3.1 INTRODUCTION

A general class of absolutely continuous distributions (which includes the exponential, gamma, half-normal, chi, Weibull and many more important distributions as special cases) was studied by Stacy (1962). The probability density function of the generalized gamma distribution is denoted by \( g(x; \alpha, \beta, \delta) \), and is given by the following.

\[
g(x; \alpha, \beta, \delta) = \beta^{-\alpha \beta} \left\{ \Gamma(\alpha) \right\}^{-1} x^{\alpha \beta - 1} \exp\left(-\frac{x}{\delta}\right)^\beta, \quad x > 0
\]

where \( \alpha, \beta \) and \( \delta \) are positive real numbers. We may call the generalized gamma distribution with probability density function (3.1.1) as the \( \text{GG}(\alpha, \beta, \delta) \) distribution.

Let \( X_1 < X_2 < \cdots < X_n \) be the order statistics of a random sample of size \( n \) drawn from \( \text{GG}(\alpha, \beta, \delta) \). The moment of order \( k \) of the \( m \)-th order statistic \( X_{m:n; \alpha, \beta, \delta} \) is denoted by \( h_{m:n; \alpha, \beta, \delta}^{(k)} \), \( (1 \leq m \leq n) \) and is defined by the following equation.

\[
h_{m:n; \alpha, \beta, \delta}^{(k)} = m(m) \int_0^\infty x^{k} \{G(x; \alpha, \beta, \delta)\}^{m-1} \{1-G(x; \alpha, \beta, \delta)\} \cdot g(x; \alpha, \beta, \delta)dx
\]

where \( G(x; \alpha, \beta, \delta) \) is the distribution function of (3.1.1) and \( k=0,1,2,\ldots; \) where we may define \( h_{m:n; \alpha, \beta, \delta}^{(0)} = 1 \).
Explicit expression for the moments of order statistics arising from $GG(a, \beta, \delta)$ is not found in the available literature. The integral defining the moment $h_m^{(k)}_{m:n,a,\beta,\delta}$ is also not seen tractable easily to get an explicit closed form expression for it. Recurrence relations specialized to the moments of order statistics arising from $GG(a, \beta, \delta)$ are also not seen derived in the literature. Hence the recurrence relations and identities available on the moments of order statistics arising from an arbitrary continuous distribution alone can be used presently, in the evaluation of the moments $h_m^{(k)}_{m:n,a,\beta,\delta}$, $m=1,2,\ldots,n$ and $k=1,2,\ldots$. Notice that from the available recurrence relations for any continuous distribution (such as relations (1.2.8) to (1.2.10)), to obtain the single moments of order $k$ of the order statistics, one require the direct evaluation of one single integral for each $k$, provided the moments of order statistics arising from lower sample sizes are also known. For the above reasons, there is a need for further study on the moments of order statistics arising from $GG(a, \beta, \delta)$. This chapter focuses on such a study.

By adopting a technique similar to that in Thomas (1991), Thomas and Moothathu (1991) have derived certain recurrence relations connecting the moments of different orders of the largest order statistic arising from a gamma distribution. Proceeding as in the above cases, the author has again arrived at, the same type of recurrence relations on the moments of the largest order statistic arising from a chi distribution with $p$ degrees of freedom. All these similar results obtained for the gamma and chi family of distributions lead the author to establish the first type of recurrence relation on the moments
of different orders of the largest order statistic arising from a
generalised gamma distribution. The result obtained by this approach
is given in theorem 3.2.1.

As pointed out already, many authors have studied the moment
problem of order statistics by utilizing certain functional relations
that exist between the distribution function and its probability density
function, for example see, Balakrishnan and Joshi (1981,1982), Joshi
(1978,1982) and Shah (1966,1970). However in all these cases and in
most of the other available recurrence relations on the moments of
order statistics, either moments of different orders of the order sta­
tistics of the same sample or moments of order statistics of
different samples of a fixed population are connected. But it may be
advantageous to have relations which connect a moment of an order
statistic arising from a distribution belonging to a family of distrib­
utions with the moments of order statistics arising from some other
distribution belonging to the same family.

In continuation of the research initiated by the author (see, Thomas, 1991 and Thomas and Moothathu, 1991) and based on an iden­
tity connecting the distribution functions of gamma distributions with
shape parameters $\alpha$ and $\alpha-1$ (for $\alpha>1$), we could establish another type
of recurrence relation connecting the moments of largest order statistics
arising from these distributions. As the same pattern of results were
obtained for the chi family of distributions also, the author was
motivated then, to extend the above type of relation to a larger class
of distributions, namely generalized gamma distributions. This result
is presented in theorem 3.2.2.
Obviously the recurrence relations on moments of order statistics arising from $GG(\alpha, \beta, \delta)$ are also true for the moments of order statistics arising from the subclass of distributions such as gamma, chi and chi-square distributions. But further modifications or improvements on the results are possible to these subclass of distributions. In section 3.3, the recurrence relations pertaining to the moments of the largest order statistic arising from the gamma distribution and their applications are presented. In section 3.4, we use the fact that chi distribution with one degree of freedom is a truncated form of the standard normal distribution, to obtain the recurrence relations on the moments of the largest order statistic arising from this distribution.

In section 3.5 we discuss, how the already existing recurrence relations for the moments of order statistics together with new results presented in sections 3.2 and 3.4 can be utilized to minimize the number of direct evaluation of the integrals for the moments of order statistics of the chi distribution with $p$ degrees of freedom and of the chi-square distribution with $p$ degrees of freedom.

3.2 RECURRENCE RELATIONS FOR MOMENTS OF EXTREMES FROM GENERALIZED GAMMA DISTRIBUTION

For the generalized gamma distribution with probability density function $g(x; \alpha, \beta, \delta)$ and distribution function $G(x; \alpha, \beta, \delta)$, we define

$$M(u, v, w; \alpha, \beta, \delta) = \int_{0}^{\infty} x^{u} [G(x; \alpha, \beta, \delta)]^{v} \{g(x; \alpha, \beta, \delta)\}^{w} \, dx \quad (3.2.1)$$

where $u$ is a non-negative real number, $v$ and $w$ are integers such
\( v \geq 0 \) and \( w \geq 1 \). To establish the recurrence relations we need the following two lemmas.

**Lemma 3.2.1.** If \( M(u,v,w;\alpha,\beta,\delta) \) is defined as in (3.2.1), then it is finite for those values of \( u \) and \( w \) such that \( u+(\alpha \beta-1)w+1 > 0 \).

**Proof.** Since \( G(x;\alpha,\beta,\delta) \leq 1 \) for all real \( x \), it is obvious that,
\[
M(u,v,w,\alpha,\beta,\delta) \leq \int_0^\infty x^u \{ g(x;\alpha,\beta,\delta) \}^w \, dx
\]
\[
= \frac{\beta^w}{\delta^{\alpha \beta w}} \left\{ \Gamma(\alpha) \right\}^w \int_0^\infty x^{u+(\alpha \beta-1)w} e^{-w(x/\delta)^\beta} \, dx
\]
(by the monotone property of integrals, as \( w \geq 1 \)). Then we have,
\[
M(u,v,w;\alpha,\beta,\delta) \leq \beta^{w-1} \delta^{-w+1} \int_0^\infty t \left\{ [u+(\alpha \beta-1)w+1]/\beta \right\}^{-1} e^{-t} \, dt
\]
which is finite as \( u+(\alpha \beta-1)w+1 > 0 \). This proves the lemma.

**Remark 3.2.1.** Notice that a sufficient condition for the integral on the right side of (3.2.1) to be finite is that \( u \geq w-1 \), which is obvious from the above lemma.

**Lemma 3.2.2.** If \( M(r,s,t;\alpha,\beta,\delta) \) is defined as in (3.2.1) where \( r \) is a positive real number, \( s \) and \( t \) are integers such that \( r \geq t-1 \), \( s \geq 0 \) and \( t \geq 2 \), then for \( k=1,2,\ldots, t-1 \), we have the following.
\[ M(r, s, t; \alpha, \beta, \delta) = \frac{k+1}{\sum_{j=1}^{k} J(k, j)} \cdot M(r-k+j \beta-\beta, s+k, t-k; \alpha, \beta, \delta) \quad (3.2.2) \]

where, \( J(0, 1)_{r, s, t; \alpha, \beta, \delta} = 1 \)

\[ J(k, 1)_{r, s, t; \alpha, \beta, \delta} = -(s+k)^{-1}[a_{\alpha}(t-k)+r+1-t] \cdot J(k-1, 1)_{r, s, t; \alpha, \beta, \delta} \]

\[ J(k, j)_{r, s, t; \alpha, \beta, \delta} = (s+k)^{-1}(t-k)\delta^{-\beta} \cdot J(k-1, j-1)_{r, s, t; \alpha, \beta, \delta} - (s+k)^{-1}[\beta(at+j-ak-1) + r+1-t] \cdot J(k-1, j)_{r, s, t; \alpha, \beta, \delta} \]

for \( j=2, 3, \ldots, k \) and

\[ J(k, k+1)_{r, s, t; \alpha, \beta, \delta} = (s+k)^{-1}(t-k)\delta^{-\beta} \cdot J(k-1, k)_{r, s, t; \alpha, \beta, \delta} \]

**Proof.** Since by assumption \( r \geq t-1 \), all \( M(u, v, w; \alpha, \beta, \delta) \) involved on the right and left sides of equation (3.2.2) for appropriate values of \( u, v \) and \( w \) satisfy the condition \( u \geq w-1 \) of remark 3.2.1. Hence all \( M(u, v, w; \alpha, \beta, \delta) \) involved in (3.2.2) are finite. If \( a \) is any real number and \( b \) is a positive integer, then it is straightforward to verify the following result involving the probability density function \( g(x; \alpha, \beta, \delta) \).

\[ \frac{d}{dx} \left\{ x^a \left[ g(x; \alpha, \beta, \delta) \right]^b \right\} = \left\{ a(a+b-1)b - \beta \delta^{-\beta} b x^\delta \right\} x^{a-1} \left\{ g(x; \alpha, \beta, \delta) \right\}^b \quad (3.2.3) \]

Also we may write \( M(r, s, t; \alpha, \beta, \delta) \) in the following form.

\[ M(r, s, t; \alpha, \beta, \delta) = \int_0^\infty \{ G(x; \alpha, \beta, \delta) \}^s g(x; \alpha, \beta, \delta) \{ x^r [ g(x; \alpha, \beta, \delta) ]^{t-1} \} \]
On integrating by parts the expression in the right side of the above equation we get the following relation.

\[
M(r,s,t;\alpha,\beta,\delta) = -(s+1)^{-1} \int_0^\infty \{G(x;\alpha,\beta,\delta)\}^{s+1} \times \frac{d}{dx} \{x^r\{g(x;\alpha,\beta,\delta)\}^{t-1}\} \, dx
\]  

(3.2.4)

The relations (3.2.4), (3.2.3) and (3.2.1) together imply the following equation.

\[
M(r,s,t;\alpha,\beta,\delta) = -(s+1)^{-1} [(r+(\alpha \beta-1)(t-1)) M(r-1,s+1,t-1;\alpha,\beta,\delta) \\
+ (t-1) \beta \delta^{-\beta} (s+1)^{-1} M(r+\beta-1,s+1,t-1;\alpha,\beta,\delta)]
\]  

(3.2.5)

Equation (3.2.5) can be rewritten as.

\[
M(r,s,t;\alpha,\beta,\delta) = J(1,1) \frac{d}{dr} (r,s,t;\alpha,\beta,\delta) M(r-1,s+1,t-1;\alpha,\beta,\delta) \\
+ J(1,2) \frac{d}{dr} (r,s,t;\alpha,\beta,\delta) M(r+\beta-1,s+1,t-1;\alpha,\beta,\delta)
\]  

(3.2.6)

where the J's are as defined in the lemma. Now we shall complete the proof of (3.2.2) by transfinite induction. To prove the result assume that (3.2.2) is true for \( k=i \), where \( 1 \leq i \leq t-2 \). Then we write, the following.

\[
M(r,s,t;\alpha,\beta,\delta) = \sum_{j=1}^{i+1} J^{(i,j)} \frac{d}{dr} (r,s,t;\alpha,\beta,\delta) M(r-i+\beta j - \beta,s+1,t-1;\alpha,\beta,\delta)
\]  

(3.2.7)

Now on using (3.2.5) in each term in the right side of (3.2.7) we get the result (3.2.2) for \( k=i+1 \). Clearly (3.2.6) is the result (3.2.2) for \( k=1 \). This completes the proof of the lemma.

\section*{Theorem 3.2.1.}

Let \( X_{n:n;\alpha,\beta,\delta} \) denote the largest order statistic of a sample of size \( n \geq 2 \) drawn from the generalized gamma distribution (3.1.1) and for \( a \geq 0 \) define,
\[ h(a) = E[X_n^{a}] \]

Then for any finite real \( r > n-1 \), we have the following result.

\[ h(n-r+(n-1)(\beta-1)) = \sum_{j=1}^{n-1} \binom{n-1}{i} h(r-n+j\beta-\beta+1) \]

where the \( J \)'s are constants defined in lemma 3.2.2.

**Proof.** When \( r > n-1 \), by direct evaluation of the integral defined by \( M(r,0,n;\alpha,\beta,\delta) \) we obtain the following.

\[ M(r,0,n;\alpha,\beta,\delta) = \frac{\beta^{n-1} \delta^{r-n+1} \Gamma((\alpha \beta n-n+r+1)/\beta)}{[\Gamma(\alpha)]^n n^{(\alpha \beta n-n+r+1-\beta)/\beta}} \]

In (3.2.2) put \( s=0, t=n \) and \( k=n-1 \) to obtain the following equation.

\[ M(r,0,n;\alpha,\beta,\delta) = \sum_{j=1}^{r} \binom{n-1}{j} M(r-n+j\beta-\beta,n-1,1;\alpha,\beta,\delta) \]

But from the definition of \( M(u,v,w;\alpha,\beta,\delta) \), we have the following relation.

\[ (v+1) M(u,v,1;\alpha,\beta,\delta) = h(u)_{v+1;v+1;\alpha,\beta,\delta} \]

where \( v \) is any non-negative integer. The relations (3.2.9) to (3.2.11) together imply the following equation.

\[ \frac{\beta^{n-1} \delta^{r-n+1} \Gamma((\alpha \beta n-n+r+1)/\beta)}{[\Gamma(\alpha)]^n n^{(\alpha \beta n-n+r+1-\beta)/\beta}} = \sum_{j=1}^{n} \binom{n-1}{j} h(r-n+j\beta-\beta+1) \]

From (3.2.12), we get the required result (3.2.8).

Now we once again use lemma (3.2.8) to prove the following theorem.
Theorem 3.2.2. Let $n \geq 2$ be an integer, $k$ be a real number such that $k \geq -\beta$ and define $h_{n:n;\alpha,\beta,\delta}^{(k)} = E(X_{n:n;\alpha,\beta,\delta}^k)$. Then for every $n \in \mathbb{N}$ we have the following result.

$$h_{n:n;\alpha,\beta,\delta}^{(k)} = \delta^{-\beta} \sum_{i=0}^{n-1} \left( \frac{1}{i+1} \right) (\beta)^{i-1} (\alpha-1)^{-i-1} \sum_{j=1}^{n-i-1} \{J(1,j)_{i+k+\beta,n-1-1,i+1,\alpha-1,\beta,\delta} \}$$

where the $J$'s are constants defined as in lemma 3.2.2.

Proof. For $x > 0$, integrate by parts the integral defining the distribution function $G(x; \alpha-1, \beta, \delta)$ corresponding to $g(x; \alpha-1, \beta, \delta)$ in (3.1.1) with $\alpha > 1$ to obtain the following.

$$G(x; \alpha, \beta, \delta) = G(x, \alpha-1, \beta, \delta) - (\alpha-1)^{-1} \beta x g(x; \alpha-1, \beta, \delta)$$

(3.2.14)

It is direct to verify the following.

$$g(x; \alpha, \beta, \delta) = (\alpha-1)^{-1} \delta^{-\beta} x^\beta g(x; \alpha-1, \beta, \delta)$$

(3.2.15)

To prove the theorem, consider the following.

$$h_{n:n;\alpha,\beta,\delta}^{(k)} = n \int_0^\infty x^k \left[ G(x; \alpha, \beta, \delta) \right]^{n-1} g(x; \alpha, \beta, \delta) \, dx$$

(3.2.16)

By using (3.2.14) and (3.2.15) in (3.2.16) we get the following.

$$h_{n:n;\alpha,\beta,\delta}^{(k)} = \frac{n}{(\alpha-1)^{\delta-\beta}} \int_0^\infty \left\{ x^k \right\} [G(x; \alpha-1, \beta, \delta) - \frac{x}{(\alpha-1)\beta} g(x; \alpha-1, \beta, \delta)]^{n-1} \times g(x; \alpha-1, \beta, \delta) \, dx$$

(3.2.17)

If we use the binomial expansion for the expression within the square bracket of (3.2.17) and use (3.2.1), we get the following equation.

$$h_{n:n;\alpha,\beta,\delta}^{(k)} = n \delta^{-\beta} \sum_{i=0}^{n-1} \left( \frac{1}{i+1} \right) (\beta)^{i-1} (\alpha-1)^{-i-1} \sum_{j=1}^{n-i-1} \{J(1,j)_{i+k+\beta,n-1-1,i+1,\alpha-1,\beta,\delta} \}$$

(3.2.18)
But under the condition \( k > -\beta \) and from lemma 3.2.1, each \( M(i+k+\beta, n-i-1, i+1; a-1, \beta, \delta) \) involved on the right side of (3.2.18) is finite. Then by using the result of lemma 3.2.2, we get the following relation (i.e., put \( r = i+k+\beta, s = n-i-1, t = i+1, k = 1 \) and replace \( a \) by \( a-1 \) in (3.2.2)).

\[
M(i+k+\beta, n-i-1, i+1; a-1, \beta, \delta) = \sum_{j=1}^{i+1} J^{(i,j)}_{i+k+\beta, n-i-1, i+1; a-1, \beta, \delta} M(k+j\beta, n-1, 1; a-1, \beta, \delta) \tag{3.2.19}
\]

Now the required result (3.2.13) follows by using (3.2.11) and (3.2.19) in (3.2.18).

The relations (3.2.8) and (3.2.13) are very useful in the evaluation of the moments of order statistics arising from several distributions belonging to the generalized gamma family of distributions. Note that if \( \beta \) is not an integer, then both (3.2.8) and (3.2.13) include the moments of fractional orders of the largest order statistics arising from the generalized gamma distributions.

But when \( \beta \) is an integer, these relations connect the moments of integer orders and hence can be used for the evaluation of moments of order statistics arising from \( g(x; a, \beta, \delta) \). In this case the relation (3.2.8) helps us to evaluate the moments of order \( k \) of \( X_{n:n; a, \beta, \delta} \) for \( k \geq (n-1)\beta \) in terms of its moments of lower orders. Hence an implication of the result (3.2.8) is that, in order to obtain the moments of any order of each order statistic of a sample of size \( n \) arising from \( g(x; a, \beta, \delta) \) for an integer value of \( \beta \), one requires the direct evaluation of at most \( (n-1)\beta - 1 \) integrals, provided the moments of order statistics arising from lower sample sizes are also known.
Clearly when $\beta$ is an integer and $\alpha > 1$, the relation (3.2.13) expresses the moment $h_{n:n;\alpha,\beta,\delta}^{(k)}$ in terms of the moments $h_{n:n;\alpha - 1,\beta,\delta}^{(k+j\beta)}$ for $j = 1, 2, \ldots, n$. Notice that when both relations (3.2.8) and (3.2.13) can be applied to evaluate any moment $h_{n:n;\alpha,\beta,\delta}^{(k)}$, we prefer (3.2.8) because of the following reasons. (i) The relation (3.2.8) includes only lower order moments $h_{n:n;\alpha,\beta,\delta}^{(k)}$ for $k_1 < k$. (ii) The number of $J$ coefficients involved in (3.2.8) is less when compared with (3.2.13). (iii) In (3.2.13) certain combinatorial coefficients are involved. Thus for known values of $\alpha$, $\beta$, and $\delta$ such that $\alpha > 1$, $\delta > 0$ and $\beta$ is an integer, we may proceed to evaluate the moments $h_{m:n;\alpha,\beta,\delta}^{(k)}$ for $1 \leq m \leq n$ and $k = 1, 2, \ldots$ in the following manner.

Let $\alpha_1 = \begin{cases} 1, & \text{if } \alpha \text{ is an integer} \\ \alpha - [\alpha], & \text{if } \alpha \text{ is not an integer} \end{cases}$

where $[\cdot]$ is the usual greatest integer function. Then first we evaluate directly the integrals for the moments $h_{n_1:n_1;\alpha_1,\beta,\delta}^{(k)}$ for $n_1 = 2, 3, \ldots, n$; $k = 1, 2, \ldots, (n_1 - 1) \beta - 1$ and use the relation (3.2.8) to obtain $h_{n_1:n_1;\alpha_1,\beta,\delta}^{(k)}$ for any $k \geq (n_1 - 1) \beta$ and $n_1 = 2, 3, \ldots, n$ (i.e., without evaluating any more integral for the moments directly). Now by using the relation (1.2.9), all moments $h_{m_1:n_1;\alpha_1,\beta,\delta}^{(k)}$ for $1 \leq m_1 \leq n_1$, $n_1 = 2, 3, \ldots, n$, and $k = 1, 2, \ldots$ can be obtained in a recursive manner. In the next stage, we apply (3.2.13) to get the moments $h_{n_1:n_1;\alpha_1 + 1,\beta,\delta}^{(k)}$ based on the moments of the largest order statistic of a sample of size $n_1$ arising from $g(x; \alpha_1, \beta, \delta)$, for $n_1 = 2, 3, \ldots, n$ and $k = 1, 2, \ldots, (n_1 - 1) \beta - 1$. Then by using (3.2.8), we may obtain $h_{n_1:n_1;\alpha_1 + 1,\beta,\delta}^{(k)}$ for $k \geq (n_1 - 1) \beta$ and $n_1 = 2, 3, \ldots, n$ without evaluating any more integral directly. Now we again use (1.2.9) with the already obtained moments $h_{n_1:n_1;\alpha_1 + 1,\beta,\delta}^{(k)}$.
for $n_1=2,3,\ldots,n$ and $k=1,2,\ldots$ to get all moments $h^{(k)}_{m_1:n_1;\alpha_1+1,\beta,\delta}$ for $1 \leq m_1 \leq n_1$, $n_1=2,3,\ldots,n$ and $k=1,2,\ldots$ of the order statistics arising from $g(x;\alpha_1+1,\beta,\delta)$. This procedure may be recursively followed till we get all moments $h^{(k)}_{m_2:n_2;\alpha,\beta,\delta}$ for $1 \leq m_2 \leq n_2$, $n_2=2,3,\ldots,n$ and $k=1,2,\ldots$ for any $\alpha>1$.

Note that in the process of obtaining the moments $h^{(k)}_{m:n;\alpha,\beta,\delta}$ for any $\alpha>1$ one need evaluate directly the integrals for the moments $h^{(k)}_{n_1:n_1;\alpha_1,\beta,\delta}$ for $k=1,2,\ldots,(n_1-1)\beta-1$ and $n_1=2,3,\ldots,n$ only. Thus for fixed values of $\beta$ and $\delta$ such that $\beta$ is an integer, if the moments of order statistics arising from $g(x;\alpha,\beta,\delta)$ are to be obtained for various sample sizes and for different values of $\alpha$, then the relations (3.2.8) and (3.2.13) can be used in a systematic and recursive manner as explained above, so as to get those moments with the direct evaluation of the above mentioned minimum number of integrals.

**Remark 3.2.2.** Since the Weibull distribution is the generalized gamma distribution (3.1.1) with $\alpha=1$, we cannot apply the recurrence relation given in theorem 3.2.2 to obtain the moments of Weibull order statistics. However if the parameters $\beta$ and $\delta$ are known and $\beta$ is an integer, then the relation given in theorem 3.2.1 applies for the moments of the largest order statistic arising from $g(x;1,\beta,\delta)$, (i.e., the Weibull distribution). In this case, using theorem 3.2.1, we arrive at the conclusion that in order to obtain the moments of any order of each order statistic arising from the Weibull distribution $g(x;1,\beta,\delta)$, one requires the direct evaluation of at most $(n-1)\beta-1$ integrals for the
moments, provided the moments of order statistics arising from lower sample sizes are also known.

3.3. ON MOMENTS OF GAMMA ORDER STATISTICS

Gamma distribution is well known in statistics for its wide range of applications. Bain (1983) has listed a number of fields such as life testing, Bayesian analysis, acceptance sampling based on life tests, weather analysis and queueing theory where gamma distribution is applied. The probability density function of a two parameter gamma distribution can be obtained by putting $\beta=1$, in $g(x; \alpha, \beta, \delta)$ defined in (3.1.1). Since our interest is only with the moments of gamma order statistics, it is enough to derive the results for the special case that the scale parameter $\delta$ equals unity. In this case we may write $g(x; \alpha)$ to denote the probability density function $g(x; \alpha, 1, 1)$ which is the following.

$$g(x, \alpha) = \{\Gamma(\alpha)\}^{-1} x^{\alpha-1} e^{-x}, x > 0$$  \hspace{1cm} (3.3.1)

where $\alpha > 0$ is the shape parameter of the distribution. We may also similarly write $X_{m:n; \alpha}$ and $h_{m:n; \alpha}^{(k)}$ to denote $X_{m:n; \alpha, 1, 1}$ and $h_{m:n; \alpha, 1, 1}^{(k)}$ respectively for $1 \leq m < n$ and $k=1,2,\ldots$.

Explicit expressions for the moments of the order statistics $X_{m:n; \alpha}$ have been obtained by Gupta (1960) for integral value of $\alpha$, and by Krishnaiah and Rizvi (1967) for a general value of $\alpha$. For the gamma distribution with integral value for $\alpha$, Joshi (1979) has established the following recurrence relations on the moments of different orders of the order statistics.
\[
\begin{align*}
\hat{h}_{1:n;\alpha}^{(k)} &= \frac{(k/n)\Gamma(\alpha)}{\Gamma(\alpha)} \sum_{j=0}^{\alpha-1} (j!)^{-1} \hat{h}_{1:n;\alpha}^{(j+k-\alpha)} \\
\hat{h}_{m:n;\alpha}^{(k)} &= \hat{h}_{m-1:n-1;\alpha}^{(k)} + \frac{(k/n)\Gamma(\alpha)}{\Gamma(\alpha)} \sum_{j=0}^{\alpha-1} (j!)^{-1} \hat{h}_{m:n;\alpha}^{(k+j-\alpha)}
\end{align*}
\]

for \(k=1,2,\ldots\) and \(2 \leq m \leq n\). Now by using (1.2.10) and (3.3.2) it is clear that, if the negative moments of orders \(-1, -2, \ldots, -1\) of the smallest order statistic in samples of size \(n\) are known, then one could obtain all moments of the smallest order statistic for \(1 \leq m \leq n\) and \(k=1,2,\ldots\).

Balakrishnan, Malik and Ahmed (1988) have reviewed several recurrence relations and identities involving the moments of gamma order statistics.

There is more interest with the standard exponential distribution, which is well known to be the special case of (3.3.1) with \(\alpha=1\). Joshi (1978) has established the following recurrence relations on the single moments of exponential order statistics.

\[
\begin{align*}
\hat{h}_{1:n;1}^{(k)} &= \frac{(k/n)\Gamma(1)}{\Gamma(1)} \hat{h}_{1:n;1}^{(k-1)} \\
\hat{h}_{m:n;1}^{(k)} &= \hat{h}_{m-1:n-1;1}^{(k)} + \frac{(k/n)\Gamma(1)}{\Gamma(1)} \hat{h}_{m:n;1}^{(k-1)}
\end{align*}
\]

where \(2 \leq m \leq n\) and \(k=1,2,\ldots\). Clearly the relations (3.3.4) and (3.3.5) help us to obtain in a simple recursive manner all single moments of order statistics of a sample of size \(n\) drawn from the standard exponential distribution. Balakrishnan, Malik and Ahmed (1988) have reviewed many recurrence relations and identities involving the moments of order statistics arising from the standard exponential distribution also. Now we shall explain how the relations (3.3.4) and (3.3.5) can be effectively combined with the results (3.2.8) and (3.2.13) in obtaining the moments of order statistics arising from a gamma distribution.
For convenience of notation we write $M(u,v,w;\alpha)$ and $J^{(k,j)}_{r,s,t;\alpha}$ to denote $M(u,v,w,\alpha,1,1)$ and $J^{(k,j)}_{r,s,t;\alpha,1,1}$ respectively, where $M(u,v,w;\alpha,\beta,\delta)$ is defined in (3.2.1) and $J^{(k,j)}_{r,s,t;\alpha,\beta,\delta}$ is defined in lemma 3.2.2. Then from lemma 3.2.2, we have the following.

\begin{align*}
J^{(0,1)}_{r,s,t;\alpha} &= 1 \quad (3.3.6) \\
J^{(k,1)}_{r,s,t;\alpha} &= -(s+k)^{-1}(at-ak+r+1-t)J^{(k-1,1)}_{r,s,t;\alpha} \quad (3.3.7) \\
J^{(k,j)}_{r,s,t;\alpha} &= (s+k)^{-1}(t-k)J^{(k-1,j-1)}_{r,s,t;\alpha} - (s+k)^{-1}(at-ak+r+j-t)J^{(k-1,j)}_{r,s,t;\alpha} \quad (3.3.8)
\end{align*}

for $j=2,3,\ldots,k$ and

\begin{equation}
J^{(k,k+1)}_{r,s,t;\alpha} = (s+k)^{-1}(t-k)J^{(k-1,k)}_{r,s,t;\alpha} \quad (3.3.9)
\end{equation}

Then we get the following corollaries of theorems 3.2.1 and 3.2.2 respectively.

**Corollary 3.3.1.** Let $X_{n:n;\alpha}$ denote the largest order statistic of a sample of size $n>2$ drawn from the gamma distribution (3.3.1) and for $a > 0$ define $h^{(a)}_{n:n;\alpha} = E(X_{n:n;\alpha}^a)$. Then for any finite real $r \geq n-1$, we have the following relation.

\begin{equation}
\begin{align*}
h^{(r)}_{n:n;\alpha} &= \{J^{(n-1,n)}_{r,0,n;\alpha}\}^{-1} \frac{\Gamma((n-r+1)+1)}{\Gamma(a)} \sum_{j=1}^{n-1} J^{(n-1,j)}_{r,0,n;\alpha} h^{(r-n+j)}_{n:n;\alpha} \quad (3.3.10)
\end{align*}
\end{equation}

where the $J$'s are constants defined as in (3.3.6) to (3.3.9).

This result is available in Thomas and Moothathu (1991).
Corollary 3.3.2. Let \( n \geq 2 \) be an integer, \( k \) be a real number such that \( k \geq -1 \) and define \( h^{(k)}_{n:n;\alpha} = E(X_k^{n:n;\alpha}) \). Then for every \( \alpha > 1 \), we have the following result.

\[
h^{(k)}_{n:n;\alpha} = \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i (\alpha-1)^{i+1} \sum_{j=1}^{i} J(i,j) h^{(k+j)}_{n-i-1,i+1,\alpha-1}
\]

(3.3.11)

where the \( J \)'s are constants defined as in (3.3.6) to (3.3.9).

To obtain all single moments \( h^{(k)}_{m:n;\alpha} \) for any integral value of \( \alpha \), \( 1 \leq m \leq n; \ k=1,2,\ldots \) if we use the recurrence relations (3.3.2) and (3.3.3) established by Joshi (1979), then we need the direct evaluation of \( \alpha-1 \) integrals for the moments of orders

\(-\alpha-1, -\alpha-2, \ldots, -1\) of \( X_1^{n;\alpha} \) and the knowledge of the moments of order statistics arising from lower sample sizes, or the direct evaluation for the moments of orders \(-\alpha-1, -\alpha-2, \ldots, -1\) of \( X_1^{n_1;\alpha} \) for each \( n_1 \leq n \). But for the same purpose our recurrence relation (3.3.10) requires the direct evaluation of the integrals for the first \( n-2 \) single moments of \( X_n^{n;n;\alpha} \) and the knowledge of the moments of order statistics arising from lower sample sizes or the direct evaluation of the integrals for the first \( n-2 \) moments of the largest order statistic \( X_1^{n_1;n_1;\alpha} \) for each \( n_1 \leq n \). Thus it is obvious that if the shape parameter \( \alpha \) is a known integer greater than \( n-1 \), our result (3.3.10) leads to a reduction in the number of integrals to be evaluated directly, when compared with that required if one uses the results of Joshi (1979). Also the major advantage of our result (3.3.10) is that (unlike Joshi's result), it can be applied to the evaluation of the moments of order statistics arising from a gamma distribution with any positive real value for the shape parameter \( \alpha \).
The recurrence relation (3.3.11) helps us to obtain the moments of order statistics arising from the gamma distribution with shape parameter \( \alpha > 1 \), with the knowledge of the moments of order statistics arising from the gamma distribution with shape parameter \( \alpha - 1 \). The recurrence relations (3.3.4) and (3.3.5) established by Joshi (1978) for the moments of order statistics arising from the exponential distribution can now be combined with our result (3.3.11) to obtain the moments of any order of the order statistic \( X_{n:n;\alpha} \), when \( \alpha \) is any positive integer. From (3.3.4) and (3.3.5) we can evaluate the moments of any order of each exponential order statistic \( X_{m:n;1} \) for \( 1 < m < n \); \( n=2,3,... \) without evaluating any integral for the moments directly. Then using the values for \( h_{n_1:n_1,1}^{(r)} \) for \( r=k+1,k+2,...,k+n_1 \) we may use our relation (3.3.11) to obtain \( h_{n_1:n_1;2}^{(k)} \) for \( n_1=2,3,...,n \) and \( k=1,2,... \). Now based on the values obtained for \( h_{n_1:n_1;2}^{(k)} \) we again use the known relation (1.2.9) to obtain the moments \( h_{m_1:n_1;2}^{(k)} \) for \( 1 < m_1 < n_1 \); \( n_1=2,3,...,n \) and \( k=1,2,... \). Then for any integer value of \( \alpha \), the above procedure can be recursively continued to obtain the moments \( h_{m:n;\alpha}^{(k)} \) for \( 1 < m < n \) and \( k=1,2,... \). Thus we have proved the following theorem.

**Theorem 3.3.1.** If the shape parameter \( \alpha \) of the gamma distribution with probability density function (3.3.1) is a known integer, then the moments of any order of each order statistic of a sample of size \( n \) arising from this distribution can be obtained without directly evaluating any integral at all for the moments.

Our result (3.3.11) is advantageous than that of Joshi (1979) for the following reasons: (1) when \( \alpha > 1 \) is an integer, because of the
Theorem 3.3.1, if we use (3.3.11), to evaluate $h_{m:n}^{(k)}$ for any $1 \leq m \leq n$, $k=1,2,\ldots$ then we need not evaluate directly any integral and (ii) if $a>1$ is not an integer, then also (3.3.11) is applicable.

More specifically, if $a>1$ is not an integer, and if $a_1 = a - \lfloor a \rfloor$ where $\lfloor \cdot \rfloor$ is the usual greatest integer function, then by directly evaluating the integrals for the moments $h_{n_1:n_1}^{(k)}$ for $k=1,2,\ldots,n_1-2$; $n_1=2,3,\ldots,n$, the moments of any order of each order statistic $X_{m_1:n_1}^{(a_1)}$ for $1 \leq m_1 \leq n_1$ and $n_1=2,3,\ldots,n$ can be obtained by using (3.3.10) without evaluating directly any further moments. Then by using any of our relations (3.3.11) and (3.3.10) and the known recurrence relation (1.2.9), the moments of any order of each order statistic $X_{m:n}^{(a_1)}$ for $1 \leq m \leq n$ can be obtained without evaluating directly any integral for the moments. If we proceed in the same manner recursively all moments $h_{m:n}^{(k)}$ for $1 \leq m \leq n$ and $k=1,2,\ldots$ can be determined without evaluating any integral for the moments, other than those integrals evaluated directly for the moments of order statistics arising from $g(x;\alpha_1)$. Thus we have proved the following theorem.

**Theorem 3.3.2.** If the shape parameter $\alpha$ is a known positive real number other than an integer and $\alpha_1 = \alpha - \lfloor \alpha \rfloor$, where $\lfloor \cdot \rfloor$ is the usual greatest integer function, then with the knowledge of the moments $h_{n_1:n_1}^{(k)}$ for $k=1,2,\ldots,n_1-2$ and $n_1=2,3,\ldots,n$ the moments of any order of each order statistic $X_{m_1:n_1}^{(a_1)}$ for $1 \leq m_1 \leq n_1$ and $n_1=2,3,\ldots,n$ can be obtained without evaluating directly any further integral for the moments (it is assumed here that $n \geq 2$).
Notice that when $\alpha$ is greater than one, both relations (3.3.10) and (3.3.11) can be applied to evaluate the moments $h_{n:n;\alpha}^{(k)}$ for all $k > n-2$. Clearly the moment of order $k$ for $k > n-2$ of $X_{n:n;\alpha}$ is expressed, in terms of the moments of lower orders of the same statistic in (3.3.10) and in terms of certain higher order moments of $X_{n:n;\alpha}$ in (3.3.11). But in (3.3.11) certain combinatorial coefficients are involved and the number of $J$ coefficients to be obtained is also larger, when compared with those involved in (3.3.10). Hence for easier evaluation of the moments $h_{n:n;\alpha}^{(k)}$ one may use (3.3.11) for $k < n-2$ and (3.3.10) for $k > n-2$.

The results (3.3.10) and (3.3.11) established in this section for the moments of gamma order statistics can also be used to check the accuracy in the values of the moments of order statistics obtained by other methods. For example if we put $n=5$, $\alpha=3$ and $r=4$ in (3.3.10), then it leads us to write the following.

\[
\frac{\Gamma(15)}{\Gamma(3)} 5^{-14} = 81 - \frac{(455/3)}{h_{5:5;3}^{(1)}} + \frac{(497/6)}{h_{5:5;3}^{(2)}} \\
- \frac{(49/3)}{h_{5:5;3}^{(3)}} + \frac{(455/3)}{h_{5:5;3}^{(4)}}
\]

(3.3.12)

If we use the tabulated values of Gupta (1960) of the moments $h_{5:5;3}^{(k)}$ for $k=1,2,3,4$, then the right side value of (3.3.12) simplifies to 0.90833. But it is obvious that the expression on the left side of (3.3.12) simplifies to 0.44635. This leads us to the conclusion that there is inaccuracy in at least one of the tabulated values of $h_{5:5;3}^{(k)}$, $k=1,2,3,4$ given by Gupta (1960). For similar comments regarding the inaccuracy of the tabulated values of the moments of gamma order statistics by Gupta (1960) see, Harter (1970, 1988).
3.4 ON MOMENTS OF ORDER STATISTICS FROM CHI DISTRIBUTION WITH ONE DEGREE OF FREEDOM

The chi distribution with one degree of freedom (df) has density \( g_1(x) = g(x; \frac{1}{2}, 2, \sqrt{2}) \), where \( g(x, \alpha, \beta, \delta) \) is as given in (3.1.1) (which corresponds to the generalized gamma distribution). Then \( g_1(x) \) is as given below.

\[
g_1(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{-\frac{x^2}{2}}, \quad x > 0
\]  

(3.4.1)

The chi distribution with one df is also known as the half-normal distribution. The density function \( g_1(x) \) can also be viewed as the left truncated standard normal distribution with probability density function \( \varphi(x) \) defined in (2.4.3) with \( c = 0 \).

Govindarajulu (1963b) has established certain relations connecting the moments of order statistics arising from a distribution which is symmetric about zero and those of the order statistics arising from the distribution obtained by folding the former at the origin. Making use of these relations Balakrishnan, Malik and Ahmed (1988) have considered the relations connecting the moments of order statistics arising from the chi distribution with one df and the standard normal distribution. They have further reviewed many more identities and recurrence relations on the moments of order statistics arising from \( g_1(x) \).

Clearly theorem 3.2.1 with \( \alpha = \frac{1}{2}, \beta = 2, \delta = \sqrt{2} \) and theorem 2.4.1 with \( c = 0 \) give recurrence relations on moments of order statistics arising from \( g_1(x) \). But with regard to the amount of computations involved and the number of direct evaluation of integrals needed,
we recommend the use of relations in theorem 2.4.1 for the easier evaluation of the moments of order statistics arising from $g_1(x)$.

The problem of the evaluation of moments of order statistics arising from the chi distribution with one df acquires importance, since by theorem 3.2.2, these moments can be utilized in the evaluation of moments of order statistics arising from a chi distribution with $(2p+1)$ df (where $p>0$ and is an integer). Therefore we state below the special case of the theorem 2.4.1 as corollary 3.4.1, in terms of the following simpler notation. Let the $m$th order statistic $X_{m:n;1}$ of a sample of size $n$ arising from (3.4.1) be denoted by $X_{m:n;(1)}$. We may also write $b^{(k)}_{m:n;1}$ to denote the moment $b^{(k)}_{m:n;1}$, the $k$th moment of $X_{m:n;(1)}$.

If we obtain the expression for $q(x)$ from (2.4.3), then by using (2.4.11) we have the following

$$I(b,n) = \int_{0}^{\infty} x^{b} \{ q(x) \}^{n} \, dx$$

$$= \int_{0}^{\infty} x^{b} \{ g_1(x) \}^{n} \, dx$$

$$= 2^{(n+b-1)/2} \Gamma((b+1)/2) \, \pi^{-n/2} \, n^{-(b+1)/2} \quad (3.4.2)$$

where $n \geq 1$ and $b > 0$ are integers. Now by using (3.4.2) and theorem 2.4.1 we have the following corollary.

**Corollary 3.4.1.** If $b^{(r)}_{n:n;1}$ denotes the moment of order $r$ of the largest order statistic $X_{n:n;(1)}$ of a random sample of size $n$ drawn from the chi distribution with one df and if $r \geq n-1$, then $b^{(r)}_{n:n;1}$ can be determined from the following relations.
If $n$ is even, say $n=2m$, then

$$b_{2m:2m;1}^{(2m-1)} = \{A_{2m-1,m}^{(2m)}\}^{-1} \{2m \sum_{j=1}^{m-1} A_{2m-1;j}^{(2m)} b_{2m:2m;1}^{(2j-1)}\} (3.4.3)$$

If $n$ is odd, say $n=2m+1$, then

$$b_{2m+1:2m+1;1}^{(2m)} = \{A_{2m+1,m+1}^{(2m+1)}\}^{-1} \{(2m+1) \sum_{j=1}^{m} A_{2m,j}^{(2m+1)} b_{2m+1:2m+1;1}^{(2j-2)}\} (3.4.4)$$

If $1 \leq k < n-1$ is such that $n=k+2m$, then

$$b_{n:n;1}^{(k+n-1)} = \{D_{k+2m-1,k+m}^{(k,n)}\}^{-1} \{n \sum_{j=1}^{k+m-1} D_{k+2m-1,j}^{(k,n)} b_{n:n;1}^{(2j-1)}\} (3.4.5)$$

If $1 \leq k < n-1$ is such that $n=k+2m+1$, then

$$b_{n:n;1}^{(k+n-1)} = \{D_{k+2m,k+m+1}^{(k,n)}\}^{-1} \{n \sum_{j=1}^{k+m} D_{k+2m,j}^{(k,n)} b_{n:n;1}^{(2j-2)}\} (3.4.6)$$

If $s$ is an integer such that $s \geq n-1$, then

$$b_{n:n;1}^{(s+n-1)} = \{B_{n-1,n}^{(s,n)}\}^{-1} \{n \sum_{j=1}^{n-1} B_{n-1,j}^{(s,n)} b_{n:n;1}^{(s-n+2j-1)}\} (3.4.7)$$

where the constants $A$'s are defined in lemma 2.2.1, $B$'s in lemma 2.2.2, $D$'s in lemma 2.2.3 and $O I(b,n)$ in (3.4.2).

Clearly the relations (3.4.3) to (3.4.7) express the moments of order $r$, for $r \geq n-1$ of $X_{n:n;1}$ in terms of its lower order even moments if $r$ is even and lower order odd moments if $r$ is odd. Thus if the integrals for the first $n-2$ moments of the order statistic $X_{n:n;1}$ are evaluated directly, then using the recurrence relations (3.4.3) to (3.4.7) and based on the moments of order statistics arising from lower sample sizes, the moments of any order of each order
statistic $X_{m:n}^{(1)}$ for $1 \leq m \leq n$ can be obtained without evaluating directly any further integral for the moments.

3.5 MOMENTS OF ORDER STATISTICS FROM THE CHI DISTRIBUTION WITH $p$ DEGREES OF FREEDOM

The probability density function of the chi distribution with $p$ degrees of freedom is denoted by $g_p(x)$ and is given as below.

$$g_p(x) = \frac{1}{\Gamma(p/2)} \frac{1}{2^{(p/2) - 1}} x^{p-1} e^{-x^2/2}, \quad x > 0 \quad (3.5.1)$$

It is obvious that $g_p(x)$ can be obtained as a special case of $g(x; a, \beta, \delta)$ defined in (3.1.1) with $a = p/2$, $\beta = 2$ and $\delta = \sqrt{2}$. It may be also noted that the probability density functions of half-normal, Rayleigh and Maxwell distributions are obtained by putting $p=1, 2$ and $3$ respectively in (3.5.1). Some practical applications of the chi distribution in communication engineering, reliability theory and analysis of two-level factorial experiments have been given by Dyer (1973). Since $\beta$ and $\delta$ involved in $g(x; a, \beta, \delta)$ are fixed for a chi distribution, we may use the simpler notation $X_{m:n}^{(p)}$ instead of $X_{m:n; p/2, 2, \sqrt{2}}$ for the $m$th order statistic of a sample of size $n$ arising from the distribution (3.5.1). Similarly we use the simpler notation $b_{m:n}^{(k)}$ for the moment of order $k$ of $X_{m:n}^{(p)}$.

In section 3.4, we have described the results already available on the moments of order statistics arising from the chi distribution with one df, and the advantage of our results in evaluating these moments. For the Rayleigh distribution (which is the chi
distribution with two df), Dyer and Wishenand (1973) have obtained the exact and explicit expressions for the first two single moments and product moments of order statistics. Balakrishnan, Malik and Ahmed (1988) have derived the following recurrence relations on the moments of order statistics arising from Rayleigh distribution.

\[
b_{1:n;2}^{(k+2)} = (k+2)n^{-1} b_{1:n;2}^{(k)}
\]
\[
b_{n:n;2}^{(k+2)} = b_{n-1:n;2}^{(k+2)} + (k+2)(n-m+1)^{-1} b_{m:n;2}^{(k)}
\]
for \(2 \leq m \leq n\) and \(k=0,1,2,...\).

In this section we will make use of relations (3.4.3) to (3.4.7) for the largest order statistic arising from \(g_1(x)\) and results (3.5.2) and (3.5.3), to simplify the evaluation of the moments of order statistics arising from \(g_p(x)\) for any integer \(p \geq 2\).

For convenience of notation we write \(L^{(k,j)}_{r,s,t;p} \) to denote \(L_{r,s,t;p/2,2,\sqrt{2}}\) where \(L^{(k,j)}_{r,s,t;\alpha,\beta,\delta}\) is defined in lemma 3.2.2. Then from lemma 3.2.2, we have the following.

\[
L^{(0,1)}_{r,s,t;p} = 1
\]
\[
L^{(k,1)}_{r,s,t;p} = -(s+k)^{-1}(pt-pk-t+r+1) L^{(k-1,1)}_{r,s,t;p}
\]
\[
L^{(k,j)}_{r,s,t;p} = (s+k)^{-1}(t-k) L^{(k-1,j-1)}_{r,s,t;p}
\]
\[
- (s+k)^{-1}(pt-pk-t+2j+r-1) L^{(k-1,j)}_{r,s,t;p}
\]
for \(j=2,3,...,k\) and
Then we get the following corollaries of theorems 3.2.1 and 3.2.2 respectively.

**Corollary 3.5.1.** Let \( X_{n:n;(p)} \) be the largest order statistic of a sample of size \( n \geq 2 \), drawn from the chi distribution with \( p \) df as defined in (3.5.1) and for \( a > 0 \) define \( b^{(a)}_{n:n;p} = E\{X_{n:n;(p)}^a\} \). Then for any finite real \( r \geq n-1 \), we have the following.

\[
b^{(r+n-1)}_{n:n;p} = \left\{ L^{(n-1,n)}_{r,0,n;p} \right\}^{-1} \left\{ 2^{(n+1)/2} [\Gamma(np-n+r+1)/2][\Gamma(p/2)]^{-n^{-}(np-n+r-1)/2} \right\}^{n^{-1}} \sum_{j=1}^{n} L^{(n-1,j)}_{r,0,n,p} b^{(r+n-j-1)}_{n:n;p}
\]

where the \( L \)'s are constants defined as in (3.5.4) to (3.5.7).

**Corollary 3.5.2.** Let \( n \geq 2 \) be an integer, \( k \) be a real number such that \( k \geq -2 \) and define \( b^{(k)}_{n:n;p} = E(X_{n:n;(p)}^k) \). Then for any \( p > 2 \), we have the following result.

\[
b^{(k)}_{n:n;p} = \sum_{i=0}^{n-1} (-1)^i (p-2)^{-i-1} \sum_{j=1}^{i+1} L^{(i,j)}_{1+k+2,n-1,1+1;p-2} b^{(k+j)}_{n:n;p-2}
\]

where the \( L \)'s are constants defined as in (3.5.4) to (3.5.7).

Note that for every positive integer \( k \geq 2(n-1) \), (3.5.8) expresses the moment of order \( k \) of the largest order statistic of a sample of size \( n \) arising from the chi distribution with \( p \) df in terms of its next immediate \( n-1 \) lower order moments, of even order if \( k \) is even and odd order if \( k \) is odd. Thus if the integrals for the first \( n-1 \) odd moments of \( X_{n:n;(p)} \) are evaluated directly then its other
higher order odd moments can be obtained recursively by using (3.5.8). Similarly based on the directly evaluated values of the integrals for the first \( n-2 \) even moments of \( X_{n:n;(p)} \), its all other higher order even moments can be obtained using (3.5.8) in a recursive manner. The importance of the relation (3.5.9) is that, if \( p > 2 \), it relates the moment of order \( k \) of the largest order statistic \( X_{n:n;(p)} \) with the moments of orders \( k+2, k+4, \ldots, k+2n \) of the order statistic \( X_{n:n;(p-2)} \).

From the explicit expression obtained by Dyer and Whisenand (1973) for the mean of the order statistics arising from the chi distribution with two df and the recurrence relations (3.5.2) and (3.5.3) established by Balakrishnan, Malik and Ahmed (1988), it follows that the moments of any order of each order statistic arising from \( g_2(x) \) can be obtained in a recursive manner without evaluating directly any integral for the moments. Then our recurrence relation (3.5.9) with \( p = 4 \) gives the moments \( b_{n:n;4}^{(k)} \) of any order \( k \) of \( X_{n:n;(4)} \) in terms of the moments of the order statistic \( X_{n:n;(2)} \). Thus for all \( n \geq 2 \), we can determine \( b_{n:n;4}^{(k)} \) in terms of the evaluated values of the moments of the largest order statistic \( X_{n:n;(2)} \) obtained by the known recurrence relations (3.5.2) and (3.5.3). Now (1.2.9) can be used to obtain the moments of any order of each order statistic \( X_{m:n;(4)} \) for \( 1 \leq m \leq n \) without evaluating directly any integral for the moments. Extending this argument further it follows that, when the df of the chi distribution \( g_p(x) \) is even, the moments of any order of each order statistic of a sample taken from this distribution can be obtained in a systematic and recursive
manner without evaluating directly any integral at all for the moments.

By using our relations (3.5.8), (3.5.9) and by a similar argument we arrive at the conclusion that, when \( p \) is an odd integer greater than one, the moments of any order of each order statistic \( X_{m:n(p)} \), \( (1 \leq m \leq n) \) can be obtained in a systematic and recursive manner without evaluating directly any integral for the moments, provided the moments of order statistics arising from the chi distribution with one df are known. This being so, in the light of our recurrence relations (3.4.3) to (3.4.7), we need the direct evaluation of integrals for the moments \( b_{n_1:n_1:1}^{(k)} \), \( k=1,2,\ldots,n_1-2; n_1=2,3,\ldots,n \) (where \( n \geq 2 \) is the sample size) so as to obtain \( b_{m_1:n_1:p}^{(k)} \), \( p \geq 1 \) is an odd integer, \( 1 \leq m_1 \leq n_1 \); \( n_1=2,3,\ldots,n \); \( k=1,2,\ldots \). Thus the total number of direct evaluation of the integrals needed is \( 2^{-1}(n-2)(n-1) \). Thus we have proved the following theorem.

**Theorem 3.5.1.** In order to find the moments of any order of each order statistic of a sample of size \( n \geq 2 \) arising from the chi distribution with \( p \) df, one has to evaluate directly (i) no integral at all if \( p \) is even and (ii) at most \( 2^{-1}(n-2)(n-1) \) integrals if \( p \) is odd. In the latter case the integrals to be evaluated directly are those regarding \( b_{n_1:n_1:1}^{(k)} \) for \( k=1,2,\ldots,n_1-2 \) and \( n_1=2,3,\ldots,n \).

Notice that when \( p \) is greater than two, both the relations (3.5.8) and (3.5.9) can be applied to evaluate \( b_{n:n:p}^{(k)} \) for all
$k \geq 2(n-1)$. Clearly the moment of order $k$ for $k \geq 2(n-1)$ of $x_{n:n;}^{(p)}$ is expressed in terms of the moments of lower orders of the same statistic in (3.5.8) and in terms of certain higher order moments of $x_{n:n;}^{(p-2)}$ in (3.5.9). As in the case of a gamma distribution here again some combinatorial coefficients are involved in (3.5.9) and the number of $L$ coefficients involved in (3.5.9) is large when compared with those involved in (3.5.8). Hence for easier evaluation of the moments $b_{n:n;}^{(k)}$ one may use (3.5.9) when $k < 2(n-1)$ and (3.5.8) when $k \geq 2(n-1)$.

Remark 3.5.1. The results established in sections 3.2 to 3.5 have some connections with the moments of order statistics arising from a chi-square distribution with $p$ degrees of freedom. The probability density function of this distribution is given as below.

$$q_p(x) = [\Gamma(p/2)]^{-1} 2^{-p/2} e^{-x/2} x^{(p/2)-1}, x > 0 \tag{3.5.10}$$

Obviously $q_p(x)$ can be obtained also as a special case of $g(x, a, \beta, \delta)$ defined in (3.1.1) with $a = p/2$, $\beta = 1$ and $\delta = 2$. Let $d_{m:n;}^{(k)}$ denote the $k$th moment of the $m$th order statistic of a sample of size $n$ arising from (3.5.9). It is easy to verify that if $x$ follows a chi-square distribution with density $q_p(x)$, then $X/2$ follows a one parameter gamma distribution with shape parameter $(p/2)$, whose density in our notation in section 3.3 is $g(x;p/2)$ and $\sqrt{X}$ follows a chi distribution with $p$ df with density $g_p(x)$ given in (3.5.1). From these relations it follows that $d_{m:n;}^{(k)}$ can be viewed as either

(i) the $2^k$ times the $k$th moment of the $m$th order statistic of a
sample of size $n$ arising from $g(x;p/2)$ or (ii) the $(2k)$th moment of the $m$th order statistic in a sample of size $n$ arising from $g_p(x)$. Hence regarding $d_{m:n;p}^{(k)}$, the material we presented in sections 3.3, 3.4 and 3.5 pertaining to the gamma and chi distributions are valid with appropriate obvious modifications. But if we compare the recurrence relations that we have derived for the moments of gamma order statistics and those for chi order statistics, then it is clear that, we get the maximum reduction in the number of direct evaluation of the integrals for moments, if we use the results on the moments of chi order statistics in obtaining $d_{m:n;p}^{(k)}$. Hence it is recommended to use the moment relations in corollaries 3.4.1, 3.5.1 and 3.5.2 connecting the even moments of chi order statistics in the evaluation of $d_{m:n;p}^{(k)}$.

The discussion in remark 3.5.1 also leads to the following corollary to theorem 3.5.1.

**Corollary 3.5.3.** In order to find the moments of any order of each order statistic of a sample of size $n$ arising from the chi-square distribution with $p$ degrees of freedom, one has to evaluate directly (i) no integral at all if $p$ is even and (ii) at most the integrals for $d_{n_1:n_1:1}^{(k)}$ for $k=1,2,...,[n_1/2]-1$; $n_1=2,3,...,n$ if $p$ is odd, where $[.]$ is the usual greatest integer function.