_n}{b_n} \) are u.a.n. Let \( S_n = \sum_{k=1}^{n} X_k \), \( n \geq 1 \). Then the class of laws which appear as the weak limit of the distribution of the sums \( \frac{S_n}{b_n} - a_n \), \( n \geq 1 \) where \( a_n \) and \( b_n \) are suitably chosen constants such that \( b_n > 0 \) is known as the class L.

If the distribution function which is the limit of the distribution functions of the sums \( \frac{S_n}{b_n} - a_n \), \( n \geq 1 \) of independent infinitesimal summands \( X_n^k = \frac{b_n}{b_n} \) is proper, then as \( n \to \infty \), \( b_n \to \infty \) and \( \frac{b_{n+1}}{b_n} \to 1 \).

In terms of characteristic functions the class L property can be stated as follows:

For a distribution function \( F \) with characteristic function \( f \) to belong to the class L, there should exist a characteristic function \( f_\alpha \) for every \( \alpha, 0 < \alpha < 1 \) such that \( f(t) = f(\alpha t) f_\alpha(t) \). Here \( f_\alpha(t) \) is the characteristic function of an infinitely divisible distribution. (See Gnedenko and Kolmogrov (1968), page 149).

1.4 DISCRETE CLASS L DISTRIBUTIONS

Steutel and van Harn (1979) proposed analogues for the concept of class L property for distributions on the non-negative integers. The discrete class L distributions share the basic properties with their continuous counterparts.
DEFINITION 1.4.1. A distribution $F$ on $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ with probability generating function $P$ is said to be in discrete class $L$ if there exists a probability generating function $P_\alpha$ for every $\alpha$, $\emptyset < \alpha < 1$ such that

$$P(z) = P(1-\alpha+\alpha z) P_\alpha(z) , \quad |z| \leq 1 .$$

Equation (1.4.1) can be written in terms of random variables as follows:

$$X \overset{d}{=} \alpha \circ X' + X_\alpha ,$$

where $\alpha \circ X'$ and $X_\alpha$ are independent, and $X'$ is distributed as $X$. Here $\alpha \circ X$ is defined (in distribution) by its probability generating function $P(1-\alpha+\alpha z)$, or by

$$\alpha \circ X = \sum_{j=1}^{X} N_j ,$$

where $P(N_j = 1) = 1 - P(N_j = 0) = \alpha$, all random variables being independent.

REMARK 1.4.1. $\alpha \circ X \in \mathbb{N}_0$, with $1 \circ X = X$, $\emptyset \circ X = \emptyset$ and $E(\alpha \circ X) = \alpha E(X)$ (See Steutel and van Harn (1979)). □

REMARK 1.4.2. A probability generating function $P$ is in discrete class $L$ if and only if it has the form
\[ P(z) = \exp \left\{ -c \int_{z}^{1} \frac{1-G(u)}{1-u} \, du \right\}, \]

where \( c > 0 \) and \( G \) is a (unique) probability generating function with \( G(0) = 0 \). (See Steutel and van Harn (1979)). \( \square \)

1.5 STABLE AND SEMI-STABLE DISTRIBUTIONS

Stable distributions play a constantly increasing role as a natural generalization of normal distribution.

DEFINITION 1.5.1. A distribution function \( F \) is said to be stable if for every \( b_1 > 0 \), \( b_2 > 0 \), \( c_1 \) and \( c_2 \) reals, there exist a \( b > 0 \) and a \( c \in \mathbb{R} = (-\infty, \infty) \) such that

\[ F\left(\frac{x-c_1}{b_1}\right) \ast F\left(\frac{x-c_2}{b_2}\right) = F\left(\frac{x-c}{b}\right), \]

where \( \ast \) denotes the convolution operation. \( \square \)

In order that a distribution function \( F \) to be stable, it is necessary and sufficient that its characteristic function \( f(t) \) can be expressed as

\[ \ln f(t) = i\mu t - c |t|^\alpha \left[ 1 + i\gamma \frac{t}{|t|} \omega(t, \alpha) \right] \]

where \( \mu, \alpha, \gamma \) and \( c \) are constants with \( c \geq 0 \), \( 0 < \alpha \leq 2 \), \( |\gamma| \leq 1 \), and

\[ \omega(t,\alpha) = \begin{cases} \tan \frac{\pi \alpha}{2} & \text{if } \alpha \neq 1 \\ \frac{2}{\pi} \ln|t| & \text{if } \alpha = 1. \end{cases} \]

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Here $\alpha$ is called the exponent or index of $F$. (See Gnedenko and Kolmogrov (1968), page 164).

The parameter $\alpha$ is an index of peakedness and $\gamma$ is an index of skewness. We can say that $\alpha$ and $\gamma$ together determine the shape of the distribution. $c$ is a scale parameter and without loss of generality we can let the location parameter $\mu = 0$. If $1 < \alpha \leq 2$, the mathematical expectation is finite and if $\gamma = 0$ we have a symmetric stable distribution, $\mu$ being mean, median or mode. $\mu$ has no obvious interpretation when $0 < \alpha \leq 1$ with $\gamma \neq 0$. A larger class of distributions, namely the semi-stable distributions was studied by Pillai (1971).

**DEFINITION 1.5.2.** A distribution function $F$ with characteristic function $f(t)$ is said to be semi-stable if $f(t) = (f(bt))^a$, $0 < |b| < a$, $a > 1$.

Pillai (1971) showed that the semi-stable laws are infinitely divisible.

**1.6 GEOMETRICALLY INFINITELY DIVISIBLE DISTRIBUTIONS**

Klebanov, Maniya and Melamed (1984) introduced the concept of geometric infinite divisibility. Pillai and Sandhya (1990) showed that the class of distributions with complete monotone derivative (hereafter c.m.d.) is a proper subclass of the class of geometrically infinitely divisible (hereafter g.i.d.) distributions.
DEFINITION 1.6.1. A random variable $Y$ is said to be g.i.d., if for every $p \in (0, 1)$, there is a sequence of independent and identically distributed random variables $X_1^{(p)}, X_2^{(p)}, \ldots$ such that

$$Y \overset{d}{=} \sum_{j=1}^{\nu_p^{(p)}} X_j,$$

where $P(\nu_p = k) = p (1-p)^{k-1}, k = 1, 2, \ldots$ and $Y, \nu_p$ and $X_j^{(p)} (j = 1, 2, \ldots)$ are independent.

DEFINITION 1.6.2. A random variable $Y$ is geometrically strictly stable if for every $p \in (0, 1)$ there exist a constant $c = c(p) > 0$ such that $Y \overset{d}{=} c(p) \sum_{j=1}^{\nu_p} Y_j$, $Y, Y_j (j = 1, 2, \ldots)$ are independent and identically distributed and independent of $\nu_p$, where $\nu_p$ has geometric distribution with mean $\frac{1}{p}$.

Klebanov et al. (1984) derived a criterion for identifying g.i.d. distributions.

THEOREM 1.6.1. A random variable $X$ with characteristic function $f(t)$ is g.i.d. if and only if,

$$\phi(t) = \exp \left(1 - \frac{1}{f(t)} \right)$$

is infinitely divisible.

An equivalent theorem is found in Pillai (1990b).

THEOREM 1.6.2. A random variable $X$ on $(0, \infty)$ is g.i.d. if and only if its Laplace transform is of the form $\frac{1}{1 + \psi(\lambda)}$ where $\psi(\lambda)$ has c.m.d. with $\psi(0) = 0$. 

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Pillai and Sandhya (1990) has the following theorem.

**THEOREM 1.6.3.** The class of g.i.d. distributions is a proper subclass of the class of infinitely divisible distributions.

### 1.7 MITTAG-LEFFLER DISTRIBUTIONS

Pillai (1990a) introduced the Mittag-Leffler distribution and studied its properties.

**DEFINITION 1.7.1.** A positive valued random variable $X$ is said to follow the Mittag-Leffler distribution (and write $X \sim \text{ML}(\alpha)$) if its distribution function is

$$F_\alpha(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k \alpha}{\Gamma(1 + k \alpha)}, 0 < \alpha \leq 1, x > 0.$$  \hspace{1cm} (1.7.1)

The Laplace transform of the distribution (1.7.1) is

$$\phi(\lambda) = \frac{1}{1 + \lambda^\alpha}, 0 < \alpha \leq 1, \lambda \geq 0.$$  

Mittag-Leffler distribution is a generalization of the exponential distribution in the sense that for $\alpha = 1$ we get the exponential distribution ($\text{exp}(1)$).

**REMARK 1.7.1.** The Mittag-Leffler distribution function $F_\alpha(x)$ belongs to class L. (See Pillai and Sabu George (1984)).
REMARK 1.7.2. $F_{\alpha}(x)$ is g.i.d. (See Pillai (1990a)).

REMARK 1.7.3. $F_{\alpha}(x)$ is normally attracted to the stable distribution with exponent $\alpha$, $0 < \alpha < 1$. (See Pillai (1990a)).

REMARK 1.7.4. g.i.d. distributions on $(0, \infty)$ can be obtained as a generalization of $F_{\alpha}(x)$. (See Fujita (1993)).

REMARK 1.7.5. If $U$ is an exponential random variable with Laplace transform $\frac{1}{1 + \lambda}$ and $Y$, distributed independently of $U$ as stable with Laplace transform $e^{-\lambda^\alpha}$ then $X = U^{1/\alpha} Y$ has Mittag-Leffler distribution $F_{\alpha}(x)$. (See Pillai (1988b)).

REMARK 1.7.6. If $X$ and $Y$ are independent random variables following Mittag-Leffler ($\alpha < 1$) and exponential (1) distributions respectively then both $X/Y$ and $Y/X$ are distributed as $P_{III}(1, \frac{1}{\alpha})$ (where $P_{III}$ stands for Pareto type III). (See Pillai (1988b)).

REMARK 1.7.7. For $0 < \mu < \alpha \leq 1$, $E(X^\mu) = \frac{\Gamma(1-\mu/\alpha) \Gamma(1+\mu/\alpha)}{\Gamma(1-\mu)}$. (See Pillai (1990a)).

REMARK 1.7.8. The canonical representation of $F_{\alpha}(x)$ is given by

$$-\ln \phi(\lambda) = \int_0^\infty \frac{1-e^{-\lambda x}}{x} f(x) \, dx$$
where

\[ f(x) = \alpha \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{\Gamma(1+ka)}. \] (See Pillai (1990a)).

**REMARK 1.7.9.** The two parameter Mittag-Leffler distribution is defined by the Laplace transform \( \phi(\lambda) = \frac{\alpha}{\lambda + \alpha} \), \( 0 < \alpha < 1 \), \( s > 0 \) and is denoted by \( ML(s,\alpha) \). (See Kalyanaraman, Sabu George and Pillai (1991)).

**REMARK 1.7.10.** The Mittag-Leffler distribution belongs to the class of subexponential distributions. (See Jayakumar (1989)).

1.8 GEOMETRIC SEMI-STABLE DISTRIBUTIONS

Linnik (1953) introduced the class of distributions which is defined by the characteristic function

\[ f(t) = \frac{1}{1 + |t|^{\alpha}}, \quad (1.8.1) \]

\( 0 < \alpha \leq 2 \), \( -\infty < t < \infty \). Distributions with characteristic function (1.8.1) is called \( \alpha \)-Laplace since \( \alpha = 2 \) corresponds to the Laplace distribution. Sabu George and Pillai (1987) found the expression for the densities of \( \alpha \)-Laplace distributions for rational values of \( \alpha \) in terms of Meijer’s G-function.

Pillai (1985) introduced a larger class of distributions of which \( \alpha \)-Laplace distributions form a subclass, namely ‘semi \( \alpha \)-Laplace distributions’. Based on Klebanov et.al. (1984), the
semi $\alpha$-Laplace distributions of Pillai (1985) can be called geometric semi-stable distributions and the $\alpha$-Laplace distributions can be called geometrically strictly stable distributions. Since we are concerned with strictly stable laws, hereafter we call geometrically strictly stable distributions as geometric stable distributions.

**DEFINITION 1.8.1.** The distribution function $F$ of a random variable $X$ is called geometric semi-stable if its characteristic function

$$f(t) = \frac{1}{1+\psi(t)}$$

and $\psi(t)$ has the property

$$\psi(t) = a \psi(bt), \ 0 < b < 1$$

where $a$ is the unique solution of

$$ab^\alpha = 1, \ 0 < \alpha \leq 2.$$  

The numbers $b$ and $\alpha$ are called the order and exponent of the distribution $F$ respectively. If $b_1$ and $b_2$ are the orders of the distribution function $F$ such that $\frac{\ln b_1}{\ln b_2}$ is irrational, then $F$ is reduced to $\alpha$-Laplace (geometric stable).

**REMARK 1.8.1.** For a geometric semi-stable distribution not
concentrated at the origin, there exists a > 0 such that |ψ(t)| ≥ a |t|^α. (See Pillai (1985)).

REMARK 1.8.2. For a geometric semi-stable distribution with exponent α, E|X|^δ exists for 0 ≤ δ < α. (See Pillai (1985)).

REMARK 1.8.3. For the geometric stable distributions, for 0 < δ < α,

\[ E|X|^\delta = \frac{2^\delta \Gamma(1+\delta/\alpha) \Gamma(1-\delta/\alpha) \Gamma((1+\delta)/2) \Gamma(1-\delta/2)}{\pi^{1/2}}. \]

(See Sabu George and Pillai (1987)).

REMARK 1.8.4. A random variable X with geometric stable distribution admits the representation

\[ X \overset{d}{=} U^{1/\alpha} Y \]

where U is an exponential random variable and Y is symmetric stable with exponent α, distributed independently of U. (See Feller (1966), page 439).

REMARK 1.8.5. The geometric stable distributions are normally attracted to the stable distribution with exponent α, 0 < α ≤ 2. (See Gnedenko and Kolmogrov (1968), page 181).
1.9 GENERALISED GEOMETRIC STABLE DISTRIBUTIONS

Ramachandran and Rao (1968) introduced generalised stable distributions and studied its properties. Pillai (1988a) introduced generalised geometric stable distributions as geometric version of generalised stable distributions.

DEFINITION 1.9.1. The distribution of a random variable $X$ is said to be generalised geometric stable if it has characteristic function

$$f(t) = \frac{1}{1 + \psi(t)}$$

and $\psi(t)$ has the property

$$\psi(t) = \sum_{i=1}^{r} a_i \psi(b_i t),$$

$0 < b_i < 1$, $a_i > 0$ ($i = 1, 2, \ldots, r$),

where

$$\sum_{i=1}^{r} a_i b_i^\alpha = 1, \; 0 < \alpha \leq 2.$$

REMARK 1.9.1. The class of geometric semi-stable distributions is contained in the class of generalised geometric stable distributions which in turn is contained in the class of g.i.d. distributions. (See Pillai (1988a) and Pillai and Sandhya (1990)).
Pillai (1988a) obtained the following theorem.

**Theorem 1.9.1.** Characteristic function of a generalised geometric stable distribution has the representation

\[
f(t) = \frac{1}{1 + it\mu + \sigma^2 t^2 - \int A(x) dM(x) - \int A(x) dN(x)},
\]

which is the analogue of the Levy's representation of infinitely divisible characteristic functions, where \( A(x) = e^{itx} - 1 - \frac{itx}{1+x^2} \).

Then

1. If \( \alpha \neq 2 \), then \( \sigma = 0 \)
2. If \( \alpha = 2 \), then \( M = N = 0 \)
3. If \( \alpha > 2 \) or \( \alpha \leq 0 \), then \( \sigma = 0 \), \( M = N = 0 \) and in this case
   \[
f(t) = 1
\]
4. If \( h(u) = u^\alpha N(u) \) and \( k(u) = |u|^\alpha M(u) \), then \( h \) and \( k \) satisfy the functional equation

\[
h(u) = \sum_{i=1}^{r} a_i b_1^\alpha h(u/b_1), \quad k(u) = \sum_{i=1}^{r} a_i b_1^\alpha k(u/b_1).
\]

1.10 Semi-Pareto Distributions

Pillai (1991a) introduced the semi-Pareto distributions and studied the properties of first order autoregressive process with semi-Pareto marginals.
DEFINITION 1.10.1. The distribution function $F$ of a random variable $X$ is called semi-Pareto and write $X \overset{d}{=} P_s(\alpha, p)$, if its survival function

$$
\bar{F}(x) = P(X > x) = \frac{1}{1 + \eta(x)}
$$

where $\eta(x)$ has the property

$$
\eta(x) = \frac{1}{p} \eta(p^{1/\alpha} x),
$$

$\alpha > 0$, $0 < p < 1$.

The first order autoregressive semi-Pareto (SPAR(1)) process is defined as

$$
X_n = \begin{cases} 
  p^{-1/\alpha} X_{n-1} & \text{w.p. } p \\
  \min(p^{-1/\alpha} X_{n-1}, \varepsilon_n) & \text{w.p. } (1-p)
\end{cases}
$$

where $\{\varepsilon_n, n \geq 1\}$ are independent and identically distributed as $P_s(\alpha, p)$, with $X_0 \overset{d}{=} \varepsilon_1$.

Pillai (1991a) characterized semi-Pareto distributions through maximum stability property and obtained a canonical representation for a special class of semi-Pareto.

REMARK 1.10.1. A non-degenerate distribution $F$ with positive support with $\alpha(F) = \inf \{x : F(x) > 0\} = 0$ and $\omega(F) = \sup\{x : F(x) < 1\} = \infty$ is maximum stable with respect to
The geometric distribution with parameter $p$ if and only if $F$ is semi-Pareto. (See Pillai (1991a)).

**REMARK 1.10.2.** For $0 < \alpha \leq 1$ a distribution is semi-Pareto if and only if its survival function has the representation

$$F(x) = \frac{1}{1 + \sum_{n=-\infty}^{\infty} \frac{1}{p^n} \eta_{\theta}(p^{n/\alpha} x)}$$

where

$$\eta_{\theta}(x) = \int_{p^{1/\alpha}}^{1} \frac{1-e^{-xu}}{u} dP(u),$$

and

$$P(u) = \int_{p^{1/\alpha}}^{u} \frac{du}{x^{\alpha+1}}. \quad (\text{See Pillai (1991a))}. \quad \Box$$