Chapter II

Medieval Polynomial Equations

301 - 1400 AD

Bakhshali Manuscript, a mathematical manuscript in fairly poor condition was discovered in 1881 in a village called Bakhshali in North Western India near Peshwar. Now this place is in Pakistan. It has been written in the Shāradā script in the Gāthā language. It consists of seventy leaves of birch bark. It is of uncertain origin and date estimates range from the 3rd to the 12th century AD with some scholars estimating the date by about the 3rd or 4th century AD. Most textual evidence indicates that the Bakhshali Manuscript is a commerce and revenue work.

Bakhshali Manuscript is a handbook on arithmetical and algebraic problems together with their solutions. Its main contents are problems and examples of fractions, square roots, profit and loss, interest and the like. The algebraic problems are classified into significant fields like binomial equations; quadratic equations; and simultaneous equations. In Bakhshali Manuscript the subject matter is arranged in groups of similar and presented as follows: A rule is stated and then a relevant example is given, first in words and then in notational form. The solution follows and finally we have the demonstration or “proof”. This method of presentation is quite unusual in Indian mathematics.

There is a controversy over the age of Bakhshali Manuscript. It drew the attention of H. Hoernle, G. R. Kaye and Datta. They examined the manuscript from various aspects. But they failed to come to a satisfactory conclusion in confirming its age. Hoernle assessed that the manuscript was probably a later copy of a document composed at some time in the first few
2.1 301 - 800 AD

India-Bakhshālī Manuscript

Bakhshālī Manuscript, a mathematical manuscript in fairly poor condition was discovered in 1881 in a village called Bakhshālī in North Western India near Peshwar. Now this place is in Pakistan. It has been written in the Shārada script in the Gatha language. It consists of seventy leaves of birch bark. It is of uncertain origin and date estimates range from the 3rd to the 12th century AD. Some scholars estimate the date by about the 3rd or 4th century AD. Circumstantial evidence indicates that the Bakhshālī Manuscript is a commentary of an earlier work.

The Bakhshālī Manuscript is a hand book of rules and illustrative examples together with their solutions. Its main contents are arithmetic and algebra with just a few problems on geometry and mensuration. The arithmetic examples cover fractions, square roots, profit and loss, interest and the Rule of Three. The algebraic problems cover simple equations; simultaneous linear equations; quadratic equations; arithmetic and geometric progressions.

In Bakhshālī Manuscript the subject matter is arranged in groups of sūtras and presented as follows. A rule is stated and then a relevant example is given, first in words and then in notational form. The solution follows and finally we have the demonstration or ‘proof’. This method of presentation is quite unusual in Indian mathematics.

There is a controversy over the age of Bakhshālī Manuscript. It drew the attention of H. Hoernle, G. R. Kaye and Datta. They examined the manuscript from various aspects. But they failed to come to a satisfactory conclusion in confirming its age. Hoernle assessed that the manuscript was probably a later copy of a document composed at some time in the first few
centuries of the Christian era [34]. He considered it originated in the 3rd to 4th century AD [5]. This has been accepted as plausible by many historians of mathematics such as Moritz Cantor, Bühler Cajori, B. Datta, S.N.Sen, A.K.Bag and R.C.Gupta. Hoernle’s dating was based on a number of aspects including the mathematical content, the units of money given in some examples, the use of the symbol ‘+’ for the negative sign and the lack of reference to certain topics especially the solution of indeterminate equations which appeared in works known to have been written later. But G. R. Kaye puts its date to be in the 12th century AD and he even doubts its Indian origin. To Datta, the mathematical principles, symbols and terminology used in this work would be better guides. Datta came to the conclusion that it was a work of 3rd to 4th century AD.

From the evidence of these experts the most likely conclusion is that the manuscript is a later copy of a work first composed around 400 AD. Without supporting evidence, everything points to the manuscript being a 10th century copy of original from around 400 AD.

The Bakhshall Manuscript is to bridge the long gap between the *Sūlbasūtras* of the Vedic Period (800-200 BC) and the mathematics of the Classical period (400 AD-1200).

In Bakhshali Manuscript, integers are written as fractions with the denominator one. In mixed expressions, the integral part is written above the fraction.

Thus \[ \frac{1}{3} = \frac{1}{3} \]

In the place of our ‘=’ they used the word *phalam* (फलम), abbreviated into *pha*. Addition was indicated by *yu* abbreviated from *yuta*.
Numbers to be combined were often enclosed between lines or in a rectangle.

Next, two examples are respectively from [12] and [34].

(i). Pha 12
\[
\begin{array}{c|c|c}
5 & 7 & 1 \\
1 & 1 & yu \\
\end{array}
\]
means \( \frac{5}{1} + \frac{7}{1} = 12 \)

(ii) \[
\begin{array}{c|c|c}
3 & 6 & 1 \\
1 & 1 & yu \\
\end{array}
\]
Pha 9 means \( 3 + 6 = 9 \)

In the Bakhshālī Manuscript, the sign for a negative quantity looks exactly like the present ‘plus’ symbol for addition or a positive quantity. The sign was placed after the number it qualifies.

For example in [34] \[
\begin{array}{c|c|c}
15 & 8+ & 3 \\
4 & & \\
\end{array}
\]
means \( \frac{15}{4} - \frac{8}{3} \)

Later the ‘+’ sign was placed by a dot over the number to which it referred. Also, the notation in the above example is the representation of fractions. It is similar to the present day representation in which the denominator is placed below the numerator without the line between the two numbers.

Therefore, the above representation means the unknown quantity whose value we are seeking. Without the signs of later times when the Hindus had a well developed system of symbols for the unknowns. Consider the example given in [34].

Multiplication is indicated by placing the numbers side by side. Also it was indicated by placing gu for ga\(\text{ñita}\. bhā is an abbreviation of bhāga, (भाग) meaning ‘part’. It indicates that the number preceeding it is to be treated as a denominator.

Thus bhā is the symbol for division. A square root was indicated by mū for múla after the term.
In the Bakhshāli treatise, there is no specific symbol for the unknown. Its place in an equation is left vacant and to indicate it vividly the sign of emptiness is put there.

For example,

\[
\begin{array}{cccc}
2 & 3 & 4 & \text{drs'ya} \\
1 & 1 & 1 & 1
\end{array}
\]

means \( x + 2x + 3x + 4x = 200 \) \[14\], \[38\].

The use of the zero sign to mark a vacant place is found in the arithmetical treatises of later times when the Hindus had a well developed system of symbols for the unknowns. Consider the example given in [34],

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
3+ & 3+ & 3+
\end{array}
\]

\( bhā \) 32 \( \text{phala} 108 \)

The black dot is used to denote the unknown quantity whose value we are seeking. The second column denotes 1 minus 1/3 or 2/3. Without the ‘+’ sign, it would denote 1 plus 1/3. In the above representation multiplication is \( \left( \frac{2}{3} \right) \times \left( \frac{2}{3} \right) \times \left( \frac{2}{3} \right) \) or \( \frac{8}{27} \).

Therefore, the above representation means

\[
x = \left[ \left( \frac{2}{3} \right)^3 \right]^{-1} \times 32 = \left( \frac{27}{8} \right) \times 32 = 108
\]

In other words, a certain number (unknown) is found by taking the reciprocal of 8/27 and multiplying the result by 32, that number is 108.

In. Bakhshāli Manuscript the unknown quantity was called \( yadṛcchā \) (यदृच्छा), \( vaṃḍhā \) (वांछा) or \( kāmika \) (कामिका) meaning any desired quantity \[14\]. This term was originally connected with the ‘Rule of False Position’.
In the Bakhshali treatise, the absolute term is called *drśya* (~) meaning visible. The dot is also used to denote zero. The dot indicated an empty place. Its Sanskrit name is *Śunya* (~) means ‘empty’ or ‘void’. From its Arabic translation to *sifr* comes the English word ‘cipher’ [14].

Consider the following examples from Bakhshali Manuscript present in [14].

(i).  
\[
\begin{array}{c}
0 & 1 & 1 & 1 & 1 \\
\end{array}
\]
\[x + \frac{5}{1} \]
and
\[
\begin{array}{c}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]
\[\frac{11}{1} + \frac{5}{1} \]

(ii).  
\[
\begin{array}{c}
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]
gu
\[\text{means} \]
\[3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 10 \]

(iii)  
\[
\begin{array}{c|c|c|c|c}
0 & 1 & 1 & 1 & 1 \\
\end{array}
\]
\[1 \quad 2 \quad 5 \quad 3 \quad 7 \quad 4 \quad 9 \]
gu
\[2+ \quad 2+ \quad 2+ \quad 1 \quad 1 \]

This means
\[
x \left(1 + \frac{3}{2}\right) + \left\{2x \left(1 + \frac{3}{2}\right) - \frac{5}{2} x\right\} + \left\{3x \left(1 + \frac{3}{2}\right) - \frac{7}{2} x\right\} + \left\{4x \left(1 + \frac{3}{2}\right) - \frac{9}{2} x\right\} \]

(iv)  
\[
\begin{array}{c|c|c|c|c|c}
1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]
36
\[\text{bhā} \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \]
\[2 \quad 3 \quad 4+ \quad 6 \]

\[\frac{36}{\left(1 - \frac{1}{2}\right) \left(1 + \frac{1}{3}\right) \left(1 - \frac{1}{4}\right) \left(1 + \frac{1}{6}\right)} \]

\[= \]

\[\frac{36}{\left(1 - \frac{1}{2}\right) \left(1 + \frac{1}{3}\right) \left(1 - \frac{1}{4}\right) \left(1 + \frac{1}{6}\right)} \]
Lack of an efficient symbolism is bound to give rise to a certain amount of ambiguity in the representation of an algebraic equation especially when it contains more than one unknown.

For example,

\[
\begin{array}{ccc|c}
0 & 5 & yu & 0 \\
1 & 1 & & 1 \\
\end{array} \quad \begin{array}{ccc|c}
0 & 7+ & mu & 0 \\
1 & 1 & & 1 \\
\end{array}
\]

which means $\sqrt{x+5} = s$ and $\sqrt{x-7} = t$. Here, different unknowns have to be assumed at different vacant places [14], [70].

It is given in [14] that, to avoid such ambiguity, in one occasion which contains as many as five unknowns, the abbreviations of ordinal numbers such as prā (from prathama, प्रथमं) first), dvi (from dvitīya, द्वितीया), second) tr (from tritīya, तृतीया third), ca (from caturtha, चतुर्थं fourth) and paṁ (from pañcama, पञ्चमं, fifth) have been used to represent the unknowns.
Example 2.1 [14], [70]

<table>
<thead>
<tr>
<th>$9 \text{ pra}$</th>
<th>$7 \text{ dvi}$</th>
<th>$10 \text{ tr}$</th>
<th>$8 \text{ ca}$</th>
<th>$11 \text{ pam}$</th>
<th>$9 \text{ pra}$</th>
</tr>
</thead>
</table>

This means in symbols as,

\[
x_1(=9) + x_2(=7) = 16
\]
\[
x_2(=7) + x_3(=k) = 17
\]
\[
x_3(=10) + x_4(=8) = 18
\]
\[
x_4(=8) + x_5(=11) = 19
\]
\[
x_5(=11) + x_6(=9) = 20
\]

An algebraic solution to a linear equation in one unknown appears for the first time in the Bakhshali Manuscript. The method used was an inversion method where one works backwards from given information.

According to [5], the first systematic treatment of the Rule of Three is found in the Bakhshali Manuscript. The term used in ancient India for it is triśatikā (त्रैशःतिक). It has appeared in the Bakhshali Manuscript, Āryabhaṭīa of Āryabhata-I and in all other later mathematical works. It was earlier discussed in a problem of Chiu Chang in [34]. The rule may be stated in the following form:

If P (argument or pramāṇa) (प्रमाणम्) yields f (fruit or phala) (फलम्), what will i (requisitor or iccha) (इच्छा) yield? [34]

The three quantities are set down as follows.

| $p$ | $f$ | $i$ |

$p$ and $i$ are of the same denominations and $f$ is of a different denomination. For the result, the middle quantity is to be multiplied by the last
quantity and divided by the first, to give the result \( \frac{f_i}{p} \). The following examples from the Bakhshali Manuscript [53],[34] illustrate the application of this rule.

**Example 2.2**

"Two Page-boys are attendants of a King. For their services one gets \( \frac{13}{6} \) Dinara a day and the other \( \frac{3}{2} \). The first owes the second 10 Dinaras. Calculate when they have equal amounts".

Take the denominators 6 and 2, together with the numbers 10 that the first has to give. The lowest common multiple of 2, 6 and 10 is 30. Therefore 30 is the \( \text{iccha} \). Now apply the rule of three

<table>
<thead>
<tr>
<th>p (day)</th>
<th>f (Dinara)</th>
<th>i (day)</th>
<th>Required result</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{13}{6} )</td>
<td>30</td>
<td>( \frac{f_i}{p} = 65 \text{ dinaras} )</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{3}{2} )</td>
<td>30</td>
<td>( \frac{f_i}{p} = 45 \text{ dinaras} )</td>
</tr>
</tbody>
</table>

If the 1st page boy gives 10 Dinara to the second, both will be left with 55 Dinaras.

**Example 2.3** [5]

"A certain lame person goes to a distance \( \frac{1}{8} \) of a yojana in \( \frac{1}{3} \) of a day. Say in how much time will he be going to a distance of 100 yojanas?"

Here \( p = \frac{1}{8} \) of a yojana

\( f = \frac{1}{3} \) of a day.

\( i = 100 \) yojanas
Required result = \( \frac{f_i}{p} = \frac{1}{3} \times 100 = \frac{100}{3} = 266 \frac{2}{3} \) days = 8 months and 26 2/3 days.

Recorde calls this rule, ‘The Rule of Proposition’. It was also known as the ‘Golden Rule’.

The method of solving simple linear equations of the type \( ax + b = 0 \) by substituting guess values, say \( g_1, g_2 \), etc. was known to Indians [5]. The problem of this type dealt with in the Sthānāṅga-sūtra (स्थानाङ्गसूत्रम्) using the term yāvat-tāvat (ऽ्यावतः – वात) for the unknown quantities has been discussed. A solution to this problem also occurs in the Bakhshālī Manuscript.

Example 2.4 [14]

“The amount given to the first is not known. The second is given twice as much as the first; the third thrice as much as the second; and the fourth four times as much as the third. The total amount distributed is 132. What is the amount of the first?”

In modern terms, if \( x \) is the amount given to the first person then

\[
x + 2x + 6x + 24x = 132
\]

‘Putting any desired quantity in the vacant place’; any desired quantity is \( \| 1 \|; \); ‘then construct the series’
Some set of problems in the Bakhshālī Manuscript leads to an equation of the form \( ax + b = p \). The method for its solution is to put any arbitrary value for \( x \), so that \( ag + b = p' \) (say)

Then the correct value is, \( x = \frac{p - p'}{a} + g \)

This method of solution of linear equation was known in the middle ages among Arab and European algebraists as the ‘Rule of False Position’. The terms yadrccchā, vāṅchā and kāmika’ of the Bakhshālī treatise are equivalent to the term yāvat-tāvat. So the origin of the term yāvat-tāvat seems to be connected with the ‘Rule of False Position’.

Another problem from Manuscript is:

Two persons start with different initial velocities \((a_1, a_2)\); travel on successive days, distances increasing at different rates \((b_1, b_2)\). But they cover the same distance after the same period of time. What is the period? [14]

They give the solution as:

Let \( x \) be the period. Then,

\[
a_1 + (a_1 + b_1) + (a_1 + 2b_1) + \ldots \ldots \text{ to } x \text{ terms} = a_2 + (a_2 + b_2) + (a_2 + 2b_2) + \ldots \ldots \text{ to } x \text{ terms}
\]
\[ \left\{ a_1 + \left( \frac{x-1}{2} \right)b_1 \right\}x = \left\{ a_2 + \left( \frac{x-1}{2} \right)b_2 \right\}x, \]

from it, \[ x = \frac{2(a_2 - a_1)}{b_1 - b_2} + 1 \]

This is the solution given in the Bakhshālī Manuscript.

“Twice the difference of the initial terms divided by the difference of the common differences is increased by unity. The result will be the number of days in which the distance moved will be the same.”

The earliest Hindu treatment on systems of linear equations in several unknowns is found in the Bakhshālī Manuscript. They belong to a system of linear equation of the type,

\[ x_1 + x_2 = a_1, x_2 + x_3 = a_2, \ldots, x_n + x_1 = a_n, \text{ } n \text{ being odd. } \ldots \ldots \ldots \ldots \ldots \text{ (A)} \]

The solution is given by the 'Method of False Position.'

Assume an arbitrary value \( p \) for \( x_1 \) and then calculate the values of \( x_2, x_3, \ldots, x_n \) corresponding to it. Let the final calculated value of \( x_n + x_1 \) be equal to \( b \) (say). Then \( x_1 \) is obtained by the formula \( x_1 = p + \frac{1}{2}(a_n - b) \)

Then, we can calculate the corresponding other values.

**Example 2.5** [14],[38]

“Three persons possess a certain amount of riches each. The riches of the first and the second taken together amount to 13; the riches of the second and the third taken together are 14; and the riches of the first and the third mixed are known to be 15. What are the riches of each?”
In modern terms, if \( x_1, x_2, x_3 \) be the riches of the three persons respectively,

Then \( x_1 + x_3 = 13, x_2 + x_3 = 14, x_2 + x_1 = 15 \). Assume the arbitrary value \( x_1 = 5 \).

Then \( x_2 = 8, x_3 = 6 \) and \( x_2 + x_1 = 11 \)

Then, \( p = 5, a_n = 15, b = 11 \)

Then the correct values are

\[
x_1 = p + \frac{1}{2} (a_n - b) = 5 + \frac{(15 - 11)}{2} = 7;
\]

\( x_2 = 6 \) and \( x_3 = 8 \)

Another example is,

Five persons possess a certain amount of riches each. The riches of the first and the second mixed together amount to 16; the riches of the second and third taken together are known to be 17; the riches of the third and the fourth taken together are known to be 18; the riches of the fourth and fifth mixed together are 19; and the riches of the first and fifth together amount to 20. Tell me what the amount of each is? [4], [14], [38].

In modern terms, \( x_1 + x_2 = 16, x_2 + x_3 = 17, x_3 + x_4 = 18, x_4 + x_5 = 19, x_5 + x_1 = 20 \).

The 'Rule of False Position' points at an early stage of development of the algebra, when there was no symbol for the unknown. It naturally disappears with the introduction of system of notations [68].

An interesting example is,
Five merchants together buy a jewel. Its price is equal to half the money possessed by the first together with the money possessed by the others or one third the money possessed by the second together with the moneys of the others or 1/4 th the money possessed by the third together with the moneys of the others or 1/5 th the money possessed by the fourth together with the moneys of the others or 1/6 th the money possessed by the fifth together with the moneys of the others. Find the price of the jewel, and the money possessed by each merchant [70].

Let \( x_1, x_2, x_3, x_4, x_5 \) be the money possessed by the merchants respectively. Let \( p \) be the price of the jewel. Then

\[
\frac{1}{2} x_1 + x_2 + x_3 + x_4 + x_5 = x_1 + \frac{1}{3} x_2 + x_3 + x_4 + x_5
\]

\[
= x_1 + x_2 + \frac{1}{4} x_3 + x_4 + x_5 = x_1 + x_2 + x_3 + \frac{1}{5} x_4 + x_5
\]

\[
= x_1 + x_2 + x_3 + x_4 + \frac{1}{6} x_5 = p
\]

Hence we have, \( \frac{1}{2} x_1 = \frac{2}{3} x_2 = \frac{3}{4} x_3 = \frac{4}{5} x_4 = \frac{5}{6} x_5 = q \) (say)

Put this in anyone of the above equations,

We get \( \frac{377}{60} q = p \).

To get integral solutions, we take \( q = 60 \) m and then \( p = 377 \) m, where ‘m’ is any integer. The answer given in the Bakhshālī Manuscript is \( p = 377, x_1 = 120, x_2 = 90, x_3 = 80, x_4 = 75 \) and \( x_5 = 72 \).

There is some resemblance between the Bakhshālī Manuscript and the Chinese Chiu Chang in the topics discussed and in the style of presentation of results. Chiu Chang, written a few centuries earlier is far more wide ranging and ‘advanced’ than the Bakhshālī Manuscript.
Example 2.6 [17], [25], [67]

“A merchant pays duty at three different places on the goods that he carries. At the first place he gives 1/3 of the goods, at the second 1/4 of the remainder, at the third 1/5 of the remainder. He pays a total duty of 24 pieces. What is the original quantity of the good?”

This problem leads to a single linear equation in one unknown.

\[ \frac{x}{3} + \left( x - \frac{x}{3} \right) / 4 + (x - x / 3 - x / 6) / 5 = 24, \]

\[ x \] stands for the original quantity of the good.

The solution of the quadratic equation was known to the author of Bakhshâli Manuscript. In it there are some problems of the following type.

A certain person travels ‘s’ yojana (योजना) on the first day and ‘b’ yojana more on each successive day. Another who travels at the uniform rate of S yojana per day has a start of ‘t’ days. When will the first man overtake the second? [14].

Let \( x \) be the number of days after which the first overtakes the second. Then

\[ S(t + x) = x \left\{ s + \left( \frac{x - 1}{2} \right) b \right\} \]

Or

\[ bx^2 - \{2(S - s) + b\} x = 2tS \]

Therefore,

\[ x = \frac{\sqrt{\{2(S - s) + b\}^2 + 8bs} + \{2(S - s) + b\}}{2b} \]

(1)

This agrees exactly with the solution stated in Bakhshâli Manuscript.
The daily travel \([S]\) diminished by the march of the first day \([s]\) is doubled; this is increased by the common increment \([b]\). That (sum) multiplied by itself is designated \{as the \(K\text{ə}p\text{a}\) quantity\}. The product of the daily travel and the start \([t]\) being multiplied by eight times the common increment, the \(K\text{ə}p\text{a}\) quantity is added. The square root of this is increased by the \(K\text{ə}p\text{a}\) quantity; the sum divided by twice the common increment will give the required number of days [14].

If \(S = 5\), \(t = 6\), \(s = 3\) and \(b = 4\), is preserved by (1)

\[
x = \frac{40}{8} = 5
\]

If \(S = 7\), \(t = 5\), \(s = 5\), \(b = 3\) then by (1)

\[
x = \frac{7 + \sqrt{889}}{6}
\]

The Bakhshali Manuscript gives the solution of a problem in the form \(ax^2 + bx - c = 0\) which reduces to \(x = \frac{\sqrt{b^2 - 4ac} - b}{2a}\) [5], [34], [49].

The Indian Mathematicians of the medieval period were basically algebraists. Their works are full of rules for calculating positive and negative numbers, with fractions and algebraic expressions. They teach how to solve linear and quadratic equations in one and several variables. They gave rules for the summation of arithmetic and geometric expressions as well as squares, cubes, and triangular numbers. They gave procedures for dealing with surds. All their rules and methods are correct. In most cases, there is no indication of how the methods and rules are derived or why they are true.

Formulas are indicated which are not strictly true but are only approximations. But it is not mentioned, they are only approximations. There is no written proofs of any type, no arguments to convince one of the correctness of the results [36], [75].
In the development of early mathematics, when the symbols for operation began to be used, a new branch evolved which named was algebra. It came to be separated from arithmetic and geometry. The differentiation of algebra as a distinct branch of mathematics generally took place during the time of Brahmagupta (628 AD). This began with the techniques of indeterminate analysis. Brahmagupta used the term Kuṭṭaka, (कुटटक) Kuṭṭaka-Gaṇita (कुटटक गणित) to denote algebra. The term Bijaganita (बीज गणित) means science of calculation with unknown quantities was hinted by Prthūdakasvāmī (860 AD).

Āryabhaṭa -I

Our knowledge of the history of Indian mathematics prior to 499 AD is very imperfect and not continuous. Barring Jaina mathematics and Bakhshāli manuscript we are ignorant as to the mathematicians and their works who lived prior to Āryabhaṭa-I. Even Āryabhaṭa-I’s work, Āryabhaṭīa (आर्यभट्टियाः) written in 499 AD had been lost. In 1864, the Indian scholar Bhau Daji got a copy of it.

The number of mathematicians who lived by the name Āryabhaṭa-I, has not been definitely answered. There were at least two Āryabhaṭas. The Āryabhaṭa of Kusuma Pura wrote the work Āryabhaṭīa in 499 AD. He is known as Āryabhaṭa-I. Another Āryabhaṭa wrote Āryasiddhānta (आर्यसिद्धान्त) in about 950 AD. Some call it, the Mahā Āryasiddhānta (महा आर्यसिद्धान्त). The Muslim historian Al-Biruni in his history written in 1030 AD speaks about two Āryabhaṭas.
Aryabhata of Kusumapura says that he wrote his work in his twenty-third year when the 3600th year of Kaliyuga was running. This means that he was born in 476 AD. He belonged to the Kusumapura School. But it seems probable that he was a native of Kerala. Even now the calendar based upon the system given in the *Aryabhatia* is followed to some extent in Kerala.

The *Aryabhatia* is a very small work. The book is very concise. There is considerable difficulty in understanding the meaning in several places.

The text *Aryabhatia* consists of four parts or *Pādas* viz; *daśa gītikā* (दश गीतिका), *gaṇita* (गणित), *kālakriyā* (कालक्रिया) and *gōla* (गोल) [32].

The first part, *gītikā-pāda* (गीतिका पाद), explains the basic definitions, important astronomical parameters and tables. The second part, *gaṇita pāda* (गणित पाद) deals with geometrical figures, their properties and mensuration; problems on the shadow of the gnomon, arithmetic progression, geometric progression; simple and compound interest; simple, simultaneous quadratic and linear indeterminate equations; square-root, cube root, the ‘Rule of Three’ etc. In this part, he presents a method of solving a first order indeterminate equation in two unknowns of the type \( ax + c = by \). The method...
of solving these equations has been [75] called *Kuṭṭaka* by later mathematicians. This equation is wrongly called ‘Diophantine equation’. The third part *Kālakriyā pāda* (कालक्रियापाद) contains 25 *slokās* (श्लेष्म) explaining various units of time and the method of determination of the positions of planets for any given day. The fourth part, *Gōlādhyāya* (गोलाध्याय) contains 50 stanzas, explaining the aspects of the celestial sphere.

Āryabhaṭa-I got the solution \( x = \frac{b-a}{m-n} \) from the relation \( mx + a = nx + b \). He tackled the problem of inverse operation. For example, \( ax = pq \) and \( ax ± b = p \), he gave the solutions \( x = \frac{pq}{a} \) and \( ax = p ± b \).

According to [14], Āryabhaṭa-I says: The difference of the known “amounts” relating to the two persons should be divided by the difference of the coefficients of the unknown. The quotient will be the value of the unknown, if their possessions be equal.

This rule gives a problem of the following kind.

Two persons, who are equally rich, possess respectively \( a, b \) times a certain unknown amount together with \( c, d \) units of money in cash. What is that amount?

In modern terms, let \( x \) be the unknown amount. Then by the problem, \( ax + c = bx + d \). Therefore \( x = \frac{d-c}{a-b} \).

As stated already the earliest Hindu treatment of systems of linear equations involving several unknowns is found in Bakhshālī Manuscript.

Let us consider the system of equations:

\[
\Sigma x - x_1 = a_1, \Sigma x - x_2 = a_2, \ldots, \Sigma x - x_n = a_n \quad \ldots \ldots \ldots \quad (*)
\]
where $\Sigma x = x_1 + x_2 + \ldots + x_n$. This type of equations has been solved by Āryabhaṭa-1 and Mahāvīra [32].

Āryabhaṭa-1 says:

“The (given) sums of certain (unknown) members, leaving out one number in succession, are added together separately and divided by the number of terms less one; that (quotient) will be the value of the whole” [14].

In modern symbols $\Sigma x = \sum_{i=1}^{n} a_i / (n - 1)$

In Āryabhaṭiya of Āryabhaṭa-1, we meet problems on simple interest that involve quadratic equations. He indicated his knowledge of quadratic equation and their solutions. He gave the solution in connection with an interest problem as follows:

$$x = \frac{-p + \sqrt{p^2 + 4tpq}}{2t}$$

where $p =$ Principal, $t =$ time, $q =$ sum of interest on principal and interest on this interest in time $t$, $x =$ interest on principal in unit time [5],[14].

Example 2.7 [77]

“One hundred is loaned for one month, and the interest received is loaned for six months. The total of the interest and the interest on the interest is 16. Find the interest on the principal.”

This problem gives a quadratic equation.

Āryabhaṭa-1’s solution is as follows.

Add the interest on the principal and interest on this interest, and multiply this by the time and by the principal, the result is 9600. Add this to the square of half the principal 2500, making 12100. Find the square root 110.
Then subtract half the principal and divide by the time, getting 10. This is the interest on the principal.

Āryabhaṭa-I calculated the value of unknown numbers of terms ‘n’ for an arithmetic progression series when its sum S, first term ‘a’, common difference ‘d’ are known with the help of quadratic equation as follows:

\[ n = \frac{1}{2} \left( \frac{\sqrt{8dS + (2a - d)^2} - 2a + 1}{d} \right) \] [5],[6],[36]

This was also given by Brahmagupta, Mahāvīra and Bhāskara-II.

According to [14], [36], the following two stanzas of Āryabhaṭiya lead to the sum and the \( n^{th} \) term of an arithmetic progression.

The first one is:

The desired number of terms minus one, halved... multiplied by the common difference between the terms, plus the first term, is the middle term. This multiplied by the number of terms desired is the sum of the desired number of terms. Or the sum of the first and last terms is multiplied by half the number of terms.

Here Āryabhaṭa-I gives as a formula for the sum \( S \) of an arithmetic progression with initial term ‘a’ and common difference ‘d’. The formula is

\[ S = n \left( \frac{n-1}{2} d + a \right) = \frac{n}{2} \left[ a + (a + (n-1)d) \right] \]

The second one is:

Multiply the sum of the progression by eight times the common difference add the square of the difference between twice the first term and the common difference, take the square root of this, subtract twice the first term, divide by the common difference, add one, divide by two. The result will be the number of terms.
This is the same as the formula given above for ‘n’ terms of an arithmetic progression.

If we rewrite the formula in the first one as a quadratic equation, we get
\[ a n^2 + (2a - d) n - 2S = 0. \]

Then the value of ‘n’ in the formula of second one follows from the quadratic formula. Here, Āryabhaṭa-I does not give a quadratic formula. But Brahmagupta wrote out the formula in the form needed. This leads to the inference that the formula was known to Āryabhaṭa-I as well.

Āryabhaṭa-I has not mentioned anywhere any method of solving the quadratic. But from the above two forms it follows that he followed two different methods to make the unknown side of the equation \[ ax^2 + bx = c, \] a perfect square. In one case, he multiplied both the sides of the equation by \( 4a \) and in the other simply by \( 'a' \) [14].

**Brahmagupta**

Brahmagupta belongs to the Ujjain school. He was born in 598 AD. Probably, he was a native of Sind, now a part of Pakistan. He was a resident of Bhinmâla situated on the northern boarder of Gujarat. This place was formerly known Bhilamâla or Sriamâla. He wrote his *Brāhma-sphuta-siddhānta* (ब्रह्म स्फुट सिद्धान्त) in his thirtieth year (628 AD).

This book is a short volume on astronomy of which \( 4 \frac{1}{2} \) chapters are devoted to pure mathematics. He called the twelfth chapter as *Gaṇita* (Arithmetic) and the eighteenth chapter as *Kūṭṭaka*. The word *Kūṭṭaka* literally means pulverizer. This may be compared with the word ‘analysis’ and denotes algebra. We don’t know exactly when the word *Bījagaṇīta* came to be used to represent algebra. This word occurs for the first time in the writings of Prthūdakasvāmī (860 AD). From the time of Bhâskara-II the word *Kūṭṭaka* has
been used to denote the solution of indeterminate equation $ax - by = c$ only [75].

Through the *Brāhma-sphuṭa-siddhānta* the Arabic became familiar with Indian Astronomy. The King Khalif Abbasid Al-Mansoor (712-775), of Baghdad invited Kanka, a scholar of Ujjain who reached Baghdad by about 770 AD. He explained to the Arabs, the Hindu system of arithmetic and astronomy. *Brāhma-sphuṭa-siddhānta* was translated into Arabic by Al-Fazārī [9]. It was named ‘Sind Hind’ or ‘Hind Sind’. It was in general use among the Arabs for a long time. Dr. Sachau, in his translation of Al-Biruni’s ‘India’ says that the Arabs learnt their astronomy from *Brāhma-sphuṭa-siddhānta* before they came to know about Ptolemy. *Alarkand*, another translation of Indian astronomy also was known to the Arabs. This may have been a translation of *Sūryasiddhānta* (सूर्य सिद्धान्त). The *Aryabahar* (Āryabhatiya) does not seem to have reached the Arabs.

Brahmagupta presents the rules for operations on positive and negative numbers:

The sum of two positive quantities is positive; of two negative is negative; of a positive and a negative is their difference; or if they are equal, zero. In subtraction the less is to be taken from the greater, positive from positive; negative from negative. When the greater, however is subtracted from the less, the difference is reserved. When positive is to be subtracted from negative, and negative from positive, they must be thrown together. The product of a negative quantity and a positive is negative; of two negative is positive; of two positive is positive. Positive divided by positive or negative by negative is positive. Positive divided by negative is negative. Negative divided by positive is negative [36].

Brahmagupta gave the following classification of equations.

(i) *Eka-varṇa-samīkarana* (एक वर्ण समीकरण) - equations in one unknown, both linear and quadratic equations.
(ii) Aneka-varṇa-samīkarana (अनेक वर्ण समीकरण) - equations in several unknowns.

(iii) bhāvita (भवित) – equations involving products of unknowns.

This classification of unknowns was further elaborated by Prthudakasvāmī and Bhāskara II [49].

Brahmagupta used Sanskrit letters, the abbreviations of names of different colours, to represent several unknown quantities. A letter kā (क) stood for Kālaka (कलङ्क) meaning black. The letter nī (नी) for nīlaka (नीलङ्क) meaning blue. A general term for any unknown was yāvat-tāvat which was shortened to the algebraic symbol yā (य). For the sign of equality, the terms drīya (दृश्य) (visible) and rūpa (रूप) were used.

Brahmagupta wrote another book Khanda-khādyaka (खण्डकाश्यक) in 665 AD. This is an expository book on astronomy.

He also gives a rule for interpolation with date at unequal intervals. The meaning of the rule is not clear. Interpolation theory was just initiated by Brahmagupta. But neither he nor any succeeding mathematician of India developed the subject further. He is the world’s first mathematician to use second order differences.

Hindus had a new plan of writing an equation in which the two sides are written, one below the other without any sign of equality. In this plan, terms of similar denominations are usually written one below the other. The terms of absent denominations on either side are expressed by putting zeros as their coefficients. This new plan is found as early as the algebra of Brahmagupta.

As per [14] regarding the solution of linear equations, Brahmagupta says:
"In a (linear) equation in one unknown, the difference of the known terms taken in the reverse order, divided by the difference of the coefficients of the unknown (is the value of the unknown)."

The following problem from Brahmagupta illustrates:

"Tell the number of elapsed days for the time when four times the twelfth part of the residual degrees increased by one, plus eight will be equal to the residual degrees plus one."

Almost all Hindu writers discussed Saṅkramaṇa (concurrence) Nārāyaṇa called it Saṅkrama. Brahmagupta and Śrīpati include it in algebra while others in arithmetic.

Saṅkramaṇa is the solution of the simultaneous equations.

\[ x + y = a \]
\[ x - y = b \]

Brahmagupta [14] gives the following rule:

"The sum is increased and diminished by the difference and divided by two; (the result will be the two unknown quantities) (this is) concurrence."

Brahmagupta restates this rule on a different occasion in the form of a problem and its solution.

"The sum and difference of the residues of two (heavenly bodies) are known in degrees and minutes. What are the residues? The difference is both added to and subtracted from the sum, and halved; (the results are) the residues." [14]

A generalized system of another type of linear equation in modern terms will be
Therefore \( \Sigma x = \frac{\Sigma (a/c)}{\Sigma (b/c) - 1} \)

Hence \( x_r = \frac{b_r \Sigma (a/c)}{c_r \Sigma (b/c) - 1} - \frac{a_r}{c_r} \) \((**)\)

where \( r = 1, 2, 3 \ldots n \)

But Brahmagupta gave a rule for the particular case that \( b_1 = b_2 = \ldots = b_n = 1 \) and \( c_1 = c_2 = \ldots = c_n = c \).

The total value (of the unknown quantities) plus or minus the individual values (of the unknowns) multiplied by an optional number being severally (given), the sum (of the given quantities) divided by the number of unknowns increased or decreased by the multiplier will be the total value; then the rest (can be determined) \([14]\).  

In modern symbols,

\[
\Sigma x \pm cx_i = a_1, \quad \Sigma x \pm cx_2 = a_2, \ldots, \quad \Sigma x \pm cx_n = a_n
\]

Therefore \( \Sigma x = \frac{a_1 + a_2 + \ldots + a_n}{n \pm c} \)

Hence \( x_i = \frac{1}{c} \left( \pm a_i \pm \frac{a_1 + a_2 + \ldots + a_n}{n \pm c} \right) \) and so on.

Brahmagupta gives another rule as follows,

Removing the other unknowns from (the side of) the first unknown and dividing by the coefficient of the first unknown; the value of the first unknown (is obtained). In the case of more (values of the first unknown) two and two (of them) should be considered after reducing them to common denominators. And (so on) repeatedly. If more unknowns remain (in the final equation) the method of the pulveriser (should be employed).
Then proceeding reversely (the values of other unknowns can be found) [14].

Later Prthūdakasvāmī gave an explanation for this.

The following is an example [17], [25], [67], of Brahmagupta which leads to a quadratic equation.

Two ascetics lived at the top of a cliff of height 100 yojanas, whose base was at a distance of 200 yojanas from a neighboring village. One descended the cliff and walked to the village. The other, being a wizard, first flew up to a certain height above the cliff, and from that height flew in a straight line to the village. The total distance traversed by each was the same. Find the height to which the second ascetic ascended.

In modern terms the problem can be expressed as,

\[(100 + x)^2 + 200^2 = (300 - x)^2,\]

where \(x\) is the height ascended. By simplification, this quadratic equation reduces to a linear equation.

Next is an ‘interest’ problem of Brahmagupta, which leads to a quadratic equation.

The interest accumulating on 500 for 4 months is lent out at the same rate for 10 months and the amount is 78. What is the rate of interest? [70]

Let \(x\) be the interest on 500 for 4 months. This gives the equation,

\[\frac{x^2}{200} + x = 78.\]

Therefore \(x = 60\) or the rate of interest is 3% per month.

Brahmagupta has given a quadratic solution arising out of an interest problem. He has also given a solution of the problem of quadratic equations in connection with the determination of the number of terms ‘\(n\)’ in an arithmetic progression.
Brahmagupta presented the quadratic formula virtually in the same form as we know it.

Take absolute number on the side opposite to that on which the square and simple unknown are. To the absolute number multiplied by four times the [coefficient of the] square, add the square of the [coefficient of the] unknown; the square root of the same, less the [coefficient of the] unknown, being divided by twice the [coefficient of the] square is the [value of the] unknown.

We can translate these words into the formula \( x = \frac{\sqrt{4ac + b^2} - b}{2a} \), for finding one solution of the equation \( ax^2 + bx = c \) \([14],[36],[67],[71]\).

Example 2.8 \([36],[69]\)

Brahmagupta presented the following equation.

\[ yā v1 \quad yā -10 \quad rū - 9 \]

He used some symbolism for the unknown. \( yā \) is an abbreviation for \( yāvata-tāvat \) (how much), \( rū \) is for \( rūpa \) (absolute number), \( yā v \) for \( yāvata-tāvat-varga \), (square of the unknown). In modern terms this is the same as \( x^2 - 10x = -9 \)

This equation can be solved by the above formula as follows:

Now the absolute number (-9) multiplied by four times the [coefficient of the] square (-36), and added to the square (100) of the [coefficient of the] unknown (making 64), the square-root being extracted (8), and lessened by [the coefficient of the] unknown (-10), the remainder 18 divided by twice the [coefficient of the] square (2) yields the value of the unknown 9.

The given equation has a second positive solution, corresponding to taking the negative of the square root less the coefficient of the unknown. He did not mention this second solution.
The second rule of Brahmagupta [36], [71] is as: “The absolute term multiplied by the coefficient of the square of the unknown is increased by the square of half the coefficient of the unknown; the square root of the result diminished by half.”

Brahmagupta uses a third formula which is similar to the one now commonly used. It has not been expressed in any rule. But we find its application in a few cases. One of them is as follows:

“A certain sum ‘p’ is lent out for a period ‘t_1’; the interest accumulated ‘x’ is lent out again at this rate of interest for another period t_2 and the total amount is ‘A.’ Find x” [14].

The equation for determining x is \( \frac{t_2}{pt_1} x^2 + x = A \)

Then \( x = \sqrt{\left( \frac{pt_1}{2t_2} \right)^2 + \frac{Apt_1}{t_2} - \frac{pt_1}{2t_2}} \)

This is exactly the form in which Brahmagupta states the result.

In *Brāhma-sphuta-siddhānta*, there is a certain astronomical problem involving the quadratic equation given in [14] as follows:

\[
(72 + a^2)x^2 + 24ax = 144\left( \frac{R^2}{2} - p^2 \right),
\]

where \( a = \text{agrā} \) (the sine of the amplitude of the sun), \( \text{b = Palabhā} \) (the equinotical shadow of a gnomon 12 anguli long), \( R = \text{radius} \) and \( x = \text{Konaśanku} \) (the sine of the altitude of the sun when its altitude is 45°).

When divided by \((72 + a^2)\), we get \( x^2 + 2mx = n \) where
\[ m = \frac{12ap}{72 + a^2}, \quad n = \frac{144\left(R^2 / 2 - p^2\right)}{72a^2} \]

Therefore, to him \( x = \sqrt{m^2 + n} \pm m \)

This result is also given in *Sūryasiddhānta* and also by Sripati.

According to [14], Brahmagupta knew the existence of two roots of quadratic equation. In his illustration of rules for the solution of quadratic equation, he has stated two examples involving practically the same equation.

(i) "The square root of the residue of the revolution of the sun less 2 is diminished by 1, multiplied by 10 and added by 2: when will this be equal to the residue of the revolution of the sun less 1, on Wednesday?"

(ii) "When will the square of one-fourth the residue of the exceeding months less there be equal to the residue of the exceeding months?"

Sridhara

There are several theories about his life time. To some historians his period was around 750 AD. Some believed that he lived at the end of the 9th century. It is certain that his life time was some time between 7th and 10th centuries.
Sridhara used the method of solving a quadratic equation by completing the square. His work on algebra is now lost. But the relevant portion of it is kept in quotations by Bhāskara- II and others. Sridhara’s method is:

“Multiply both the sides (of an equation) by a known quantity equal to four times the coefficient of the square of the unknowns; add to both sides a known quantity equal to the square of the (original) coefficient of the unknown; then extract the root” [5], [14], [34], [64], [69].

We can express this in modern terms as follows:

To solve \( ax^2 + bx = c \)

We have \( 4a^2 x^2 + 4abx = 4ac \)

Or \( (2ax + b)^2 = 4ac + b^2 \)

Then \( 2ax + b = \sqrt{4ac + b^2} \)

Therefore, \( x = \frac{\sqrt{4ac + b^2} - b}{2a} \)
An application of this rule is found in Śrīdhara’s *Trisatikā* in connection with finding the number of terms of an arithmetic progression.

That is, \[ n = \frac{\sqrt{8dS + (2a-d)^2} - 2a + d}{2d} \] where ‘a’ is the first term, ‘d’ the common difference ‘S’ the sum of ‘n’ terms.

Bhāskara-II’s quotes of Śrīdhara’s work presents no evidence that Śrīdhara used both signs of the radical [34], [64].

2.2 801-1200 AD

Arabs

The most important contributions of Islamic mathematicians are in the area of algebra. They took the mathematics already developed by the Babylonians, combined it with the classical Greek geometry and produced a new algebra. Also they tried to extend it. In the Islamic world, the chief Greek mathematical classics were well known by the time of 9th century. Islamic scholars studied them and made commentaries on them. The notion of proof was the most important idea they got from their study of Greek works. They got the idea that one could not consider a mathematical problem solved unless one could demonstrate that the solution was valid. The only real proofs were geometric. Islamic scholars set themselves the tasks of justifying algebraic rules, either from the ancient Babylon or new ones which they themselves discovered. This justification was through geometry.
The main contribution to the solution of linear equations by the Arabs was the definite recognition of the application of the axioms to the transposition of terms and the reduction of an implicit function of $x$ to an explicit function. This is given by Al-Khwārizmi.

The Arabs did not know about the advances of the Hindus. So the Arabs had neither negative quantities nor abbreviations for their unknown. However Al-Khwārizmi gave a classification of different types of quadratic, which are numerical examples of each. The different types arise since he had no zero or negatives.

**Al-Khwārizmi**

Al-Khwārizmi (780-850) was a scholar at the Dār-al-Hikma (House of Wisdom) when the Caliph Al-Ma’mun (809-833) reigned Baghdad. He was a notable scholar who wrote on various subjects. He worked on the numbering system of the Hindus, probably based on the work of Brahmagupta. A Latin translation of Al-Khwārizmi’s, ‘On the Indian Numbers’, became one of the means by which Europe learned of the new system of numeration.
Around 825 AD, he wrote Al-kitāb al-muhtasar fi hisāb al-jabr wa-l-muqābala which is translated as ‘The Condensed Book of Completion and Restoration’. The Arabic equivalent of ‘completion’ is al-jabr, from which the word algebra has been deduced [9].

In his work, notation is not used and numbers are spelled out. He and his successors solved problems by ‘reducing’ a problem to one of just a few fundamental types. Their solutions are straightforward. His work shifted the emphasis from solving equations to transforming equations using what is called the rules of algebra. His text consists of a large collection of problems, many of which involve these manipulation and most of which result in quadratic equations. His main contribution was in quadratic equations.

Al-Khwārizmi explained how to multiply the quantities including the unknowns. The application of the arithmetic techniques to the unknown quantities in algebra was an early step in arithmetization of algebra. He used the term ‘thing’ to represent the unknown quantity.

He observed that the numbers which are required in calculating by Completion and Reduction are of three kinds, namely, roots, squares and simple numbers relative to neither root nor square. A root is any quantity which
is to be multiplied by itself, consisting of units or numbers ascending, or fractions descending. A square is the whole amount of the root multiplied by it. A simple number is any number which may be pronounced without reference to root or square. *Dirhems* were used to indicate them.

The algebra of Al-Khwārizmi is neither purely Indian nor purely Greek.

Al-Khwārizmi’s work ‘The Condensed Book of Completion and Restoration’ consists of three parts. In the first part, he explains, the solution of six types to which all linear and quadratic equations can be reduced. As in [9], [50],[34],[61],[76],[81],[82] the six types are

(i) Squares equal to roots, that is, $ax^2 = bx$

(ii) Squares equal to numbers, that is, $ax^2 = b$

(iii) Roots equal to numbers, that is, $ax = b$

(iv) Square and roots equal to numbers, that is, $ax^2 + bx = c$

(v) Square and numbers equal to roots, that is, $ax^2 + c = bx$

(vi) Roots and numbers equal to square, that is, $ax^2 = bx + c$ where $a$, $b$, $c$ are given positive numbers.

He called the first power of the unknown as a ‘root’, the second power as a ‘square’

He gave rules to solve these equations, demonstrated and illustrated them with examples.

After disposing the first three simple equations, he proceeded to more complicated equations, combining squares, roots and simple numbers.
Al-Khwārizmi did not use any symbols and presented everything in words, including numbers in his examples [10], [34], [36], [61], [72], [77], [82].

Example 2.9 [50], [36], [77]

“What must be the square which when increased by ten of its own roots, equals to thirty nine?”

A simplified version of Al-Khwārizmi’s method to solve the problem is as follows:

Take the half of the number of roots (i.e. 5).

Multiply this number by itself (i.e. 25).

Add this number to 39 (i.e. 64).

Obtain the root of this (i.e. 8)

Subtract from the above, half the number of roots.

The result is 3.

In modern terms, the problem can be expressed as $x^2 + 10x = 39$.

Demonstration

Al-Khwārizmi gave two demonstrations of the validity of the method of solution in the case of a ‘square and ten roots equals thirty-nine dirhems’. A geometrical version is given in figure 2.5. In both cases he started with a square $ABCD$ whose sides are unknown. The process of adding ten roots is the key to the procedure. According to him, adding ten roots is the same as adding a rectangle, one side of which is the side of the square and the other side is the number of roots to be added.
He added ten roots to the square ABCD. For this, he took half of ten, to get five and constructed rectangles BCEF and CDGH with one side, equal to the side of the square ABCD and the other side having a length five. The resulting figure is the square ABCD and ten roots BCEF and CDGH. It is short of a square ECHI of side five. Therefore, square ECHI is twenty-five.

![Figure 2.5](image)

If the square ECHI is added to the square and ten sides, the result is a square. Numerically the square and ten sides are equal to thirty-nine. The square ABCD, ten sides and square ECHI are altogether sixty-four. Thus the large square, FAGI is sixty-four. Its side is eight. That is, side of FAGI minus side of ECHI = 8-5 =3, side of the original square. From [34] and [71] we know that, Euclid and Al-Khwārizmi did not admit negative lengths. So the solution $x = -13$ to $x^2 + 10x = 39$ does not appear.

Al-Khwārizmi combined a rule with its example. Generalization was left to the reader. The geometric verification of this procedure reveals his Babylonian indebtedness.

Problems like 'two squares and ten roots are equal to forty-eight dirhems' were used to 'one square and five roots are equal to twenty-four dirhems'. He solved it in the same way as in the previous example.
In [36], it is given that, in solution of type 5 for the case \( x^2 + c = bx \), Al-Khwārizmi’s procedure corresponds to the formula

\[
x = b/2 \pm \left( \frac{b}{2} \right)^2 - c \right)^{1/2}
\]

**Example 2.10** [9], [76]

A square and twenty-one in numbers are equal to ten roots of the same square.

Take half the number of roots which is five, multiply by itself, making twenty-five. Subtract this from twenty-one (the number with the square), which is four. Find the root. Subtract this from half the roots, leaving three. This is the root of the square and the square is nine. Or add the root to half the roots, which is seven, and the root of the square, whose square is forty-nine.

**Demonstration**

Begin with figure 2.6, a square ABDC of unknown side length as in [77]. To this unknown, add the rectangle HABN representing twenty-one *dirhems*, with sides AB, NB. Thus the complete figure HCDN represents a square and twenty-one *dirhems*, which is equal to 10 roots. Since a root is the side of the square ACDB the side HC must be ten.

![Figure 2.6](image-url)
Divide HC into equal parts at G. Draw perpendicular GT. Add GK, equal to the difference between CG. GT (i.e. GK = GA) making TK = KM. Thus MKTN is a square. Since HG is half of ten, then MKTN is square of five, or twenty-five, From KM, take a line KL equal to the rectangle TBAG.

Since the rectangle HABN represents 21 numbers, so do the rectangles HGTN, MLRH. Since MKTN is twenty-five, the square KGRL is four and thus RG is equal to two, as GK and consequently GA. As GC is half of ten, or five, and GA is two, then AC must be three, which is the side of the original square.

Most of his problems result in quadratic equations.

Example 2.11

One problem in [36], explained that Al-Khwārizmi had divided ten into two parts, and was having multiplied each part by itself, he had put them together, and have added to them the difference of the two parts previously to their multiplication, and amount of all this as fifty-four.

In modern terms this may be shown as \((10-x)^2 + x^2 + (10-x)-x = 54\) and may be reduced to \(x^2 + 28 = 11x\). By using his method for type 5, \(x = 4\). But he ignores the second root \(x = 7\), because it violates the condition of the problem.

He promised in the preface of the book that he would write about what is useful. But very few of his problems leading to quadratic equations deal with any practical ideas. Many of them are similar to the example and begin with the one in which he had divided ten into two parts. There are few problems concerned with dividing money among certain number of men. But even in such cases practical sense is lacking.
As in [59], Al-Khwārizmi undertakes a brief study of some properties of the application of elementary laws of arithmetic to the simplest algebraic expressions. In this way he studies the products of the type $(a \pm bx)(c \pm dx)$ where $a$, $b$, $c$, $d$ are positive rational numbers.

We might ask from where the inspiration for Arabic algebra came? We can’t give any categorical answer to this question. But the arbitrariness of the rules and the strictly numerical form of the first six chapters remind us the ancient Babylonian and medieval Indian mathematics. The exclusion of indeterminate analysis and the avoidance of any syncopation, such as is found in Brahmagupta might suggest Mesopotamia as more likely a source than India. As we read beyond the sixth chapter, we see the influence of Greek rather than Babylonian or Indian mathematics. Therefore, there are three main schools of thought on the origin of Arabic algebra. Hindu influences, Mesopotamian or Syriac-Persian tradition and the Greek inspiration [10].

In the time of Al-Khwārizmi and immediately afterwards, there was an expansion of research already begun by him. It was on the theory of quadratic equations, algebraic calculation, indeterminate analysis and the application of algebra to the problems of inheritance, partition etc. The first one was already opened up by Al-Khwārizmi himself.

**Ibn Turk**

The section ‘Logical Necessities in Mixed Equations’ from a longer work *Kitāb al-jabr wa-l-muqābala* by Abd al-Hamid ibn Wāsī ibn Turk al-Jili is extant. He is a contemporary of Al-Khwārizmi about whom very little is known. The sources even differ as to whether he was from Iran, Afghanistan or Syria.

As per [36], the extant chapter deals with quadratic equations of Al-Khwārizmi’s types i, iv, v and vi. This contains a more detailed
geometric description of the method of solution than is found in Al-Khwārizmi’s work. In the case of type v, \( ax^2 + c = bx \), Ibn Turk gave geometric versions for all possible cases. His first example \( x^2 + 21 = 10x \) is the same as that of Al-Khwārizmi. Ibn Turk’s geometric justification for one case of \( x^2 + c = bx \) is given in figure 2.7(a). But he began the geometrical demonstration by taking \( G \); the mid point of \( CH \) as shown in figure 2.7(b) may be either on the line segment \( AH \), as in Al-Khwārizmi’s diagram or on the line segment \( CA \) of Ibn Turk’s figure. Complete the squares and rectangles similar in the figure of Al-Khwārizmi. But in this case the solution \( x = AC \) is now given as \( CG + GA \).

As in [36], Al-Khwārizmi’s geometric justification for the solution of \( x^2 + c = bx \) is given in figure 2.7(b). Also he discussed what he
called the ‘intermediate case’ where the root of the square is exactly equal to half the number of roots. He gave the example; \( x^2 + 25 = 10x \). In this case the geometric diagram consists of a rectangle divided into two equal squares.

Ibn Turk noted that “There is a logical necessity of impossibility in this type of equation when the numerical quantity is greater than [the square of] half the number of roots” [36].

Example 2.12 [36]

\[ x^2 + 30 = 10x \]

He turned to a geometric argument as in figure 2.8. Assuming that G is located on the segment AH. We know that the rectangle KMNT is greater than the rectangle HABN. The conditions of the problem show that the rectangle HABN equals 30 and the rectangle KMNT only equals 25. When G is on CA similar argument exists.


Abū Kāmil

Abū Kāmil ibn Aslam (850-930) was an Egyptian Mathematician. *Kitāb fi al-jabr wa-l-muqābala* is his own algebra text. He was sometimes referred to as the “Rockoner from Egypt”. His work is primarily a commentary on Al-Khwārizmi. Many of the problems are the same. Abū Kāmil followed
his discussion of the various forms of quadratic equations by treatment of various algebraic rules and a large selection of problems.

He takes greater case in proving algebraic identities. For example [77], a problem from Al-Khwārizmi is:

"I have divided ten into two parts; and have divided the first by the second, and the second by the first, and the sum of the quotient is two ‘dirhems’ and one-sixth."

Al-Khwārizmi introduced the identity without proof as follows:

“If you multiply each part by itself, and add the products together, then their sum is equal to one of the parts multiplied by the other, and again by the quotient which is two and one-sixth.”

If \( \frac{a}{b} + \frac{b}{a} = c \) then \( a^2 + b^2 = abc \)

First Abū Kāmil proved, if A divided by B is G, then A times A divided by B times A is also G in the following manner [77].

That is \( \frac{A}{B} = \frac{A^2}{AB} \)

Let A times A be H.

Let B times A be D.

“Since A divided by B is G, B is contained in A as many times as there are units in G. Since A multiplied by A is H, and B multiplied by A is D, D is in H as B is in A. And B is in A as many times as there are units in G.”

i.e. If \( \frac{A}{B} = G, A = BG \) Since \( PA = H, BA = D \), then \( \frac{H}{D} = \frac{A}{B} = G \)
Then he proved the result: “Given two numbers A and B so that A divided by B is D and B divided by A is Z, A multiplied by itself is G; B multiplied by itself is H, and A multiplied by B is E. If the sum of G and H is divided by E, it comes to the sum of Z and D.” That is if \( \frac{A}{B} = D \) and \( \frac{B}{A} = Z \), then \( \frac{AA + BB}{AB} = D + Z \) as follows.

A divided by B is D.

This is equal to G divided by E

\[
\frac{B}{A} = Z, \text{ which by the previous proposition is equal to } H \text{ divided by E.}
\]

Therefore, \( \frac{G + H}{E} = D + Z \)

Or when we multiply the sum of D and Z by E, we get the sum of G and H.

\[
\frac{A^2}{B} = \frac{G}{E} = D = \frac{Z}{E}
\]

Similarly, \( \frac{B^2}{A} = \frac{H}{E} = Z = \frac{E}{E} \)

Thus \( D + Z = \frac{G}{E} + \frac{H}{E} = \frac{G + H}{E} \)

Or \( E(D + Z) = G + H \)

Abū Kāmil dealt with irrationals. He used them in his problems.

For example,
"If one says that 10 is divided into two parts, and one part is multiplied by itself and the other by the root of 8, and subtract the quantity of the product of one part times the root of 8 from the product of the other part multiplied by itself, it gives 40" [36].

In this case the equation is \((10 - x)(10 - x) - x\sqrt{8} = 40\)

This is same as \(x^2 + 60 = 20x + \sqrt{8}x^2\)

i.e. \(x^2 + 60 = (20 + \sqrt{8})x\)

As in [36], he used the algorithm for squares and numbers equal to roots and got that

\[x = 10 + \sqrt{2} - \sqrt{42 + \sqrt{800}}\] and that

\[10 - x = \sqrt{42 + \sqrt{800}} - \sqrt{2},\] the other part.

Abū Kāmil used substitutions to simplify problems. He dealt with equations of degree higher than 2 as long as they were quadratic in form. For example in [36],

"One says that ten is divided into two parts, each of which is divided by the other, and when each of the quotients is multiplied by itself and the smaller is subtracted from the larger, then there remains 2."

The above equation is \(\left(\frac{x}{10-x}\right)^2 - \left(\frac{10-x}{x}\right)^2 = 2\)

Abū Kāmil made a new thing \(y\) equal to \(\frac{10-x}{x}\)

Then he got a new equation, \(\frac{1}{y^2} = y^2 + 2\)
Multiply by $y^2$ on both sides, we get a quadratic equation in $y^2$.

$$(y^2)^2 + 2y^2 = 1$$

Its solution is $y^2 = \sqrt{2} - 1$

Hence $y = \sqrt{\sqrt{2} - 1}$

Squaring on both sides and simplifying, we get

$$x = 10 + \sqrt{50} - \sqrt{50 + \sqrt{20000} - \sqrt{5000}}$$

As in [19] we give Abū Kāmil’s geometrical interpretation of the solution of $x^2 + 21 = 10x$ as in figure 2.9.

Figure 2.9

Take the number, which is together with the square, 21. It is more than the square. Construct the square ABGD. Add 21 to it; then we get ABHL. This is larger than the ABGD by construction. Line BL is greater than BD. Then HLGD is 10 roots of the square ABGD. Then line LD is 10 and ABHL is 21. It is equal to the product of LB by BD for BD is equal to BA. Let C be the mid point of LD.
LD is divided into two unequal parts at B. Thus the product of LB by BD added to the square on CB is equal to the square on CD. But the product of line CD by itself is 25, because its length is 5.

Line LB times BD is 21.

The square on line CB is 4, since its side is 2.

But the line BD is 3

This is the root of the square. The square is 9.

Abū Kāmil contributed to the theory of equations and to the extension of algebraic calculation to the field of rational numbers and to the set of irrational numbers. He gave the geometrical justification of algebraic solutions of quadratics using *Elements* of Euclid.

We find in the works of Abū Kāmil, transformations of expression with irrational numbers as follows:

\[
\frac{x}{10-x} + \frac{10-x}{x} = \sqrt{5}
\]

Divide 10 into two parts \(x\) and \(10-x\) to get \(\frac{x}{10-x} + \frac{10-x}{x} = \sqrt{5}\)

The corresponding quadratic equation is

\[
(2+\sqrt{5})x^2 + 100 = (20+\sqrt{500})x
\]

Multiply by \(\sqrt{5}-2\)

\[
x^2 + \sqrt{50000} - 200 = 10x
\]

Abū Kāmil gives another simpler solution by taking \(y = \frac{10-x}{x}\)

Then he got the equation, \(y^2 + 1 = \sqrt{5}y\)
This has the solution \( y = \sqrt{1 + \frac{1}{4} - \frac{1}{2}} \)

We arrive at the linear equation, \( \frac{10 - x}{x} = \sqrt{1 + \frac{1}{4} - \frac{1}{2}} \)

i.e, \( \frac{10}{x} - 1 = \sqrt{1 + \frac{1}{4} - \frac{1}{2}} \)

From this we can find \( x \). But this gives a result with an irrational denominator.

He suggests, \( 10 - x = \sqrt{1 + \frac{1}{4} - \frac{1}{2}} x \)

That is, \( 10 - \frac{x}{2} = \sqrt{1 + \frac{1}{4} x} \)

Squaring both sides and simplifying, we get \( x^2 + 10x = 100 \).

We get the solution \( x = \sqrt{125} - 5 \)

\( \text{Thabit ibn Qurra} \)

\( \text{Thabit ibn Qurra (830-890) was born in Harran, in Southern Turkey. He was brought to Baghdad, the House of Wisdom in about 870 by Mohammed ben Musa. After the works of Al-Khwārizmi and Ibn Turk, the Islamic mathematicians had decided that the necessary geometric foundations to the algebraic solution of quadratic equations should be based on the work of Euclid rather than on the ancient traditions. He is the first to distinguish clearly between the algebraic and geometric methods. He tried to show that both methods lead to the same result.} \)
Among his many writing on mathematical topics is a short work entitled *Qawl fi tashish masāil al-jabr bi-barāhin al handasiya* ('On the Verification of Problems of Algebra by Geometrical Proofs'). To solve the equation $x^2 + bx = c$, he gave the following procedure as in figure 2.11 [36], [77], [81].

Let $AB = x$ and square ABCD represents $x^2$.

Let $BE = b$.

The Rectangle DAEF = AB x EA = c

Let W be the mid point of BE.

*Elements* II-6 of Euclid implies that $EA \times AB + BW^2 = AW^2$

But $EA \times AB = c$ and $BW^2 = \left(\frac{b}{2}\right)^2$
Then $AW^2$ and $AW$ are known

Then $x = AB = AW - BW$ is determined.

Thabit noted that the geometric procedure of *Elements* II-6 is completely similar to the algorithm stated by Al-Khwārizmi.

This gives the necessary justification.

Figure 2.12: Pythagorean Theorem in Thābit Ibn Qurra’s Translation of *Elements*.

Thābit also showed how to use the same proposition to solve $x^2 = bx + c$ and how to use Proposition 5 of Book II of *Elements* to solve $x^2 + c = bx$ [36], [59], [81].

There are two opposite trends among the mathematicians and astronomers of Baghdad. One was represented by Al-Khwārizmi who used Indian and Persian sources. On the other hand, we have “the Greek School working for the reception of Greek science by the Arabs”.
Al-Karaji

Al-Karaji (953-1029) took an important step in the development of algebra. He wrote a work on algebra *Al-Fakhrī* ('The Marvelous'). He began by making a systematic study of the algebra of exponents. He dealt with the algebra of higher powers of the variables than the cube. Al-Karaji named higher powers as shown in the following table [77]. This was an early step in the development of modern algebraic notation.

<table>
<thead>
<tr>
<th>Al-Karaji</th>
<th>Modern</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cube – cube</td>
<td>$x^6$</td>
</tr>
<tr>
<td>Square-cube</td>
<td>$x^5$</td>
</tr>
<tr>
<td>Square – square</td>
<td>$x^4$</td>
</tr>
<tr>
<td>Cube</td>
<td>$x^3$</td>
</tr>
<tr>
<td>Square</td>
<td>$x^2$</td>
</tr>
<tr>
<td>Thing</td>
<td>$x$</td>
</tr>
<tr>
<td>Unit</td>
<td>1</td>
</tr>
</tbody>
</table>
Al-Karaji continued the process of arithmetization of algebra, begun by Al-Khwārizmi. He treated the higher degree terms in a polynomial exactly like the digits of a multidigit number. To add two polynomials such as “three cube-cubes plus four squares plus three units” and “two square-cubes plus three square-squares plus two-squares plus five things plus two units”, he lined up the terms of the same power and then added the coefficients.

Al-Karaji failed to provide a complete system of arithmetic because there was no generalized concept of a negative number. In particular, the procedure of subtracting a negative from a negative had no analog among arithmetical algorithms [77].

Once the powers were understood, Al-Karaji could establish general procedures for adding, subtracting and multiplying monomials and polynomials. However, he could only deal with division of monomials by monomials and of polynomials by monomials. This is because of his inability
to incorporate rules for negative numbers into his theory and the lack of symbolism.

Omar Khayyam

Omar Khayyam (1048-1131) is very well known through the translation of his poem the *Rubáiyát*. He was a poet, philosopher, astronomer and mathematician. He lived in the 11th century. In algebra, his work was devoted to solving quadratic and cubic equations. This work was through geometrical constructions. The solutions are obtained as the co-ordinates of the points where certain curves meet, such as a circle and a parabola, or a circle and hyperbola or a parabola and a hyperbola.

![Figure 2.13: Omar Khayyam](image)

The algebra of Omar Khayyam is mainly geometric. He first solves the linear and quadratic equations by the geometrical methods contained in Euclid’s *Elements*.

Omar began with the problem [77], “A square and ten roots equal the number thirty-nine”. However, his solution depended on Euclid, Book II of *Elements*. His demonstration is given in figure 2.14 as the same as that given by Ibn Qurra.

Let the square ABCD be the unknown. Let ten times its root represent the rectangle CDEF. So DE is ten.
Divide DE into equal parts at Z and extend it by AD.

Since the line DE has been divided into equal parts and a straight line DA added to it in a straight line, then the rectangle on EA, AD is equal to the rectangle ABFE plus the square on DZ, equals the square on ZA. The square of DZ or half the number of roots is known.

The area of the rectangle ABFE, is equal to the given number, is known.

Hence the square of ZA is known.

Thus the line ZA is known.

ZA – ZD gives the line AD.

To solve the quadratic \( x^2 + px = q \), Omar Khayyam’s rule [69] is as follows:

“Multiply half the root by itself; add the product to the number; and from the square root of this sum subtract half the root. The remainder is the root of the squares.”

This is the same as \( x = \sqrt{\frac{1}{4} p^2 + q - \frac{1}{2} p} \)
By “half the root” is meant $\frac{1}{2}p$.

By the “number” is meant $q$.

He used the example, $x^2 + 10x = 39$.

Omar obviously followed the solution on the basis of the model set by Al-Khwârizmi.

Omar gave the rule for the type, $x^2 + q = px$, being based upon the identity

$$x - \left(\frac{1}{2}p\right)^2 = \left(\frac{1}{2}p\right)^2$$

He also gave the rule for the type, $px + q = x^2$ based upon the identity

$$x(x - p) + \left(\frac{1}{2}p\right)^2 = \left(x - \frac{1}{2}p\right)^2$$

The first important attempt at a systematic classification is found in the algebra of Omar Khayyam, but the classification is not like our present one. He considers equations of the first three degree as either simple or compound. The single equations are of the type,

$$r = x, r = x^2, r = x^3, ax = x^2, ax = x^3, ax^2 = x^3$$

Compound equations are first classified as trinomials [69]. They include the following twelve forms.

(i) $x^2 + bx = c, x^2 + c = bx, bx + c = x^2$

(ii) $x^3 + bx^2 = cx, x^3 + cx = bx^2, cx + bx^2 = x^3$

$x^3 + cx = d, x^3 + d = cx, cx + d = x^3$

$x^3 + bx^2 = d, x^3 + d = bx^2, bx^2 + d = x^3$
They are then classified as quadrinomials as follows.

(iii) \( x^3 + bx^2 + cx = d, \) \( x^3 + bx^2 + d = cx \)

(iv) \( x^3 + bx^2 = cx + d, \) \( x^3 + cx = bx^2 + d \)

\[ x^3 + d = bx^2 + cx \]

Omar also discussed equations in which the unknown was raised to a negative power. His terms of powers were almost identical to Al-Karaji. He stepped with the “cube-cube” and the corresponding “part of a cube-cube”. He understood that the table of Al-Karaji could be extended indefinitely in both directions. Since there was no method for solving equations beyond the cube or part of the cube, there was no need to attempt for them.

Equations involving parts could be dealt with in one of the two ways, in [77].

(i) Terms could be replaced with their similar terms. Part of a square was similar to a square and so on. For example, “A part of square is equal to half the part of a root” could be transformed to the equation, “A square is equal to half its root”.

(ii) Replace terms with proportional succeeding terms. For example, “a root equal to unity and two parts of a root” was changed to “a square equal to a root and two units”. That is, replacing each term with the next higher term, so the ‘parts’ were eliminated. These transformations are equivalent to replacing \( \frac{1}{x} \) with \( v \) and multiplying through by \( x^n \) respectively.

Thus Omar Khayyam noted that any equation that ‘spanned’ four terms in proportion could be solved. If the equation spanned more than four terms, a solution was impossible.
Al-Samaw'ål

Al-Samaw'ål (1125-1180) was born in Baghdad to a well-educated Jewish Parents. He wrote his major mathematical work *Al-Bāhir fil-hisāb* (‘The Shining Book of Calculation’) when he was only 19.

Al-Samaw’ål introduced negative coefficients in polynomials. He gave his rules in [36] for dealing with these coefficients in his work *Al-Bāhir*.

If we subtract an additive number from any empty power $0x^n - ax^n)$, the same subtractive number remains; if we subtract the subtractive numbers from any power $(ox^n - (ax^n))$, the same additive number remains. If we subtract an additive number from a subtractive number, the remainder is their subtractive sum; if we subtract a subtractive number from a greater subtractive number, the result is their subtractive differences; if the number from which one subtracts is smaller than the number subtracted, the result is their additive difference.

Using these rules, Al-Samaw’ål could easily add and subtract polynomials by combining like terms.

He also gave a clear formulation of the Law of Exponents. Al-Karaji, Abū Kāmil and others used this law in essence. However, since the product of a square and a cube were expressed in words as a square-cube, the property of adding exponents could not be seen. Al-Samaw’ål decided that this law could be expressed by using a table consisting of columns, each column representing a different power of either a number or an unknown. He dealt with powers of $\frac{1}{x}$ as easily with powers of $x$. In his work, the columns are headed by the Arabic letters standing for the numerals, reading both ways from the central column labeled zero. Each column then has the name of the particular power or reciprocal power. For example, the column headed by 2 on the left is named ‘square’. The column headed by 5 on the left is named
For simplification we will use powers of \( x \). He put a particular number under the 1 on the left, such as 2, and then various powers of 2 in the corresponding columns [36]:

\[
\begin{array}{cccccccccccc}
7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
X^7 & x^6 & x^5 & x^4 & x^3 & x^2 & x & 1 & x^{-1} & x^{-2} & x^{-3} & x^{-4} & x^{-5} & x^{-6} & x^{-7} \\
128 & 64 & 32 & 16 & 8 & 4 & 2 & 1 & 1/2 & 1/4 & 1/8 & 1/16 & 1/32 & 1/64 & 1/128
\end{array}
\]

Then he used the chart to explain the Law of Exponents,

\[ x^m \cdot x^n = x^{m+n} \]

The distance of the order of the product of the two factors from the order of one of the two factors is equal to the distance of the order of the other factor from the unit. If the factors are in different directions then we count (the distance) from the order of the first factor towards the unit; but, if they are in the same direction, we count away from the unit.

For example, multiply \( x^3 \) by \( x^4 \)

Count four orders to the left of column 3, we get \( x^7 \)

To multiply \( x^3 \) by \( x^{-2} \), count two orders to the right from column 3, we get \( x^1 \). By these rules, Al-Samaw’al easily multiplied polynomials in \( x \) and \( \frac{1}{x} \). Also he divided polynomials by monomials.

Al-Samaw’al was also able to divide polynomials by polynomials using a similar chart. In this new chart each column stands for a given power of \( x \) or \( \frac{1}{x} \). Here the numbers in each column represent the coefficients of the various polynomials involved in the division.
Example 2.13 [36]

Divide $20x^2 + 30x$ by $6x^2 + 12$

Firstly, he arranged 20 and 30 in the columns headed by $x^2$ and $x$ respectively. Then 6 and 12 are placed below these in the columns headed by $x^3$ and 1. Since, there is an 'empty order' for the divisor in the $x$ column; he places a zero (0) there. Next, he divides $20x^2$ by $6x^2$. He got $3\frac{1}{3}$, which goes in the units column on the answer line. The product of $3\frac{1}{3}$ by $6x^2 + 12$ is $20x^2 + 40$. Next he subtracts. The remainder in the $x^2$ column is zero and that in the $x$ column is 30 and that in the unit’s column is -40. Now he writes a new chart in which 6, 0, 12 are shifted one place to the right. The directions are given to divide that into $30x - 40$. When he divides $30x$ by $6x^2$, the initial quotient is $5\frac{1}{x}$. So 5 is placed in the answer line in the column headed by $\frac{1}{x}$. Continuing the process, the following is the display of his first two charts for this division problem.

<table>
<thead>
<tr>
<th>$x^2$</th>
<th>$x$</th>
<th>1</th>
<th>$\frac{1}{x}$</th>
<th>$\frac{1}{x^2}$</th>
<th>$\frac{1}{x^3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>6</td>
<td>0</td>
<td>12</td>
<td>30</td>
</tr>
</tbody>
</table>

In terms of the quotient ending with 5.419, 
$\frac{1}{x}$ determines the first entry.

He also thought of partial results as significant. He did not so recognize explicitly that one could approximate the polynomial in $\frac{1}{x}$. He also thought of partial results as significant. He thought of the best way to recognize explicitly that one could approximate the polynomial in $\frac{1}{x}$.
In this example, the division is not exact.

He continues the process through eight steps to get

\[
\begin{array}{cccccc}
x^2 & x & 1 & \frac{1}{x} & \frac{1}{x^2} & \frac{1}{x^3} \\
3 \frac{1}{3} & 5 \\
30 & -40 \\
6 & 0 & 12 \\
\end{array}
\]

He checks the answer by multiplying it by the divisor. The product differs from the dividend by terms only in \( \frac{1}{x^6} \) and \( \frac{1}{x^7} \). He calls the result given, “the answer approximately”. That is, if \( a_n \) represents the coefficients of \( \frac{1}{x^n} \), the pattern is given by \( a_{n+2} = -2a_n \). Then he writes the next 21 terms of the quotient ending with \( 54,613 \frac{1}{3}(x^{18}) \). He gave the first example of polynomial division.

Al-Samaw’al thought of extending division of polynomials into polynomials in \( \frac{1}{x} \). He also thought of partial results as approximations. He was the first to recognize explicitly that one could approximate fractions more and more closely by calculating more and more decimal places [36].
India-Mahāvīra

Mahāvīra flourished in the present Karnataka state. In [70] it is given that Mahāvīra was a Jain by religion and *Ganita-Sāra-Saṅgraha* (गणितसारसंग्रह) was written in 850AD. Also Mahāvīra is the world’s first mathematician who gives the general formula.

\[
\binom{n}{r} = \frac{n(n-1)(n-2) \ldots (n-r+1)}{1.2.3 \ldots r}
\]

According to [6], an interesting example from *Ganita-Sara-Saṅgraha* is,

One night, in the month of the spring season a certain charming young lady was lovingly happy along with her husband on the floor of big mansion which was shining with whiteness of the moonlight and situated in a pleasure garden with trees bent down with the load of the luxuriant bunches of flowers and fruits and resonant with the sweet sounds of parrots, cuckoos and bees which were intoxicated with the honey obtained from the flowers in that garden. Then, in the love-quarrel arising between the loving youthful couple, the lady’s necklace made up of pearls became sundered and the pearls fell helter-skelter. A third of the pearls in the necklace reached the maid servant there; a sixth fell on the (soft) bed; then a half of the last fraction, again a half of the latter and so on, counting that way six times in all, of the pearls fell on the floor scattered (in groups); and there were found to remain 1161 pearls (still on the necklace). If you know how to work miscellaneous problems in fractions, give out the total number of pearls (adoring the neck of the youthful lady).

Let \( x \) be the total number of pearls in the necklace originally.

Then the above problem results in the equation,

\[
\frac{x}{3} + \frac{x}{6} + \frac{x}{12} + \frac{x}{24} + \frac{x}{48} + \frac{x}{96} + \frac{x}{192} + \frac{x}{384} + 1161 = x
\]

Solving this, we get \( x = 3456 \)
Therefore, the total number of pearls in the necklace of the lady = 3456.

As per [14], [17], [69] regarding the solution of linear equations with several unknowns, Mahávira gave the following examples, which lead to simultaneous linear equations. He gave the rules for solving each.

**Example 2.14** [14], [17], [69]

The price of 9 citrons and 7 fragrant wood-apples taken together is 107; again the price of 7 citrons and 9 fragrant wood-apples taken together is 101. O mathematician, tell me quickly the price of a citron and of a fragrant wood-apple quite separately.

The solution is,

From the larger amount of price multiplied by the (corresponding) bigger number of things subtract the smaller amount of price multiplied by the corresponding smaller number of things. (The remainder) divided by the difference of the squares of the numbers of things will be the price of each of the bigger numbers of things. The price of the other will be obtained by reversing the multipliers.

In modern terms, let $x$ be the price of a citron and $y$ be the price of a fragrant wood apple.

Then we have $9x + 7y = 107$

$7x + 9y = 101$

Generally, $ax + by = m$

$bx + ay = n$
Then, \[ x = \frac{am - bn}{a^2 - b^2}, \quad y = \frac{an - bm}{a^2 - b^2} \]

Example 2.15 [6], [14]

A wizard having powers of mystic incantations and magical medicines seeing a cock-fight gain on, spoke privately to both the owners of the cocks. To one he said, 'If your bird wins, then you give me your stake -money, but if you do not win, I shall give you two-thirds of that.' Going to the other, he promised in the same way to give three-fourths. From both of them his gain would be only 12 gold prizes. Tell me, ornament of the first-rate mathematicians, and the stake money of each of the cock-owners.

In modern symbols \[ x - \frac{3}{4}y = 12 \]

\[ y - \frac{2}{3}x = 12 \]

Generally, \( x - \frac{c}{d}y = p, \quad y - \frac{a}{b}x = p \)

The solution given in modern terms is

\[ x = \frac{b(c+d)}{(c+d)b-(a+b)c} p, \quad y = \frac{d(a+b)}{(a+b)d-(c+d)a} p \]

Then we get \( x = 42 \) and \( y = 40 \)

In connection with the solution of system of linear equations (*) Mahâvîra states:

"The stated amounts of the commodities added together should be divided by the number of men less one. The quotient will be the total value (of all the commodities). Each of the stated amounts being subtracted from that; (the value) in the hands (of each will be found)" [14].

\[ 182 \]
In [14], it is given while Mahâvîra formed this rule; the following example was in his mind.

Example 2.16 [14]

Four merchants were each asked separately by the customs officer about the total value of their commodities. The first merchant, leaving out his own investment, stated it to be 22; the second stated it to be 23, the third 24 and the fourth 27; each of them deduced his own amount in the investment. O friend, tell me separately the value of (the share of) the commodities owned by each.

\[
\begin{align*}
22 + 23 + 24 + 27 &= 324 - 1 \\
\text{Therefore, } x_1 &= 10, x_2 = 9, x_3 = 8, x_4 = 5
\end{align*}
\]

Mahâvîra gives an example which is a particular case of system of equations (**).

Example 2.17 [14]

Three merchants begged money mutually from one another. The first on begging 4 from the second and 5 from the third became twice as rich as the others. The second on having 4 from the first and 6 from the third became thrice as rich. The third man on begging 5 from the first and 6 from the second became five times as rich as the others. O mathematician, if you know the citra-kuttaka-miśra, tell me quickly what was the amount in the hand of each.

\textit{citra-kuttaka-miśra} is the name given by Mahâvîra to problems involving equations of the above type.

Mahâvîra gives the solution as follows:
The sum of the amounts begged by each person is multiplied by the multiple numbers relating to him as increased by unity. With these (products) the amounts at hand are determined according to the rule Īṣṭāgūṇa-ghna etc. They are reduced to a common denominator, and then divided by the sum diminished by unity of the multiple numbers divided by them as increased by unity. (The quotients) should be known to be the amounts in the hands of the persons.

In modern symbols, the equations are

\[ x + 4 + 5 = 2(y + z - 4 - 5) \]
\[ y + 4 + 6 = 3(z + x - 4 - 6) \]

i.e. \( z + 5 + 6 = 5(x + y - 5 - 6) \)

Or \( 2(x + y + z) - 3x = 27 \)
\[ 3(x + y + z) - 4y = 40 \]
\[ 5(x + y + z) - 6z = 66 \]

Combining the above equations with (***) we get,
\[ x = 7, y = 8, z = 9 \]

To Datta and Singh [14], Mahāvīra treated some interest problems which lead to simple simultaneous equations involving several unknowns. In these problems certain capital amounts \( (c_1, c_2, c_3, \ldots) \) are stated to have been lent out at the same rate of interest \( (r) \) for different periods of time \( (t_1, t_2, t_3, \ldots) \). If \( (i_1, i_2, \ldots) \) be the interests accumulated on several capitals,

\[ i_1 = \frac{r t_1 c_1}{100}, \quad i_2 = \frac{r t_2 c_2}{100}, \quad i_3 = \frac{r t_3 c_3}{100} \]

There are three cases:
Case (i)  \[ i_1 + i_2 + i_3 + \ldots = I, \quad c_r \text{ and } t_r \text{ be known for } r = 1,2, \ldots, \] we have

\[ i_r = \frac{I c_r t_1}{c_1 t_1 + c_2 t_2 + c_3 t_3 + \ldots} \]

with similar values for \( i_2, t_2 \ldots \)

Case (ii)  \[ c_1 + c_2 + c_3 + \ldots = C, \quad i_r \text{ and } t_r \text{ be known for } r = 1,2,3, \ldots, \] we have

\[ c_1 = \frac{C i_t / t_1}{i_1 / t_2 + i_2 / t_2 + \ldots} \]

and so on.

Case (iii)  \[ t_1 + t_2 + t_3 + \ldots = T, \quad c_r \text{ and } t_r \text{ be known then, } t_1 = \frac{T i_t / c_1}{i_1 / c_1 + i_2 / c_2 + \ldots} \]

Similarly we can find \( t_2, t_3, \ldots \)

Example 2.18 [70]

Four different sums of money equal to 40, 30, 20, and 50 respectively are lent out at the same rate of interest for 5, 4, 3, 6 units of time respectively. The total interest is 34. What is the interest that each amount fetches?

By the above case (i) we have  \[ i_r = \frac{I c_r t_1}{c_1 t_1 + c_2 t_2 + c_3 t_3 + c_4 t_4} \]

\[ = \frac{34 \times 40 \times 5}{40 \times 5 + 30 \times 4 + 20 \times 3 + 50 \times 6} = 10 \]

In a similar manner we can find the interest of other amounts.

The only available work of Mahâvîra is the Ganita-Sâra-Sarîgraha. It is mainly devoted to arithmetic. But in it, there are several problems whose solutions need knowledge of the roots of the quadratic.
Example 2.19 [14],[17]

“One-fourth of a herd of camels was seen in the forest; twice the square root of the herd had gone to the mountain-slopes; three times five camels were on the bank of the river. What was the number of those camels?”

In modern terms, if $x$ be the number of camels in the herd, then

$$\frac{1}{4}x + 2\sqrt{x} + 15 = x$$

Generally, the equation to be solved is $\frac{a}{x} + c\sqrt{x} + d = x$

That is $\left(1-\frac{a}{b}\right)x - c\sqrt{x} = d$

Mahâvîra gives [14] the following rule for the solution of this equation.

Half the coefficient of the root (of the unknown) and the absolute term should be divided by unity minus the fraction (coefficient of the unknown). The square-root of the sum of the square of the coefficient of the root (of the unknown) and the absolute term (treated as before) is added to the coefficient of the root (of the unknown treated as before). The sum squared is the (unknown) quantity in this múla type of problems.

In modern symbols,

$$x = \frac{c/2}{1-a/b} + \sqrt{\left(\frac{c/2}{1-a/b}\right)^2 + \frac{d}{1-a/b}}$$

This shows that Mahâvîra, used the modern rule for finding the root of a quadratic. He solved the interest problem of Brahmagupta. He got the same solution as that of the Brahmagupta.
Mahāvīra, gave no rule for the quadratic. But he proposed a problem involving the equation $c + \frac{c}{3} + \frac{3c^2}{3^6} = m$, adding the following statement.

"In relation to the combined sum [of the three quantities] as multiplied by 12, the quantity known as to be added 64. Of this [second] sum the square root diminished by the square root of the quantity thrown in gives rise to the measure"[69].

In modern terms, $c = \sqrt{12m + 64} - \sqrt{64}

This shows that Mahāvīra has substantially the modern rule for finding the root of a quadratic.

As per [14] Mahāvīra knows that a quadratic equation has two roots. We shall substantiate it by the following rules and illustrations from his work.

“One-Sixteenth of a collection of Peacocks multiplied by itself, was on the mango tree; $\frac{1}{9}$ of the remainder multiplied by itself together with 14 were on the tamāla tree. How many were they? ”

Let $x$ be the number of peacocks in the collection.

Then we have, $\frac{x}{16} \times \frac{x}{16} + \frac{15x}{16 \times 9} \times \frac{15x}{16 \times 9} + 14 = x$

Generally, $\frac{a}{b}x^2 - x + c = 0$

The following rule has been given for its illustration.

The quotient of its denominator divided by its numerator, less four times the remainder, is multiplied by that denominator (as divided by the numerator). The square- root of this should be
'added to' and 'subtracted from' that denominator (as divided by the numerator'); half that is the total quantity [14].

Thus \[ x = \frac{b/a \pm \sqrt{(b/a - 4c) b/a}}{2} \]

In [34], it is given that Mahâvîra was certainly aware of both signs of the radical. He gave the solution in general terms of \( ax^2 + bx = c \) as

\[ x = \frac{-b/a \pm \sqrt{(b/a - 4c) b/a}}{2} \]

There are some other problems that lead to the equations of the form \( \left( \frac{a}{b} x \pm d \right)^2 + c = x \)

The solution is given as follows in [14],

Half the denominator divided by its numerator is increased or diminished by the quantity to be subtracted or added. The square of this is diminished by the square of the quantity to be subtracted or added and the remainder. The square root of the result ‘added to’ or ‘subtracted from’ the square root (of the square obtained before) and divided by the fractional part, will be the value (of the unknown).

In modern symbols, \( x = \left\{ \left( \frac{b}{2a} \pm d \right) \pm \sqrt{\left( \frac{b}{2a} \pm d \right)^2 - d^2 - c} \right\} \div \frac{a}{b} \)

These illustrations prove that Mahâvîra recognized both roots of the quadratic equation.

However, there are a few problems in which Mahâvîra considered only one root of the equation.
For example, consider the equation in Example 2.19

$$\frac{1}{4}x + 2\sqrt{x} + 15 = x$$

Or $$\frac{3}{4}x - 2\sqrt{x} = 15$$

Therefore, \( \sqrt{x} = \left\{ \frac{4}{3} \pm \sqrt{\left(\frac{4}{3}\right)^2 + 4.15} \right\} \)

i.e., \( \sqrt{x} = \left( \frac{4}{3} \pm \frac{14}{3} \right) = 6 \) or \( \frac{-10}{3} \)

Therefore, \( x = 36 \)

The negative value of the radical is neglected in the rule.

According to [70], another example of Mahâvîra is:

“Out of a certain numbers of Sârâsa birds, one-fourth the number are moving about in lotus plants; \( \frac{1}{9} \) th coupled with \( \frac{1}{4} \) th as well as 7 times the square root of the number move on a hill; 56 birds remain in Vâkula trees. What is the total number of birds?”

Let \( x \) be the total number of birds. Then we have the equation

\[ x = \frac{x}{4} + \frac{x}{9} + \frac{x}{4} + 7\sqrt{x} \]

Therefore \( x = 576 \)

Another example from \( \text{Ga}ñita-Sâra-Saîngraha \) in [6] is as follows:

“One-third of a herd of elephants and three times the square root of the remaining part (of the herd) were seen on the mountain slope; and in a
Lake was seen a tusker (male elephant) along with three female elephants. How many elephants were there?”

Let \( x \) be the number of elephants in the herd. Then we have

\[
\frac{x}{3} + 3 \left( \frac{2}{3} \right)^{1/2} + 4 = x
\]

i.e. \( x - \frac{x}{3} - 4 = 3 \left( \frac{2}{3} x \right)^{1/2} \)

Squaring both sides and simplifying,

We get \((2x - 3)(x - 24) = 0\)

We get \( x = 24 \), when taking the positive integral solution for \( x \) as valid.

Prthudakasvami

Prthudakasvami (860 AD) represented the following equation \( 10x - 8 = x^2 + 1 \) as follows.

\[
\begin{array}{ccc}
yāva & 0 & yāva 10 \ yā 0 \\
rū & 8 & rū 1
\end{array}
\]

where \( x \) stands for \( yā \) which is an abbreviation for \( yāvat-tāvat \) (the unknown quantity) \( yāva \) is an abbreviation for \( yāvat avad-varga \) (the square of the unknown quantity) and \( rū \) represents \( rūpa \), the constant term. That is \( 0x^2 + 10x - 8 = x^2 + 0x + 1 \) [14],[34].

If there are several unknowns those of the same kind are written in the same column with zero coefficients, if necessary. Prthudakasvami represents the equation \( 197x - 1644y - z = 6302 \), as follows.

\[
\begin{array}{cccc}
yāva 197 & kā 1644 & nī 1 & rū 0 \\
yāva 0 & kā 0 & nī 0 & rū 6302
\end{array}
\]
Here $x$ for $yā$, $y$ for $kā$ and $z$ for $ni$.

That is $197x - 1644y - z + 0 = 0x + 0y + 0z + 6302$

The use of the old method of writing equations is sometimes met within later works also. For example, Datta and Singh in [14] quoted that the manuscript of Prthūdakasvāmī’s commentary on the *Brāhma-sphuta-siddhānta* contains the following:

“First side $yā vargah 1 yā vakah 200 ru 0$;

Second side $yā vargah 0 yā vakah 0 ru 1500”

This is the same as $x^2 + 200x + 0 = 0x^2 + 0x + 1500$.

Prthūdakasvāmī solved the following problem of Brahmagupta as follows:

“Tell the number of elapsed days for the time when four times the twelfth part of the residual degrees increased by one, plus eight will be equal to the residual degrees plus one.”

Here the residual degrees are (put as) $yāvah – tāvat, yā$; increased by one $yā 1 ru 1$; twelfth part of it $yā 1 ru 1 12$; four times this, $yā 1 ru 1 3$; plus the absolute quantity eight, $yā 1 ru 25 3$

This is equal to the residual degrees plus unity. The statement of both sides tripled is

$yā 1 ru 25$

$yā 3 ru 3$
The difference between the coefficients of the unknown is 2. By this, the difference of the absolute terms, namely 22 being divided, is produced the residual of the degrees of the sun, 11. These residual degrees should be known to be irreducible. The elapsed days can be deduced then, (proceeding) as before.

In modern terms, we have to solve the equation,

$$\frac{4}{12}(x+1)+8=x+1$$

This gives \(x + 25 = 3x + 3\)

i.e., \(2x = 22\) this implies \(x = 11\)

Prthûdakasvâmî has explained Bramhagupta's rule given in page 145 as follows:

In an example in which there are two or more unknown quantities, colours such as yâvat-tâvat, etc., should be assumed for their value. Upon them should be performed all operations conformably to the statement of the example and thus should be carefully framed two or more sides and also equations. Equi-clearance should be made first between two and two of them and so on to the last; from one side one unknown should be cleared, other unknowns reduced to a common denominator and also the absolute numbers should be cleared from the side opposite. The residue of other unknowns being divided by the residual coefficient of the first unknown will give the value of the first unknown. If there be obtained several such values, then with two and two of them, equations should be formed after reduction to common denominators. Proceeding in this way to the end, find out the value of one unknown. If that value be (in terms of) another unknown then the coefficients of those two will be reciprocally the values of the two unknowns. If however, there be present more unknowns in that value, the method of the pulveriser should be employed. Arbitrary values may then be assumed for some of the unknowns [14].
This rule accepts the determinate as well as indeterminate equations. All the illustrations of this rule by Brahmagupta are of indeterminate equations.

Sripati

Sripati lived about 1039 AD. According to [14], regarding the solution of linear equations, Sripati says:

“First remove the unknown from anyone of the sides (of the equation) leaving the known term; the reverse (should be done) on the other side. The difference of the absolute terms taken in the reverse order divided by the difference of the coefficients of the unknown will be the value of the unknown.”

He gives two methods of solving the quadratic equation. But it can be identified to be the same as that of Srîdhara. According to Datta and Singh, they are:

“Multiply by four times the coefficient of the square of the unknown and add the square of the coefficient of the unknown; (then extract) the square root divided by twice the coefficient of the square of the unknown, is said to be (the value of) the unknown.” Or

Multiplying by the coefficient of the square of the unknown and adding the square of half the coefficient of the unknown, (extract) the square root. Then proceeding as before, it is diminished by half the coefficient of the unknown and divided by the coefficient of the square of the unknown. This (quotient) is said to be (the value of) the unknown.

In symbols, \( ax^2 + bx = c \)

\[ a^2x^2 + abx + (b/2)^2 = ac + (b/2)^2 \]
Therefore \((ax + b/2)^2 = ac + (b/2)^2\)

\[
ax + b/2 = \sqrt{ac + (b/2)^2}
\]

Therefore \(x = \frac{\sqrt{ac + (b/2)^2} - b}{2a}\)

Bhāskara- II

Bhāskarachārya is sometimes referred to as Bhāskara-II, because there was an earlier mathematician of the same name. He was born in Bijjada Bida of present Karnataka state. His period is from 1114 to 1185. He served much of his adult life as the head of the astronomical observatory at Ujjain. His major work, the *Siddhāntasiromāṇi* (सिद्धांत शिरोमणि) was a treatise on astronomy. Mathematics concepts are found in only two chapters, one as *Lilāvati* (लीलावती), named after his daughter, on arithmetic and the other as *Bijaṅgaṇita* on algebra.

Bhāskara-II’s work is essentially a textbook. He himself acknowledges that he has collected and condensed the material available in the algebraical works of Brahmagupta, Sridhara and Padmanābha. The algebraical works of Sridhara and Padmanābha are not available now. He does not make specific mention of Mahāvīra. But there are plenty of instances in his work, which Bhāskara-II has received from Mahāvīra. We can infer that Bhāskara-II was quite aware of the works of almost all his predecessors. A rough conception of infinity, that any other number when divided by zero, gives infinity, first occurs in his work. He treats zero and infinity in the first chapter of *Bijaṅgaṇita*.

The following are two examples from the *Bijaṅgaṇita* of Bhāskara-II as per [14] regarding the representation of equations.
Taking \( x \) for \( y\), \( y \) for \( k\) and \( z \) for \( n\), we get,

\[
5x + 8y + 7z + 90 = 7x + 9y + 6z + 62
\]

\[
\begin{align*}
\text{yâ gha 8} & \quad \text{yâ va 4} & \quad \text{kâ va yâ. bhâ 10} \\
\text{yâ gha 4} & \quad \text{yâ va 0} & \quad \text{kâ va yâ. bhâ 12}
\end{align*}
\]

This is the same as \( 8x^3 + 4x^2 + 10y^2x = 4x^3 + 0x^2 + 12y^2x \)

i.e. \( 8x^3 + 4x^2 + 10y^2x = 4x^3 + 12y^2x \)

Here the terms are ordered according to the descending powers of the unknowns. Numerical coefficients are placed after the unknowns. If the coefficient is unity, it is noted particularly. If the coefficient is a fraction, it is kept distinct from the unknown. The denominator is not to come under the unknown. The minus sign is put over the numerical coefficient. The absolute term is put last on either side.

Figure 2.15: Palm-leaf manuscript of the Lilâvati

In connection with the solution of linear equations Bhâskara-II, says:
"Subtract the unknown on one side from that on the other and the absolute term on the second from that on the first side. The residual absolute number should be divided by the residual coefficient of the unknown; thus the value of the unknown becomes known."

Bhāskara-II has studied examples which are recreative problems given by his predecessors Śridhara, Śripati, Mahāvīra, Brahmagupta and Prthūdakasvāmī. He simplified, improved and given these in his work.

Figure 2.16 : From Bhāskara II’s Līlavati from a manuscript of 1600 AD

Example 2.20 [6]

A beautiful pearl necklace of a lady was torn in a love quarrel (Mithuna Kalāhe) and the pearls were all scattered on the floor. One-third of the number of pearls was on the bed. One-sixth was found by the pretty lady, one-tenth was collected by the love and six were seen hanging in the thread. Tell me the total number of pearls in the necklace.

Let \( x \) be the number of pearls in the necklace initially.

From the problem, we have the equation,

\[
\frac{x}{3} + \frac{x}{5} + \frac{x}{6} + \frac{x}{10} + 6 = x
\]

We get the solution \( x = 30 \)
Example 2.21 [14]

"One person has three hundred coins and six horses. Another has ten horses (each) of similar value and he has further a debt of hundred coins. But they are of equal worth. What is the price of a horse?"

Here the statement is equivalent to \[ 6x + 300 = 10x - 100 \]

Now, by the rule,

Subtract the unknown on one side from that on the other etc', unknown on the first side being subtracted from the unknown on the other side, the remainder is \(4x\). The absolute term on the second side being subtracted from the absolute term on the first side, the remainder is 400. The residual known number 400 being divided by the coefficient of the residual unknown \(4x\), the quotient is recognized to be the value of \(x\), (namely) 100.

Bhāskara-II, gives a method called Īśa-karma or 'operation with an optional number'. This is illustrated as follows:

"What is that number, which multiplied by five, diminished by its third part and (then) divided by ten, will become together, with its one-third, half and one-fourth parts, equal to seventy minus two?" [14]

In modern terms, \[ \frac{5x - 5x/3}{10} + \frac{x}{3} + \frac{x}{2} + \frac{x}{4} = 70 - 2 \]

He assumes \(x = 3\) and then calculates,

\[ \frac{5 \times 3 - 5 \times 3/3}{10} + \frac{3}{3} + \frac{3}{2} + \frac{3}{4} = \frac{17}{4} \]

Then, \(x = 68 \times 3 + \frac{17}{4} = 48\)

He observes:
In every example, by whatever the (required) number is multiplied or divided, by whatever fraction of the number it is found to have been increased or diminished, assuming an optional number, on it perform the same operations in accordance with the statement of the problem; by that which results, divide the known number multiplied by the assumed number; the quotient will be the (required) number [14].

The following example and a solution [14] are given by Bhāskara-II in his *Bījagāṇita* in connection with the solution of linear equation with several unknowns.

**Example 2.22**

“One says, ‘Give me a hundred, friend, I shall then becomes twice as rich as you’ The other replies , ‘If you give me ten, I shall be six times as rich as you’. Tell me what the amount of their (respective) capitals is?”

In modern terms, the equation is

\[ x + 100 = 2(y - 100) \]  \hspace{1cm} (1)

\[ y + 10 = 6(x - 10) \]  \hspace{1cm} (2)

Bhāskara-II, gives two methods of solving these equations. They are as follows.

**First Method:**

Assume \( x = 2z - 100, \ y = z + 100 \)

So that (1) is identically satisfied.

Put these values in (2) we get

\[ z + 110 = 12z - 660 \]

Then \( z = 70 \)
Therefore \( x = 40, y = 170 \)

Second Method:

From (1) we get \( x = 2y - 300 \)

From (2) we get \( x = \frac{1}{6}(y + 70) \)

Equate these two values of \( x \), we get \( 2y - 300 = \frac{1}{6}(y + 70) \)

i.e. \( 12y - 1800 = y + 70 \)

Then \( y = 170 \)

Put this value of \( y \) in any of the two expressions for \( x \), we get \( x = 40 \).

The rules of Bhāskara-II for the solution of simultaneous linear equations involving several unknowns are the same as that of Brahmagupta.

Example 2.23 [14]

Eight rubies, ten emeralds and a hundred pearls which are in thy ear-ring were purchased by me for thee at an equal amount; the sum of the price rates of the three sorts of gems in thee less than the half of a hundred. Tell me, O dear auspicious lady, if thou be skilled in mathematics, the price of each.

Let \( x, y, z \) be the prices of a ruby, emerald and pearl respectively.

Then

\[
8x = 10y = 100z
\]

\[
x + y + z = 47
\]

Assume the equal amount to be \( w \).
If these are put in the next equation, we get $w = 200$

Therefore, $x = 25, \ y = 20, \ z = 2$

**Example 2.24** [14]  

"Tell the three numbers which become equal when added with their half, one-fifth and one-ninth parts, and each of which, when diminished by those parts of the other two, leaves sixty as remainder."

The equations are

$$x + \frac{x}{2} = y + \frac{y}{5} = z + \frac{z}{9} \quad \ldots \ldots \ (1)$$

$$x - \frac{y}{5} - \frac{z}{9} = y - \frac{z}{2} - \frac{x}{5} = z - \frac{x}{2} - \frac{y}{9} = 60 \quad \ldots \ldots \ (2)$$

Bhāskara-II takes $w$ as each of the equal quantities in (1)

Then $x = \frac{2}{3}w, \ y = \frac{5}{6}w, \ z = \frac{9}{10}w$

When put these values in (2), we get $\frac{2}{3}w = 60$

Therefore, $w = 150$

Therefore, $x = 100, \ y = 125, \ z = 135$

**Example 2.25** [14]  

Three portions $(x, y, z)$ of a sum of money ‘c’ were lent out at three different rates of interest $(r_1, r_2, r_3$ percent per month) for three different
periods \((t_1, t_2, t_3\) months). The interests accrued on them severally were the same. What were those portions?

The equations are, \(x + y + z = c\)  

\[
\frac{xr_1}{100} = \frac{yr_2}{100} = \frac{zr_3}{100} = I
\]

\(............ \ (2)\)

From (2)  

\[x = \frac{100I}{r_1t_1}, y = \frac{100I}{r_2t_2}, z = \frac{100I}{r_3t_3}\]

Equation (1) becomes  

\[100I \left( \frac{1}{r_1t_1} + \frac{1}{r_2t_2} + \frac{1}{r_3t_3} \right) = c\]

Therefore,  

\[I = \frac{c}{100 \left( \frac{1}{r_1t_1} + \frac{1}{r_2t_2} + \frac{1}{r_3t_3} \right)}\]

\[100 \times \frac{1}{c} \times c\]

Therefore,  

\[x = \frac{r_1t_1}{100 \times 1 + 100 \times 1 + 100 \times 1} \times c\]

\[\frac{r_1t_1}{r_1t_1 + r_2t_2 + r_3t_3}\]

We can find similar values for \(y\) and \(z\).

Bhāskara-II says:

“The arguments multiplied by their respective times are divided by the fruit taken into elapsed times. They being divided by their sum and multiplied by the total amount give the portions severally lent out.”

As per [14], Bhāskara-II says,

When the square of the unknown, etc remain, then multiplying the two sides (of the equation) by some suitable quantities, other suitable quantities should be added to them so that the side containing the unknown becomes capable of yielding a root (Pada-prada). The equation should be then be
formed again with the root of this side and the root of the known side. Thus the value of the unknown is obtained from that equation.

He explains this rule further:

When after perfect clearance of two sides, there remain on one side of the square etc. of the unknown and on the other side of the absolute term only, then, both the sides should be multiplied or divided by some suitable optional quantity; some equal quantities should further be added to or subtracted from both the sides so that the unknown side will become capable of yielding a root. The root of that side must be equal to the root of the absolute terms on the other side. For, by simultaneous equal additions, etc., to the two equal sides the equality remains. So forming an equation again with those roots the value of the unknown is found.

The Hindus understood early that the quadratic equation has generally two roots. Relating this, Bhâskara-II has quoted the following rule of ancient writer Padmanâbha whose treatise on algebra is not available now.

“If (after extracting roots) the square root of the absolute side (of the quadratic) be less than the negative absolute term on the other side, then taking it negative as well as positive, two values (of the unknown) are found.”

Bhâskara-II illustrated that though these double roots of the quadratic are theoretically correct, they are not suitable in certain occasions. So they should not always be accepted. He modified the rule as given below:

“If the square root of the known side (of the quadratic) be less than the negative absolute term occurring in the square root of the unknown side, then making it negative as well as positive, two values of the unknown should be determined. This is (to be done) occasionally.”
Example 2.26 [14],[34],[70]

The eighth part of a troop of monkeys, squared was skipping inside the forest, being delightfully attached to it. Twelve were seen on the hill delighting in screaming and rescreaming. How many were they?

Here the troop of monkeys is $x$. The square the eighth part of this together with 12 is equal to the troop. So the sides are

$$\frac{1}{64}x^2 + 0x + 12 = 0x^2 + x + 0$$

Reducing these to a common denominator, and then deleting the denominator, and also making clearance, the two sides become

$$x^2 - 64x + 0 = 0x^2 + 0x - 768$$

Adding the square of 32 to both sides and (extracting) square roots, we get $x - 32 = \pm(0x + 16)$

In this instance, the absolute term on the known side is smaller than the negative absolute term on the side of the unknown; hence it is taken positive as well as negative; the two values of $x$ are found to be 48, 16.

Example 2.27 [14], [36], [70], [77]

The fifth part of a troop of monkeys, leaving out three, squared, has entered a cane; one is seen to have climbed on the branch of a tree. Tell how many are they?

In this rule value of the troop is $x$; its fifth part less three is $\frac{1}{3}x - 3$; squared $\frac{1}{25}x^2 - \frac{6}{5}x + 9$; this added with the visible (number of monkeys) $\frac{1}{25}x^2 - \frac{6}{5}x + 10$, is equal to the troop.

Reducing to a common denominator, then deleting the denominator and making clearance, the two sides become

$$x^2 - 55x + 0 = 0x^2 + 0x - 250$$
Multiplying these by 4, adding the square of 55, and extracting the roots, we get \(2x - 55 = \pm (0x + 45)\)

In this case also, as in the previous, two values are obtained: 50, 5. But in this case, the second (value) should not be accepted, as it is inapplicable. ‘People have no faith in the known becoming negative.’

Example 2.28 [14],[63]

The shadow of a gnomon of twelve fingers being diminished by a third part of the hypotenuse becomes equal to fourteen fingers. O mathematician, tell it quickly.

Here the shadow is (taken to be) \(x\), this being diminished by a third part of the hypotenuse becomes equal to fourteen fingers. Hence conversely, fourteen being subtracted from it, the remainder, a third of the hypotenuse is \(x - 14\). Thrice this, which is the hypotenuse, is \(3x - 4x\). The square of it, \(9x^2 - 252x + 1764\), is equal to the square of the hypotenuse, \(x^2 + 144\). On making equi-clearence, the two sides become

\[8x^2 - 252x + 0 = 0x^2 + 0x - 1620\]

Multiplying both sides by 2 and adding the square of 63, the roots are \(4x - 63 = \pm (0x + 27)\)

On forming an equation with these sides again, and (proceeding) as before, the values of \(x\) are \(45/2, 9\). (Thus) the value of the shadow is \(45/2\) or 9. The second value of the shadow is less than 14 so on account of impracticability, it should not be accepted. Hence it was been said ‘two fold values occasionally’. This will be an exception to what has been stated in the algebra of Padmanâba.

The method of solving the quadratic equation was known amongst the Hindu algebraists by the name madhyamâharâna or ‘The Elimination of the Middle’.
As per [28], Bhāskara-II used letters to represent unknown quantities. In his works *Lilāvati* and the *Bijagāpita*, he suggested that positive numbers have two square roots and negative numbers no roots or with modern terminology, no ‘real roots’. He never gives examples of quadratic equations having two negative roots or no real roots at all. He does not give examples of quadratic equations having irrational roots. In every example, the square root in the formula is a rational number.

Bhāskara-II deals with multiple roots. Completing the square is the technique for solving quadratic equation. He adds an appropriate number to both sides of \( ax^2 + bx = c \), so that the left hand side becomes a perfect square.

That is, it is the form \((rx - s)^2 = d\). Then he solves the equation \(rx - s = \sqrt{d}\) for \(x\). But, “if the root of the absolute side of the equation is less than the number; having the negative sign, comprised in the root of the side involving the unknown then putting it negative or positive, a two-fold value is to be found of the unknown quantity” [36].

That is if \(\sqrt{d} < s\), then there are two values for \(x\), \(\frac{s + \sqrt{d}}{r}\), and \(\frac{s - \sqrt{d}}{r}\).

**Example 2.29** [6]

“O girl! Out of a group of swans, \(7/2\) times the square root of the number is playing on the shore of a tank. The two remaining ones are playing with amorous fight, in the water. What is the total number of swans?”

Let \(x\) be the number of swans.
Then from the question, we get the equation \( \frac{7}{2}\sqrt{x} + 2 = x \)

Then \( 49x = 4(x - 2)^2 \)

Solving this, we get \( x = 16 \) and \( x = 1/4 \)

Then \( x = 16 \) is the valid solution.

Example 2.30 [6]

"O! tender girl, out of the swans in a certain lake, ten times the square root of their number went away to Manasa Sarovar on the advent of the rains, 1/8 the number went away to a forest by name Sthala padmini. Three pairs of swans remained in the tank, engaged in water sports. What is the total number of swans?"

Let \( x \) be number of swans

Then from the question, \( 10\sqrt{x} + \frac{1}{8}x + 6 = x \)

\[ 10\sqrt{x} = \frac{7}{8}x - 6, \quad 10\sqrt{x} = \frac{7x - 48}{8} \]

Squaring and cross multiplying we get,

\[ 6400x = (7x - 48)^2 \]

\[ 49x^2 - 672x + 2304 = 6400x \]

\[ 49x^2 - 7072x + 2304 = 0 \]

\[ (x - 144)(49x - 16) = 0 \]

\[ x = 144, 16/49 \]

Therefore \( x = 144 \) is the valid solution.
Therefore \( x = \frac{144}{16/49} \)

\( x = 144 \) is the valid solution.

Example 2.31 [6]

Out of a swarm of bees, a number equal to the square root of half their number went to the Mālati flowers; \( \frac{8}{9} \) of the total number also went to the same place. A male bee enticed by the fragrance of the lotus got into it. But when it was inside it, night fell, the lotus closed, and the bee was caught inside. To its buzz, its consort was replying from outside. What is the number of bees?

Let \( x \) be the number of bees.

Then from the question we have the equation \( \sqrt{\frac{x}{2} + \frac{8}{9}x + 2} = x \)

\[
\sqrt{\frac{x}{2}} = x - \frac{8}{9}x - 2
\]

\[
\sqrt{\frac{x}{2}} = \frac{x}{9} - 2 = \frac{x-18}{9}
\]

Squaring, we get \( \frac{x}{2} = \frac{(x-18)^2}{81} \)

i.e. \( 2(x-18)^2 - 81x = 0 \)

\( 2x^2 - 153x + 648 = 0 \)

\( (x-72)(2x-9) = 0 \)

\( x = 72, \frac{9}{2} \)

\( x = 72 \) in the valid answer.
There is an interesting problem with reference to the battle between Arjuna and Karṇa in the Mahābhārata. This problem from [6], [57] is an imaginary one.

"Pārtha, with rage, shot a round of arrows to kill Karṇa in the war. With half of those arrows he destroyed Karna’s arrows, then killed his horses with four times the square root, hit Śalya with six arrows, destroyed Karna’s umbrella, flag and bow with three arrows and finally beheaded Karṇa with one arrow. How many arrows did Arjuna shoot?"

Let the number of arrows used by Arjuna (Pārtha) be $x$.

Then from the question,

i. Arrows used to destroy Karṇa’s arrows = $x/2$

ii. Arrow shot to kill the horses = $4\sqrt{x}$

iii. Arrows used for killing Śalya = 6

iv. Arrows to destroy the umbrella, flag and bow of Karṇa = 3

v. Arrow to behead Karṇa = 1

The sum of these should be the total number of arrows used by Arjuna.

Therefore, $\frac{x}{2} + 4\sqrt{x} + 10 = x$

$4\sqrt{x} = x - \frac{x}{2} - 10 = \frac{x - 20}{2}$

Squaring both sides, $16x = \frac{(x-20)^2}{4}$

i.e. $x^2 - 40x + 400 = 64x$
i.e. \( x^2 - 104x + 400 = 0 \)

That is \((x - 100)(x - 4) = 0\)

Therefore, \(x = 100, 4\)

\(x = 100\) is the valid solution.

Bhāskara-II’s rule on Vargakarma is as follows. It is an operation relating to squares. It deals with the problem of finding two unknown quantities whose sum of squares less one and difference of squares less one are perfect squares so that they can afford square roots. That is it deals with the problem of finding two quantities \(x\) and \(y\) such that \(x^2 + y^2 - 1\) and \(x^2 - y^2 - 1\) are perfect squares.

In Lilavati, Bhāskara-II gave arithmetic procedure for finding both integer and fractional solutions. He also gave illustrations. The following are the three different methods for finding the two unknowns stated by Bhāskara-II in the words of Mallayya V. M. [44].

The square of an arbitrary number multiplied by eight diminished by one, halved and divided by the arbitrary numbers is one (quantity); its square halved and added to one, is the other quantity. Unity, divided by twice an arbitrary number and added to the arbitrary number is the first (quantity) and the unit is other. These are the pairs of quantities the sum and difference of whose squares less one are squares. The square of square and the cube of an ordinary number, both multiplied by eight with unity added to the first (product) are such quantities both in arithmetic and algebra.

As per this rule, the two unknowns \(x\) and \(y\) satisfying \(x^2 + y^2 - 1\) and \(x^2 - y^2 - 1\) can afford the square roots \(\frac{1}{2}\left(8a^2 - 1\right)\) and \(a\).
\[
\left( \frac{1}{2} \right) \left( \frac{\left( \frac{1}{2} \right) (8a^2 - 1)}{a} \right)^2 + 1 \text{ or } \left( \frac{1}{2a} + a \right) \text{ and } 1, \text{ or } 8a^4 + 1 \text{ and } 8a^3 \text{ where 'a' is an arbitrarily assumed quantity. The first two methods are used for finding fractional solutions and the last one is used for finding integer solutions.}
\]

**China**

Jia Xian (1050) generalized the square and cube roots procedures of the *JiuZhang Suanshu* to higher roots by using the array of numbers. Today, it is known as the Pascal’s triangle. Some of Jia Xian’s methods are discussed in a work of Yang Hui written about 1261.

The first detailed study of Jia’s method for solving equations, probably improved appears in Qin Jiushao’s *Shushu JiuZhang*.

**Europe**

The Greek heritage, as well as some of the mathematics developed in the Islamic world, were brought into Western Europe through the efforts of translators. European scholars discovered the major Greek scientific works, primarily in Arabic translation beginning in the twelfth century. They started translation of these into Latin much of which was achieved at Toledo in Spain. There was a flourishing Jewish community, many of whose members were fluent in Arabic. The translations were made in two stages, first by a Spanish Jew from Arabic into Spanish and then by a Christian scholar from Spanish into Latin. An Englishman, Robert of Chester, who lived in Spain for several years, translated the ‘Algebra’ of Al-Khwārizmi in 1145. This introduced the algebraic algorithms for solving quadratic equations to Europe.

The greatest of all translators was Gerard of Cremona (1114-1187). He was an Italian who worked primarily in Toledo. He is credited with
the translation of more than 80 works. Among Gerard’s works was a new translation of Euclid’s *Elements* from Arabic of Thābit ibn Qurra and the first translation of Ptolemy’s *Almagest* from Arabic in 1175.

By the end of the twelfth century, many of the major works of Greek mathematics and a few Islamic works were available to scholars in Europe who read Latin. During the next centuries, these works were assimilated and new mathematics began to be created by the Europeans themselves.

Euclid’s *Elements* was translated into Latin early in the twelfth century. Before this, Arabic versions were available in Spain. When Abraham bar Hiyya of Barcelona wrote his *Hibbur ha-Meshihah ve-ha-Tishboret* (‘Treatise on Mensuration and Calculation’) in 1116 to help French and Spanish Jews with the measurement of their fields, he began the work with a summary of some important definitions, axioms and theorems from Euclid.

By about the 11th century methods of solving quadratic equations were well understood in Europe.

**Abraham bar Hiyya**

Abraham (died at 1136) took over the Islamic tradition of proof, absorbed from the Greeks. He gave geometric justifications of methods for solving the algebraic problems. Abraham included in his work, the major results of *Elements* Book II on Geometric Algebra. He used them to demonstrate methods of solving quadratic equations. His work was the first in Europe to give the Islamic procedures for solving such equations. As in [36], Abraham posed the problem:

“If from the area of a square one subtracts the sum of the (four) sides and there remains 21, what is the area of the square and what is the length of each of the equal sides?”
In modern terms, the question is equivalent to the quadratic equation \( x^2 - 4x = 21 \).

He solves in the familiar way beginning with the halving of 4 to get 2. Abraham’s statement of the problem is not geometrical. He writes of subtracting a length, the sum of the sides from an area, but in his geometric justification he restates the problem to mean the cutting off a rectangle of sides 4 and \( x \) from the original square of unknown side \( x \) to leave a rectangle of area 21. Then he bisects the side of length 4 and applied the results (Elements II-6). This is to justify the algebraic procedure. Clearly, Abraham had learned his algebra not from Al-Khwārizmi but learned from an author such as Thābit ibn Qurra, who used Euclidian justifications.

In a similar way, Abraham presented the method and Euclidean proof for examples of the two other Islamic classes of mixed quadratic equations, \( x^2 + 4x = 77 \) and \( 4x - x^2 = 3 \).

In the case of \( 4x - x^2 = 3 \), he gave both positive solutions. Also, as per [36], he solved quadratic problems as the systems \( x^2 + y^2 = 100, x - y = 2 \) and \( xy = 48, x + y = 14 \).

2.3 1201 – 1400 AD

India- Nārāyaṇa

In connection with the solution of linear equations, as per [14] Nārāyaṇa states:

“From one side clear off the unknown and from the other the known quantities; then divide the residual known by the residual coefficient of the unknown. Thus will certainly become known the value of the unknown.”
For the solution of linear equations with several unknowns Nārāyaṇa states:

“The sum of the depleted amounts divided by the number of persons less one is the total amount. On subtracting from it, the stated amounts severally will be found the different amounts.”

Some historians of mathematics like Cantor, Kaye consider the above type of equations to be modification of the type considered by the Greek, Thymaridas and solved by his well known rule *Epanthema* as follows [14], [29].

\[ x + x_1 + x_2 + ... + x_{n-1} = S \]

\[ x + x_1 = a_1, \quad x + x_2 = a_2 \ldots x + x_{n-1} = a_{n-1} \]

The solution given as \[ x = \frac{(a_1 + a_2 + \ldots + a_{n-1}) - S}{n - 2} \]

But Sarada Kanta Ganguly [14] has shown that it is based upon a wrong understanding. To him in the Thymaridas type of linear equations, the value of the sum of the unknowns is given. But in the Āryabhaṭa type it is not known. Āryabhaṭa determines only that value.

Nārāyaṇa also treated a generalized system of linear equations. He says:

Multiply the sum of the monies received by each person by his multiple number plus unity. Then proceed as in the method for “the purse of discord”. Divide the multiple numbers related to each by the same as increased by unity. By the sum diminished by unity of these quotients, divide the results just obtained. The quotients will be the several amounts in their possession [14].
With the opening of the 13th century there appear two mathematicians in Europe called Leonardo of Pisa and Jordanus Nemorarius.

Leonardo

Leonardo of Pisa (1170-1240) is often known today by the name Fibonacci. His father was a Pisan merchant in Bugia on the North African coast. He spent much of his early life there learning Arabic and studying mathematics under Islamic teachers. The works which have been preserved include the *Liber abaci* (1202, 1228), the *Practica geometriae* (1220), *Flos* (1225) and the *Liber quadratorum* (1225). The sources of his works were largely in the Islamic world, which he visited during many journeys. He arranged and enlarged the materials he collected. *Liber abaci* contained various forms of problems solvable by use of quadratic equations along with other topics. A limited amount of theory is mixed among the problems such as methods for summing series and geometric justifications of the quadratic formulas.

He used many of methods in his solutions of problems. He used special procedures designed to suit a particular problem rather than more general method. Leonardo used the old Egyptian Method of False Position. He also used the methods of Al-Khwārizmi for solving quadratic equations. He often took problems from Islamic mathematicians as Al-Khwārizmi,
Abū Kāmil and Al-Karaji. Some of the problems seem to have come from China or India. Probably he learned these in Arabic translation.

About 1225, when Frederick-II held court at Pisa, the astronomer Dominicus presented Leonardo to the emperor. On that occasion, John of Palermo proposed several problems. Leonardo solved them promptly. Three of these are mentioned in Flos.

As per [10], [11], [12], [17], [69], [74], [81], [82] the following are the problems. The first problem was to find a number $x$ such that $x^2 + 5$ and $x^2 - 5$ are square numbers. The solution is,

$$x = 3 \frac{5}{12}, x^2 + 5 = \left(4 \frac{1}{12}\right)^2, x^2 - 5 = \left(2 \frac{7}{12}\right)^2.$$

It is presented without proof in the book Flos. Leonardo sent it to Frederick-II.

The second problem was the solution of a cubic equation

$$x^3 + 2x^2 + 10x = 20$$

In Flos Leonardo proved that the solution is neither an integer, a fraction, nor one of the irrationalities defined in Book X of the Elements of Euclid. He presented an approximate solution in the sexagesimal form as

$$x = 1; 22, 7, 42, 33, 40$$

This is the same as

$$x = 1 + \frac{22}{60} + \frac{7}{60^2} + \frac{42}{60^3} + \frac{33}{60^4} + \frac{4}{60^5} + \frac{40}{60^6}$$

In decimal notation, $x \approx 1.368808106$

On substituting this in the expression $y = x^3 + 2x^2 + 10x$, we get
If we write $x = 1.368808110$, we get $y \approx 20.00000005$

Thus $1.368808106 < x < 1.368808110$

In writing common fractions Brahmagupta and Bhāskara-II had used the scheme of placing the numerator above the denominator without any line of separation. At first, Arabs copied it, later it was improved by inserting a horizontal bar between the two numbers. Leonardo followed this in the *Liber abaci*. Habitually, he put the fractional part of a mixed number before the integral part, with juxtaposition used to imply their addition. His notation for it, for example is

\[
\begin{align*}
\frac{7}{10} + \frac{1}{10} + \frac{7}{10 \times 10} & \quad \text{means } 8 + \frac{1}{10} + \frac{7}{10 \times 10} \\
\frac{1}{2} + \frac{5}{6} + \frac{7}{10} & \quad \text{means } \frac{7}{10} + \frac{5}{10 \times 6} + \frac{1}{10 \times 6 \times 2}
\end{align*}
\]

The third problem asked by the John of Palermo was as follows:

Three men possess a sum of money and their shares being $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$. Each one of them takes some money from the pile until the whole sum is taken. Then the first man put back $\frac{1}{2}$ of what he had taken, the second $\frac{1}{3}$ and the third $\frac{1}{6}$. When the total so put back is divided equally amongst the three men it is found that each man then possesses what he is entitled to. What was the total sum, and how much did each man take from the original pile?

Leonardo's solution in modern terms is:
Call the total original sum as \( S \) and the third part of the total sum put back \( x \). Each man, by taking \( x \), gets what he was entitled to. That is the sums \( \frac{S}{2}, \frac{S}{3}, \frac{S}{6} \).

Therefore, three men possessed before taking \( x \),

\[
\frac{S}{2} - x, \frac{S}{3} - x, \frac{S}{6} - x
\]

They had possessed these sums after putting back \( \frac{1}{2}, \frac{1}{3}, \frac{1}{6} \) of what they had first taken.

Therefore, these sums are \( \frac{1}{2}, \frac{2}{3}, \frac{5}{6} \) of what they had first taken.

Therefore, what they had first taken were

\[
2 \left( \frac{S}{2} - x \right), 3 \left( \frac{S}{3} - x \right), 6 \left( \frac{S}{6} - x \right)
\]

These sums together make \( S \).

Hence, \( S = S - 2x + \frac{S - 3x}{2} + \frac{S - 6x}{5} \)

i.e. \( 7S = 47x \)

Therefore, this problem is indeterminate.

One solution is \( S = 47, x = 7 \). And the sums taken by the men from the original pile are 33, 13, and 1.

Leonardo called \( x \) *res* and \( S \) *tota communis pecunia*. Leonardo published these solutions in two remarkable works, the *Liber quadratarum* and the *Flos*.

In *Liber abaci*, Leonardo presents several versions of the classic problem of buying birds.
Example 2.32 \cite{36},\cite{81}

How to buy 30 birds for 30 coins if partridges costs 3 coins each, pigeons 2 coins each and sparrows 2 for 1 coin.

He begins by noting that he can buy 5 birds for 5 coins by taking 4 sparrows and 1 partridge. Similarly, two sparrows and 1 pigeon will give him 3 birds for 3 coins. By multiplying the first transaction by 3 and the second by 5, he obtains 12 sparrows and 3 partridges for 15 coins and 10 sparrows and 5 pigeons also for 15 coins. Sum of these transactions gives the required answer. That is, 22 sparrows, 5 pigeons, 3 partridges.

The problem is to solve the pair of equations $x + y + z = 30$, $3x + 2y + \frac{1}{2}z = 30$ in positive integers $x, y, z$. The solution is $x = 3, y = 5, z = 22$.
Example 2.33 [36]

The pit is 50 feet deep. The lion climbs up $1/7$ of a foot each day and then falls back $1/9$ of a foot each night. How long will it take him to climb out of the pit?

He uses a version of false position. He takes 63 as the answer because 63 is divisible by both 7 and 9. Thus in 63 days the lion will climb up 9 feet and fall down 7, for a net gain of 2 feet. By proportionality, to climb 50 feet, the lion will take 1575 days. Leonardo’s answer is incorrect. At the end of 1571 days the lion will be only $8/63$ of a foot from the top. On the next day the lion will reach the top.

Example 2.34 [36]

Suppose that two men each have some money. The first says to the second, “if you give me one *denarius*, we will each have the same amount”. The second says to the first, “if you give me one *denarius*, I will have ten times as much as you”. How much does each have?

In modern terms, let $x$ represent the amount by the first and let $y$ represent the amount of the second.

This problem leads to the system of equations

$x + 1 = y - 1, \ y + 1 = 10(x - 1)$

Leonardo introduced the new unknown the total sum of the money

That is, $z = x + y$

Then, $x + 1 = \frac{1}{2}z$ and $y + 1 = \frac{10}{11}z$
Adding these equations we get

\[ z + 2 = \frac{31}{22} z \]

From this, \( z = \frac{44}{9}, x = \frac{4}{9}, y = \frac{4}{9} \)

There is a sequence of problems concerning ‘buying a horse’
Leonardo begins with a simple case of two persons.

Example 2.35 [81]

One says to the other, “If you give me one-third of your cash, I can by the horse”. The other replies: “If you give me a quarter of your cash, I can by the horse”. How much does each have?

Let \( s \) be the price of the horse. We have two linear equations with two unknowns \( x \) and \( y \).

Then \( x + \frac{1}{3} y = s \)

\[ y + \frac{1}{4} x = s \]

Since \( s \) is not given, the problem is indeterminate. The solution is given as

\[ x = (3 - 1) \times 4 = 8 \]

\[ y = (4 - 1) \times 3 = 9 \]

\[ s = 3 \times 4 - 1 \times 1 = 11 \]

Another case of this problem leads to 3 equations in 3 unknowns.

They are:
To solve these equations, he introduces a new unknown \( t \).

\[
\begin{align*}
\frac{x}{3} + \frac{y+z}{3} &= s \\
\frac{y}{4} + \frac{x+z}{4} &= s \\
\frac{z}{5} + \frac{x+y}{5} &= s
\end{align*}
\]

\[
\begin{align*}
\left\{
\begin{array}{c}
x + y + z = t \\
\frac{x}{3} + \frac{y+z}{3} &= s \\
\frac{y}{4} + \frac{x+z}{4} &= s \\
\frac{z}{5} + \frac{x+y}{5} &= s
\end{array}
\right. 
\end{align*}
\]

Subtracting each of the three equations in (1) from (2) we get,

\[
\begin{align*}
\frac{2}{3} (y+z) &= \frac{3}{4} (x+z) \\
&= \frac{4}{5} (x+y) = t - s = D
\end{align*}
\]

Hence \( y+z = \frac{3}{2} D \)

\[
\begin{align*}
x + z &= \frac{4}{3} D \quad (3) \\
x + y &= \frac{5}{4} D \quad (4)
\end{align*}
\]

To get the integral solution, Leonardo puts \( D = 24 \),

We get, \( y+z = 36 \)

\[
\begin{align*}
x + z &= 32 \\
x + y &= 30
\end{align*}
\]

Then \( x = 13, y = 17, z = 19 \)

Example 2.36 [77]

The Lion, The Leopard, and the Bear: A Lion ate a sheep in 4 hours, a leopard in 5 hours, and a bear in 6; if one sheep is thrown to them, in how many hours will it be devoured?
Suppose they devour sheep for 60 hours. In this time, the lion devours 15, the leopard 12 and the bear 10 sheep. The total is 37 sheep. Therefore in 60 hours, they ate 37 sheep. We want to find, how long to eat one sheep? Multiply 1 by 60 and divide by 37, we get $1\frac{23}{37}$ hours.

Leonardo used the 'Rule of Double False Position' in the following example:

"Suppose 100 rotuli are worth 260 soldi; how much is 1 soldi worth?" [77].

Suppose 1 rotuli is worth 1 soldi

Then 100 rotuli are worth 100 soldi.

But there were 260 soldi.

Thus the first guess is false differs from the correct value by 160.

Next, we suppose a rotuli is worth 2 soldi.

As before, this gives a value of 200 soldi.

It is also false and differs by 60.

Now the first guess was short by 160 and the second by 60.

The difference between the first and second guesses is 1 soldi, and the corresponding difference is 100 soldi.

Therefore, 60 more are needed.

Thus multiply 1 by 60 and divide by 100. The result is added to the second guess, 2 to make $2\frac{3}{5}$ soldi.
Leonardo also deals with determinate and indeterminate problems in more than two unknowns.

**Example 2.37** [36]

There are four men such that the first, second, and third together have 27 *denarii* the second, third, and fourth together have 31, the third, fourth and first have 34, while the fourth, first and second have 37.

To determine how much each man has requires solving a system of four equations in four unknowns. He gave it by adding the four equations together to determine that four times the total sum of money equals 129 *denarii*. The amount of each man is then calculated.

Rabbit problem is the most famous problem of the *Liber abaci*.

How many pairs of rabbits can be bred in one year from one pair? A certain person places one pair of rabbits in a certain place surrounded on all sides by a wall. We want to know how many pairs can be bred from that pair in one year, assuming it is their nature that each month they give birth to another pair, and in the second month after birth, each new pair can also breed [10], [11], [17], [36], and [77].

Leonardo proceeds as follows: After the first month there will be two pairs, after the second, three pairs. In the third month two pairs will produce. So at the end of that month there will be five pairs. In the fourth month, three pairs will produce. So there will be eight. Continuing in this way, he shows that there will be 377 pairs by the end of the twelfth month. He tested the sequence 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377 in the margin. Leonardo notes that, each number is found by adding the two previous numbers. Also “Thus you can do it in order for an infinite number of months”.

Today this sequence is known as ‘Fibonacci Sequence’ [58].
Liber abaci contained no particular advance over mathematical works than that was current in the Islamic world. The main value of the work was that it gave Europe’s first comprehensive introduction to the mathematics of Islamic works. Leonardo made original contributions to mathematics which occurs in a shorter work, the Liber quadratorum (‘Book of Squares’). This appeared in the year 1225. It is a book on number theory, in which Leonardo follows in the footsteps of Diophantus.

In the final chapter of Liber abaci, Leonardo shows his complete knowledge of the algebra of his Islamic predecessors. He shows how to solve higher degree equations that reduce to quadratic equations. He discusses each of the six types of quadratic equation given by Al-Khwārizmi.

Leonardo solves the six normal forms [81],

\[
\begin{align*}
ax^2 &= bx \\
ax^2 &= c \\
bx &= c \\
ax^2 + bx &= c \\
ax^2 + c &= bx \\
ax^2 &= bx + c
\end{align*}
\]

Then he gives geometric proofs of the solution procedures for each of the three mixed cases.

The unknown quantity \(x\) is called radix; its square is called quadratus or census. The constant term \(c\) is called numerus. The methods of solution are illustrated by examples.

According to [81], the first example of Leonardo’s form for a mixed quadratic equation is \(x^2 + 10x = 39\). It is same as in the algebra of Al-Khwārizmi. The solution is illustrated by the figure 2.19.
In the other examples, one has to divide 10 into two parts \( x \) and \( 10-x \) satisfying an auxiliary condition such as

\[
\frac{x}{10-x} + \frac{10-x}{x} = \sqrt{5}
\]

Also, Leonardo includes equations that can be reduced to quadratic equations.

As per [81], the set of equations \( y = \frac{10}{x}, z = \frac{y^2}{x}, z^2 = x^2 + y^2 \) leads to a quadratic equation for \( x^4 \):

\[
x^8 + 100x^4 = 10,000
\]

Leonardo, like the Arabic mathematicians before him recognized that a quadratic equation can be satisfied by two values. But he habitually rejected negative numbers as solutions [11].

**Jordanus de Nemore**

Jordanus de Nemore is another European mathematician of the early 13\(^{th}\) century. About him nothing is known. It is believed that he taught in
Paris around 1220. His writing includes several works on arithmetic, geometry and mechanics. Here we will discuss his major work on algebra, *De numeris datis* ('On Given Numbers').

*De numeris datis* is an analytic work on algebra. It is based on the Islamic algebras that had made their way into Europe by the early thirteenth century. It appears to be modeled on Euclid's *Data* available to him in a Latin translation by Gerard of Cremona. However, the problems in *De numeris datis* are algebraic rather than geometric.

Although, many of the problems and the numerical examples were available in the Islamic algebras, he adopted them to his own purposes. He used letters to stand for arbitrary numbers. His algebra is no longer entirely rhetorical. This is not to say that his symbolism looks modern. He picks his letters in alphabetical order with distinction between letters representing known quantities and those representing unknowns. He uses no symbols for operation. Sometimes, two letters represent a single number. At other times the pair of letters ‘ab’ represents the sum of the two numbers ‘a’ and ‘b’. The basic arithmetic operations are always written in words. Jordanus does not use the new Hindu-Arabic numerals. All of his numbers are written as Roman numerals.

Jordanus writes each proposition in a standard form. The general statement is followed by a restatement in terms of letters. By use of general rules, the letters representing numbers are manipulated into a canonical form. From this the general solution can easily be found. Finally, a numerical example is calculated following the general outlines of the abstract solution. The canonical forms themselves are among the earliest of the propositions. The following examples of Jordanus are related to linear equations which are known as propositions.
Example 2.38 [19], [36], [77]

“If a given number is divided into two parts whose difference is given, then each of the parts is determined.”

Jordanus proof is as follows:

Namely, the lesser part and the difference make the greater. Thus the lesser part with itself and the difference make the whole. Subtract therefore the difference from the whole and there will remain double the lesser-given number. When divided (by two), the lesser part will be determined; and therefore also the greater part. For example, let 10 be divided into two parts of which the difference is 2. When this is subtracted from 10 there remain 8, who’s half is 4, which is thus the lesser part. The other is 6.

In modern terms, the above proposition amounts to the solution of the two equations $x + y = a, x - y = b$

Jordanus first notes that $y + b = x$ so that $2y + b = a$

Therefore, $2y = a - b$

Thus, $y = \frac{1}{2}(a - b)$ and $x = a - y$

He uses this proposition in many of the remaining problems of Book I. For example, a proposition in Book I is:

“If a given number is divided into two parts, and the product of one by the other is given, then of necessity each of the two parts is determined.” [19], [36], [77]

This proposition is the same as one of the standard Babylonian problems: $x + y = m, xy = n$. However, Jordanus’s method of solution is different from the Babylonian solution. Also he uses notation as below.
Suppose, the given number be ‘abc’. It is divided into two parts ‘ab’ and ‘c’. Suppose ‘ab’ multiplied by ‘c’ is ‘d’ and ‘abc’ multiplied by itself is ‘e’.

Let ‘f’ be the quadruple of ‘d’ and ‘g’ be the difference of ‘e’ and ‘f’. Then ‘g’ is the square of the difference between ‘ab’ and ‘c’. ‘b’ is its square root. Then, ‘b’ is the difference between ‘ab’ and ‘c’. Since both the sum and difference of ‘ab’ and ‘c’ are given, both ‘ab’ and ‘c’ are determined according to the first proposition.

The numerical example of Jordanus has 10 as the sum of two parts and 21 as the product. He notes that 84 is the quadruple of 21, that 100 is 10 squared. Then, 16 is their difference. Then the square root of 16 (i.e. 4) is the difference of the two parts of 10. Then by Example 2.38, 4 is subtracted from 10 to get 6. Then 3 is the smaller part and 7 is the larger part.

In modern terms, Jordanus solution amounts to using the identity 

\[(x - y)^2 = (x + y)^2 - 4xy = m^2 - 4n \]

to determine \(x-y\) and reduce the problem of the form proposition I of Book I.

Then the solution is \(x = m - \frac{1}{2}(m-\sqrt{m^2-4n})\),

\[y = \frac{1}{2}(m-\sqrt{m^2-4n})\].

His method appears to be new with him. He continues to use his own methods throughout the work.

As per [36], a proposition of Book I of Jordanus proposes to solve the system \(x + y = m, x^2 + y^2 = n\).

He first determines \(2xy\) from the identity,

\[2xy = (x + y)^2 - (x^2 + y^2)\]
Since \((x-y)^2 = x^2 + y^2 - 2xy\), both the sum and difference of the unknowns are determined are by a proposition of Jordanus. Then \(x\) and \(y\) can be found.

Similar methods are used in his proposition the solution of the system \(x - y = m, xy = n\) and proposition the solution of the system \(x - y = m, x^2 + y^2 = n\).

Every proposition in Book I of Jordanus deals with a number divided into two parts.

**Example 2.39** [11]

If the sum of the squares of the two parts of a given number added to their difference is known, then the two parts can be found.

The problem in modern terms is as follows.

The two equations are \(x + y = a, x^2 + y^2 + x - y = b\).

His example is \(x + y = 10, x^2 + y^2 + x - y = 62\).

Its solution is \(x = 7\) and \(y = 3\).

Jordanus de Nemore’s *De numeris datis* gives us his first example of a pure quadratic equation. It contains system \(x + b = y, x^2 + bx = a\). The second equation can be written as \((x + b)x = a\). His numerical example is \(x^2 + 6x = 40\). It has the solution \(x = 4\). Therefore, \(y\) the sum of the two parts \(x\) and \(b\) is 10 [36].

Three propositions in his Book IV are giving three standard forms of the quadratic equation. They are presented with algebraic rather than geometric justifications. However, Jordanus uses the standard Islamic algorithm. The following is an example in Jordanus’ work given by Katz V.J.
"If the square of a number added to a given number is equal to the number produced by multiplying the root and another given number, then two values are possible."

By this example, Jordanus asserts that there are two solutions to the equation \( x^2 + c = bx \). Then he gives the procedure for solving the equation as follows.

Take half of \( b \), square it to get \( f \).

Let \( g \) be the difference of \( x \) and \( \frac{1}{2} b \)

That is \( g = x - \frac{1}{2} b \)

Then \( x^2 + f = x^2 + c + g^2 \) and \( f = c + g^2 \)

His conclusion is that \( x \) may be obtained by either subtracting ‘\( g \)’ from \( \frac{b}{2} \) or by adding ‘\( g \)’ to \( \frac{b}{2} \).

His example is:

Solve \( x^2 + 8 = 6x \)

He squares half of 6, getting 9.

Then subtract 8 from 9 leaving 1.

The square root of 1 is 1.

This is the difference between \( x \) and 3.

Therefore \( x = 2 \) or \( x = 4 \).
In Book IV, the other quadratic problems Jordanus solved are the systems $xy = a, x^2 + y^2 = b$ and $xy = a, x^2 - y^2 = b$. In each case, the given example results in a positive integral solution. When he uses fractions as part of his solution, he has carefully arranged matters so that final answers are always whole numbers.

De numeris datis represents an advance from the Islamic works in the use of analysis and in some symbolization. As the first Western mathematician consistently to employ letters of the alphabet to designate quantities, known as well as unknown, Jordanus advanced the evolution of algebraic symbolism.

Qin Jiushao

During the Song dynasty, Qin Jiushao (1202-1261) published Shushu Jiuzhang (‘Mathematical Treatise in Nine Sections’ (1247)). This work was greatly influenced by the old Jiuzhang Suanshu. Like the earlier work, ‘Qin’s Mathematical Treatise’ was a collection of problems with solutions and methods. Many of these problems were similar problems of the old text and were solved by similar methods.

The Shushu Jiuzhang contains many problems in solving polynomial equations. The methods for solving these equations in it and in several other Chinese works can be considered as a generalization of the methods for solving pure quadratic equations ($x^2 = a$) detailed in the Jiuzhang Suanshu. There is one example of a mixed quadratic equation in the text. But method of solution is not given.

Qin Jiushao had three contemporaries: Li Ye (1192-1279), Yang Hui (second half of 13th century), and Zhu Shijie (late 13th century). They also made significant contributions to the mathematics of solving equations.
Li Ye wrote two major mathematical works, the *Ceyuan Haijing* (‘Sea Mirror of Circle Measurements’) in 1248 and the *Yigu Yanduan* (‘Old Mathematics in Expanded Sections’) in 1259. The *Ceyuan Haijing* dealt with the properties of circles inscribed in right triangles. It was mainly concerned with the setting up and solution of algebraic equations for dealing with these properties. The *Yigu Yanduan* dealt with geometric problems on squares, circles, rectangles, and trapezoids. But its main object was the teaching of methods for setting up the appropriate equations, particularly quadratic, for solving the problem.

Following is an example from *Yigu Yanduan*:

“There is a circular pond inside a square field and the area outside the pond is 3300 square feet. The sum of the perimeters of the square and the circle is 300 feet. Find the two perimeters” [36], [41]

Consider the geometrical representation in figure 2.20. Li Ye sets \( x \) to be the diameter of the circle and \( 3x \) to be the circumferences where they used 3 instead of \( \pi \). Then \( 300 - 3x \) is the perimeter of the square.

Squaring, he gets \( 90,000 - 1800x + 9x^2 \) as the area of 16 square fields.

\[
\text{Since } \frac{3x^2}{4} \text{ is the area of one circular pond, } 12x^2 \text{ is the area of 16 circular ponds. The difference of the two expressions,}
\]

\[90,000 - 1800x - 3x^2\] is equal to 16 portions of the area outside the pond.

Then \( 16 \times 3300 = 52,800 \)

The required equation is \(37,200 - 1800x - 3x^2 = 0\)
Li Ye now merely asserts that 20 is the root. Therefore, the
diameter is 20.

Therefore 60 is the circumference of the circle and 240 is the
perimeter of the square.

Li Ye always follows his algebraic derivation with a geometric
derivation. See the following figure. Here the side of the large square is 300,
the sum of the given perimeters. The shaded areas represent $16 \times 3300$.

Since $300x$ is the area of each long strip, $x^2$ is the area of each
small square.

$12x^3$ is the total area of the 16 circular ponds.

He derives the equation,

$$300^2 - 16 \times 3300 = 6 \times 300x - 9x^2 + 12x^2 = 1800x + 3x^2$$

i.e. $37200 = 1800x + 3x^2$

The diagram indicates the 3 small squares at the bottom right.
Yang Hui

Yang Hui, lived under the Sung dynasty in the south of China. Two of his major works are still extant: The *Xiangjie Jiushang Suanfa* (‘A Detailed Analysis of the Arithmetical Rules in the Nine Sections) of 1261 and the collection known as *Yang Hui Suanfa* (‘Yang Hui’s Methods of Computation’) of 1275. The *Yang Hui Suanfa* contained material on quadratic equations. Yang Hui gave a detailed account of his methods. He used the same method as Qin. But he also gave alternate methods. Yang presented geometric diagrams consisting of squares and rectangles illustrating various numerical methods used.

![Figure 2.21: Yang Hui](image)

According to [45], Yang Hui explains how to solve equations of the type $ax^2 + bx = c$ numerically.

**Example 2.40** [47]

“There is a rectangular field, 864 square paces in area, and whose breadth is 12 paces less than the length. Find the breadth.”

We have $A = lb = (b + d)b = b^2 + bd$

Then $x^2 + 12x = 864$
Then the breadth is obtained from 

\[-864 + 12x + x^2 = 0\]

A method had existed from old times for the solution of such an equation. Yang Hui explains, if we knew that \(b = 24\) and \(l = 36\) paces, the area of the rectangle will consist of such parts as shown in the figure 2.22.

Suppose we take \(x = 20\). Operate as in the case of the extraction of a square root, then we shall see that the area may be divided as in the figure 2.23.

If we subtract \(20^2\) and \(20 \times 12\) the remainder will be equal to

\[2 \times (20 \times 4) + 12 \times 4 + 4^2.\]

Divide the remainder roughly by \((20 \times 2 + 12)\) the next figure in the evolved root should be 4.
If we subtract $4^2$ from the above there is no remainder and we get the breadth 24.

If however there is still a remainder, continue the same process further.

Yang Hui uses two types of methods:

(i) For quadratic equations, methods based on geometry.

(ii) For an equation of degree four, a method similar to Horner’s method.

To some authors, Yang Hui explained that the second-degree equation $-x^2 + 60x = 864$ has two distinct roots [45].

Zhu Shijie

Zhu Shijie was a thirteenth century Chinese mathematician. He was probably born near Beijing. He spent most of his life as professional mathematics teacher. He wrote two major works, *Suanxue Qimeng* (‘Introduction to Mathematical Studies’) in 1299 and the *Sijuan Yujian* (‘Precious Mirror of the Four Elements’) in 1303.

He adapted Qin’s method of solving polynomial equations into a procedure for solving systems of equations. We illustrate his method by considering the first problem of the ‘Precious Mirror’.

“Given that the length of the diameter of a circle inscribed in a right triangle multiplied by the product of the lengths of the two legs equals 24, and the length of the vertical leg added to the length of the hypotenuse equals 9, what is the length of the horizontal leg?” [36]

Consider the geometrical representation in figure 2.24. In modern terms, let ‘$a$’ be the vertical leg, ‘$b$’ the horizontal leg, ‘$c$’ the hypotenuse,
and 'd' the diameter of the circle. The problem can be translated into the two equations,

\[ \begin{align*}
    d a b &= 24 \\
    a + c &= 9
\end{align*} \]

In addition, Zhu assumes as known, the two equations, \[ a^2 + b^2 = c^2, \quad d = b - (c - a), \] where the second gives the relationship between the diameter of the inscribed circle and the lengths of the sides of the triangle.

![Figure 2.24](image)

The method of solution is only briefly indicated and the fifth degree equation satisfied by 'b' is simply written down.

The Chinese mathematicians were proficient in solving many kinds of algebraic problems. Many of their methods probably stemmed from geometric considerations. But they were apparently translated into purely algebraic procedures. The Chinese scholars were primarily interested in solving problems of importance to the Chinese bureaucracy. Some incorrect methods, as contained in the *Jiuzhang Suanshu* were repeated down through the centuries. Although the thirteenth century mathematicians exploited the counting board to the maximum its very use provided limits. Due to political circumstances some of the great 13th century works were no longer studied. With the arrival of the
Jesuit priest Matteo Ricci (1552-1610) in the late 16th century, Western mathematics entered China, and then the traditional system began to disappear.