INTRODUCTION

The concept of ideals and filters in a groupoid has been developed previously by Frink and Smith [7]. The collection of all filters in a groupoid is an algebraic lattice having 0 as the smallest element [7]. The present thesis is devoted to developing a theory of ideals and filters in a groupoid (by defining an ideal (filter) to be a non-empty set). In particular we obtain a theory of prime ideals in a groupoid analogous to the theory of prime semi-ideals in a poset. This class of ideals is used in the characterisation of a special class of groupoids namely intraregular groupoids. Our work is further concerned with the study of pseudocomplements in groupoids. We also present a theory of last residue class in groupoids. Two congruences on a groupoid are defined and the quotients under these are studied. In this work we further obtain extensions of the hull-kernel topology on the set of prime ideals, due to Kist [11]† in a commutative semigroup, to the set of prime ideals in an intraregular groupoid with 0. This seems to be an analogue of Venkatanarasimhan’s construction of topology on the set of prime semi-ideals in a poset [22,24].

† Numbers enclosed in [ ] refer to the Bibliography at the end.
Our work is accommodated with four chapters. In chapter 1 we include standard notations and terminologies and also some basic definitions and properties regarding lattice, semilattice, poset and groupoid which we refer to in subsequent chapters. In the first section we include some of the results in semilattices and posets the analogues of which we obtain in groupoids in the later chapters. Section 2 of this chapter contains properties of ideals and filters in a groupoid. A modified version of Frink and Smith's result on filters [7] is contained in the first part of theorem 1.2.2. Theorem 2 [24] regarding a prime semi-ideal in a semilattice is contained in corollary 1.2.5. The characteristic property of prime ideals is embodied in theorem 1.2.6. A corresponding result for a commutative semigroup is contained in lemma 2.1 [11] and is the keystone in the determination of the hull-kernel topology on the set of prime ideals in a commutative semigroup. Theorem 1.2.7 is similar to theorem 2 of Frink and Smith [7] and is used in obtaining the join of two filters. In lemma 1.2.11 we mention some results without proof which we find in [3].

Chapter 2 is devoted to a study of pseudocomplements in groupoids. Section 1 of this chapter

† Numbers which are not enclosed in [ ] refer to the results of the present thesis. e.g. theorem 1.2.2 or (1.2.2) means the 2 result in section 2nd of chapter 1.
deals with intraregular groupoids. Theorem 2.1.2 is an analogue of theorem 12 [22], which shares comparison with Stone's existence theorem [§4, [1]]. Based on this theorem we formulate the topology on the set of prime ideals in an intraregular groupoid (chapter 4 of this thesis). A characterisation of an intraregular groupoid using prime ideals is obtained in theorem 2.1.3. When the groupoid is intraregular having the 0 element, corollary 2.1.5 gives a concrete expression to the intersection of all minimal prime ideals in the groupoid. The result is an analogue of the corresponding result in a semilattice [24]. Lemma 2.1.7 is remarkable as it is used in proving many main theorems in this work. The second section is occupied with a study of pseudocomplements. There already exists a well developed theory of pseudocomplements in lattices and particularly in distributive lattices due to Glivenko, Stone and Birkhoff. Frink [8] has obtained a generalisation of this theory for semilattices in 1962. Venkatanarasimhan extended the concept to posets. His work in this direction included extensions of some results of Frink and Balachandran. If the definition of pseudocomplement in a semilattice due to Frink [8] involves the meet operation, Venkatanarasimhan obtained the extension to a poset using the concept of a semi-ideal. The result of Gilvenko on the closed elements is extended to a semilattice...
and to a poset [23]. The theory of pseudocomplements has application in many branches of Mathematics. The concept of pseudocomplement in groupoids is contained in definition 2.2.1. The characteristic structure of the set of dense elements in an intraregular groupoid with 0 is obtained in theorem 2.2.4 which is an analogue of the result for a poset [23]. We derive the various formulas for pseudocomplements in lemmas 2.2.8, 2.2.9 and 2.2.10. The concluding section of this chapter is concerned with pseudocomplement and cutcomplement of an ideal. The ideal theory of the distributive lattice and Boolean rings was initiated by Tarski and developed by Stone. Macneille in his fundamental work on partially ordered sets has casually suggested a definition of an ideal in a partially ordered set. A study on the ideal theory of Boolean algebra is made in [17] by Pankajam making use of the concepts of comprincipal ideals and cutcomplements which have been introduced by Vaidyanathaswamy. Pankajam has shown that the notion of product complement can be expressed in terms of the cutcomplement. It is worth mentioning that the set of all ideals of a groupoid with 0 forms a completely distributive complete lattice closed for pseudocomplements. Pseudocomplement of an ideal is described in theorems 2.3.1 and 2.3.5. Theorem 2.3.4 is an analogue of a theorem for
semi-ideal of a poset [22]. The notion of comprincipal ideal and comprincipal envelope is extended to groupoids. A result on normal semi-ideal of a poset is also worked for a normal ideal of an intraregular groupoid. Pseudocomplement of a semi-ideal in a poset (semi-lattice) is characterised using prime semi-ideals (minimal prime semi-ideals) [22,21]. A similar characterisation of the pseudocomplement of an ideal in a groupoid is obtained in theorem 2.3.17 and corollary 2.3.18. A necessary and sufficient condition for a principal ideal to be normal is obtained. The result is an analogue of theorem 7 [24]. The cutcomplement of an ideal (filter) of a groupoid is described in theorem 2.3.22. The dense ideals of an intraregular groupoid are precisely determined. An analogue of theorem 7 [22] is contained in a remark after theorem 2.3.29 which shows that the pseudocomplement of an ideal and that of its comprincipal envelope are identical.

Chapter 3 deals with the theory of last residue class of a filter and ideal extension in groupoids. During 1940's it is Krishnan who first studied the theory of last residue class in a distributive lattice. In [12] Krishnan tries to solve the three problems relating to the last residue class proposed by Vaidyanathaswamy. The main tools of Krishnan's investigation are product complementation and
ideal extension. Balachandran exploit the concept of prime ideals to effect a solution of Vaidyanathaswamy’s problems relating to last residue class in a different way. We first define a congruence relation in terms of a filter in a groupoid with 0 and study the properties of the last residue class of the filter (the congruence class containing 0). We prove the analogues of some results of Krishnan and Balachandran. Section 3.1 begins with the set-theoretic determination of the last residue class of a filter. Our claim is that in a groupoid the last residue class of a filter is an ideal. As mentioned in [1], we prove that the last residue class preserves inclusion of filters. Motivated by the remarks in [1] we could obtain that complements of proper factors of a filter are factors of its last residue class. This leads to a characterisation of the last residue class of a filter. In theorem 3.1.8 we show that maximal filters are precisely those filters whose complements conincide with their last residue class. Corollary 3.1.9 deals with the corresponding result for a minimal prime ideal. We determine the largest filter whose last residue class is identical with that of a given filter in a groupoid. A similar characterisation in a distributive lattice has been obtained by Balachandran [1]. A characterisation of the last residue class of a filter in
terms of minimal prime ideals (prime ideals) is embodied in theorem 3.1.11 (corollary 3.1.12). We also derive a necessary and sufficient condition for an ideal of the groupoid to be the last residue class of some filter. One may refer theorem 8 [1] and 8.2 [12] for similar arguments.

The concept of ideal extension in a distributive lattice (semilattice) is due to Krishnan (Venkatanarasimhan). We introduce a theory of extension of ideals in an intraregular groupoid in section 3.2 and obtain results analogues to some of the results of Krishnan and Venkatanarasimhan. In chapter 2 we have proved that the set $N$ of normal elements of an intraregular groupoid $S$ closed for pseudocomplements forms a Boolean algebra. An expression of the extension of an ideal $A$ of $N$ as the union of principal ideals of $S$ generated by elements of $A$ is our first result in this section. Theorem 3.2.4 and corollary 3.2.5 are important as they relate an ideal of $N$ and its extension. The class of ideals of an intraregular semigroup $S$ which are extensions of some ideals of $N$ is discussed in lemma 3.2.10. The last residue class of a filter of an intraregular semigroup $S$ is obtained as the extension of an ideal of $N$ (corollary 3.2.12). One may read 2.9 [12] in connection with theorem 3.2.14. We prove that there is an embedding from the lattice of ideals of $N$ into the lattice
of ideals of $S$ which preserves arbitrary unions, arbitrary intersections, prime property and normality. An analogue of the result exists in semilattices [25]. The concept of elementwise pseudocomplement of a filter is dealt in section 3.3. In a distributive lattice the concept has been introduced by Krishnan. In an intraregular semigroup $S$ closed for pseudocomplements the last residue class of a filter of $S$ is the extension of its elementwise pseudocomplement. This result is used to prove that the last residue class of the intersection of any family of filters is the intersection of their last residue classes.

Sections 1, 2, 3 and 4 of chapter 4 include some congruences on a groupoid and a lattice and the quotients under these are studied. In section 1 we define two congruences $\theta_p$ and $\theta_m$ on a groupoid in terms of prime ideals. The quotients under these turn out to be semilattices. Relations between the ideals of $S$ and those of the quotients are investigated. Section 4.2 deals with additional properties of the congruences. We observe that the congruence $\theta_p$ we have defined is the least congruence such that the quotient is a semilattice. We show that $S/\theta_m$ is a disjunction semilattice. The fact that $S/\theta_m$ is a Boolean algebra is proved using theorem 12 [24]. In section 4.4 two congruences on a lattice $L$ are given for which the quotients
are distributive lattices. Theorem 4.3.2 determines the least congruence on \( L \) with this property.

Motivated by the construction of topology for prime ideals in a poset [22] we introduce a topology \( \tau \) for the set of all prime ideals \( \mathcal{P} \) in an intraregular groupoid \( S \) with 0. This forms the contents of the concluding section of the last chapter. Our results are analogues to some of the results of Venkatanarasimhan [22,24]. Analogous to theorem 23 [22], we prove that \( \mathcal{P} \) in \( T_0 \). If further \( S \) has 1, \( \mathcal{P} \) is compact and non regular. Connectedness of \( \mathcal{P} \) is proved in theorem 4.4.14. In theorem 4.4.16 we obtain a basis for the open sets in \( \tau \). The anti-\( T_1 \) points of \( \mathcal{P} \) are precisely determined in theorem 4.4.17. The result is an analogue of theorem 35 [22] for a poset. The set \( \mathcal{M} \) of minimal prime ideals (\( \mathcal{M} \) of normal prime ideals) of \( S \) are seen to be subspaces of \( \mathcal{P} \) with respect to the relative topology. Moreover as is seen in theorem 16 [24], \( \mathcal{M} \) is a \( T_3 \) space. We extend the notion of hull-kernel topology on the set of prime ideals in a commutative semigroup, due to Kist, to the set of prime ideals is an intraregular groupoid with 0. In this we use the same terminology used in [11]. We remark that the topology on \( \mathcal{P} \) we have defined, is identical with the hull-kernel topology. Also \( \mathcal{M} \) is a completely regular Hausdorff space.