ABSTRACT

Graphs are really important and probably, they are more important than we think. Graphs are among the most ubiquitous models of both natural and human-made structures and Graph theory is one of the most developing branches of Mathematics with wide applications to computer science. Graph theory is applied in diverse areas such as social sciences, linguistics, physical sciences, communication engineering and others. Graph theory also plays an important role in several areas of computer science, such as switching theory and logic design, artificial intelligence, formal languages, computer graphics, operating system, compiler writing and information organization and retrieval. Graph theory, especially trees and binary trees are used widely in the representation of data structures. Graph theory has now established itself as a discipline on its own. The advent of the computers and the use of graph theory in computer science have helped development of the subject. Many branches of Mathematics begin with sets and relations. Graph theory is on exception to this. Indeed, graphs are next only to sets.

One simple way of representing the structure of a system is to use graphs, which are simple diagrams consisting of points (vertices) and lines (edges). Graphs are useful in enhancing the understanding of the organization and behavioral characteristics of complex systems. The basic ideas of Graph theory were introduced during 18\textsuperscript{th} century by the great Mathematician Leonard Euler. In particular, the term "graph" was introduced by “Sylvester” in a paper published in 1878 in “Nature”, where he draws an analogy between "quantic invariants" and "co-variants" of algebra and molecular diagrams.

The first textbook on graph theory was written by “Denes Konig” and published in 1936. A later textbook by “Frank Harary” published in 1969 was enormously popular and enabled mathematicians, chemists, electrical engineers and social scientists to talk to each other. Graphs are among the most ubiquitous models of both natural and human-made structures. They can be used to model many types of relations and process dynamics in physical, biological and social systems. Many problems of practical
interest can be represented by graphs. Graphs are represented graphically by drawing a dot or circle for every vertex and drawing an arc between two vertices, if they are connected by an edge. If the graph is directed, the direction is indicated by drawing an arrow. A graph drawing should not be confused with the graph itself (the abstract, non-visual structure) as there are several ways to structure the graph drawing. All that matters is which vertices are connected to which others by how many edges and not the exact layout. In practice it is often difficult to decide, if two drawings represent the same graph. Depending on the problem domain some layouts may be better suited and easier to understand than others.

Graph drawing also can be said to encompass problems that deal with the crossing number and its various generalizations. The crossing number of a graph is the minimum number of intersections between edges that a drawing of the graph in the plane must contain. For a planar graph, the crossing number is zero by definition.

There are different ways to store graphs in a computer system. The data structure used depends on both the graph structure and the algorithm used for manipulating the graph. Theoretically one can distinguish between list and matrix structures but in concrete applications the best structure is often a combination of both. List structures are often preferred for sparse graphs as they have smaller memory requirements. Matrix structures on the other hand provide faster access for some applications but can consume huge amounts of memory.

In computer science, graphs are used to represent networks of communication, data organization, computational devices, the flow of computation, etc. One practical example: The link structure of a website could be represented by a directed graph. The vertices are the web pages available at the website and a directed edge from page $A$ to page $B$ exists if and only if $A$ contains a link to $B$. A similar approach can be taken to problems in travel, biology, computer chip design and many other fields. The development of algorithms to handle graphs is therefore of major interest in computer science. There, the transformation of graphs is often formalized and represented by graph rewrite systems. They are either directly used or properties of the rewrite systems
(e.g. confluence) are studied. Complementary to graph transformation systems focusing on rule-based in-memory manipulation of graphs are graph databases geared towards transaction-safe, persistent storing and querying of graph-structured data.

Trees are the most important class of graphs and they make fine modelling tools. In 1847, the German Physicist Gustav Robert Kirchoff used them to solve systems of linear equations for electrical networks. Ten years later, Arthur Cayley studied the isomers of saturated hydrocarbons $C_nH_{2n+2}$ with them. Today trees are widely used in Mathematics and Computer Science, as well as in Linguistics and the Social Sciences.

An algorithm (pronounced AL-go-rith-um) is named after the 9th century Persian Mathematician Muhammad Musa Al-Khwarizmi, who was part of the royal court in Baghdad and who lived from about 780 to 850. Al-Khwarizmi’s work is the likely source for the word algebra as well. Algorithms are used for calculation, data processing and automated reasoning. More precisely, an algorithm is an effective method expressed as a finite set of well defined instructions for calculating a function. Starting from an initial state and initial input (perhaps empty) the instructions describe a computation that when executed will proceed through a finite number of well-defined successive states, eventually producing ‘output’ and terminating at a final ending state. The use of the algorithm evolved to include all definite procedures for solving problems or performing tasks. The word algorism originally referred only to the rules of performing arithmetic using Hindu-Arabic numerals but evolved via European Latin translation of Al-Khwarizmi’s name into algorithm by the 18th century. The use of the word evolved to include all definite procedures for solving problems or performing tasks.

Graph-theoretic methods in various forms have proven particularly useful in linguistics, since natural language often lends itself well to discrete structure. Traditionally, syntax and compositional semantics follow tree-based structures, whose expressive power lies in the Principle of Compositionality, modeled in a hierarchical graph. More contemporary approaches such as Head-driven phrase structure grammar (HPSG) model syntactic constructions via the unification of typed feature structures,
which are directed acyclic graphs. Within lexical semantics, especially as applied to computers, modeling word meaning is easier when a given word is understood in terms of related words; semantic networks are therefore important in computational linguistics. Still other methods in phonology (e.g. Optimality Theory, which uses lattice graphs) and morphology (e.g. finite-state morphology, using finite-state transducers) are common in the analysis of language as a graph. Indeed, the usefulness of this area of Mathematics to linguistics has borne organizations such as Text Graphs, as well as various 'Net' projects, such as WordNet, VerbNet and others.

Graph theory is also used to study molecules in chemistry and physics. In condensed matter physics, the three dimensional structure of complicated simulated atomic structures can be studied quantitatively by gathering statistics on graph-theoretic properties related to the topology of the atoms. For example, Franzblau's shortest-path (SP) rings. In chemistry a graph makes a natural model for a molecule, where vertices represent atoms and edges represents bonds. This approach is especially used in computer processing of molecular structures ranging from chemical editors to database searching. In statistical physics, graphs can represent local connections between interacting parts of a system, as well as the dynamics of a physical process on such systems.

Likewise, graph theory is useful in biology and conservation efforts where a vertex can represent regions where certain species exist (or habitats) and the edges represent migration paths or movement between the regions. This information is important when looking at breeding patterns or tracking the spread of disease, parasites or how changes to the movement can affect other species. In Mathematics, graphs are useful in geometry and certain parts of Topology, e.g. Knot Theory. Algebraic graph theory has close links with group theory.

The year 1850, for the first time witnessed a study of domination in graphs, where the problem is to place the minimum number of queens on an \( n \times n \) chess board so that every square gets covered or dominated. It actually took over a decade, i.e., in 1960 researchers took up a serious and comprehensive study of the subject. For the first time
in 1962, the concepts were entitled ‘dominating set’ and ‘domination number’ by Ore. In 1977, E.J.Cockayne and S.T.Hedetniemi conducted a commendable and broad survey on the outcomes of the existing concepts of dominating sets in graphs at that time. The notation $\gamma(G)$ for the domination number of a graph, for the first time was applied by the pair and was accepted widely since then. The domination theory of graphs put forth by Ore and Berge has been the new area for the researchers, recently.

For finding these various types of dominating sets in different interval graphs we introduced the different types of algorithms.

In this thesis we focus our attention on interval graphs, circular - arc graphs and discuss certain graph theoretic aspects of these graphs.

Interval graphs have drawn the attention of many researchers for over 25 years. They have extensively studied and revealed their practical relevance for modeling problems arising in the real world. Since their appearance in the graph theory literature on interval graphs has grown quickly and now they form one of the most studied classes of graphs. In part, this is due to the fact that interval graphs have a lot of applications in very diverse areas such as database, artificial intelligence, biology, archaeology, genetics, psychology, traffic control, computer scheduling storage information retrieval and electronic circuit design etc. On the other hand interval graphs are famous in the graph theory community because of their huge number of properties that arise from their nice and simple structure.

Let $I = \{I_1, I_2, \ldots, I_n\}$ be the given interval family. Each interval $i$ in $I$ is represented by $[a_i, b_i]$, for $i = 1, 2, \ldots, n$. Here $a_i$ is called the left endpoint and $b_i$ the right endpoint of the interval $I_i$. Without loss of generality we may assume that all end points of the intervals in $I$ which are distinct between 1 and $2n$. The intervals are labelled in the increasing order of their right endpoints. Two intervals $i$ and $j$ are said to intersect each other, if they have non-empty intersection. Interval graphs play important role in numerous applications, many of which are scheduling problems. They are a subset of perfect graphs. A graph $G = (V, E)$ is called an interval graph if there is a one-to-one correspondence between $V$ and $I$ such that two vertices of $G$ are joined by an
edge in $E$ if and only if their corresponding intervals in $I$ intersect. That is, if $i = [a_i, b_i]$ and $j = [a_j, b_j]$, then $i$ and $j$ intersect means either $a_j < b_i$ or $a_i < b_j$. Let $G$ be a graph, with vertex set $V$ and edge set $E$.

The open neighbourhood set of a vertex $v \in V$ is $\text{nbd} \ (v) = \{u \in V \mid uv \in E\}$. The closed neighbourhood set of a vertex $v \in V$ is $\text{nbd}[v] = \text{nbd} \ (v) \cup \{v\}$. A vertex in a graph $G$ dominates itself and its neighbours.

Circular - arc graphs are introduced as generalization of Interval graphs. If we bend the real line into a circle, then any family of intervals of the real line is transformed into a family of arcs of the circle. Therefore, every interval graph is a circular - arc graph. But, the converse need not be true. However both these classes of graphs have received considerable attention in the literature in recent years and have been studied extensively. A circular - arc graph is the intersection graph of a set of arcs on the circle. It has one vertex for each arc in the set and an edge between every pair of vertices corresponding to arcs that intersect.

Let $A = \{A_1, A_2, \ldots, A_n\}$ be a circular - arc family on a circle. Where each $A_i$ is an arc. Without loss of generality assumes that the end points of all arcs are distinct and no arc covers the entire circle. Denote an arc $i$ that begins at $p$ and ends at the point $q$ in the clockwise direction by $(p, q)$. Define $p$ to be the head and $q$ to be the tail of the arc $i$ and now $i$ is denoted by $i = (p, q)$. Two arcs $j$ and $i$ are said to intersect each other, if they have non-empty intersection.

Let $G(V, E)$ be a graph. Let $A = \{A_1, A_2, \ldots, A_n\}$ be a family of arcs on a circle. Then $G$ is called a circular - arc graph, if there is a one-to-one correspondence between $V$ and $A$ such that two vertices in $V$ are adjacent if and only if their corresponding arcs in $A$ intersect.

The interval graphs and the circular - arc graphs are discussed above form the basis for our study in the forthcoming chapters of the thesis.
Split domination and non-split domination in graphs was introduced by Kulli in 1997. They have studied these parameters for various standard graphs and obtained the bounds for them. Q.M. Mahyoub and N.D. Soner initiate the split dominating set and split domination number in fuzzy graphs.

The bondage number $b(G)$ of a non-empty graph $G$ is the minimum cardinality among all sets of edges $E_1$ for which $\gamma(G - E_1) > \gamma(G)$. Here $\gamma(G)$ indicates the domination number of $G$. Thus, the bondage number of $G$ is the smallest number of edges whose removal will render every minimum dominating set in $G$, a non-dominating set in the resultant spanning subgraph. Since the domination number of every spanning subgraph of a non-empty graph $G$ is at least as great as $\gamma(G)$. This concept was introduced by Fink and they have studied this parameter for some standard graphs, trees and general bounds are obtained.

In chapter 1 necessarily graph theoretic preliminaries are presented.

In chapter 2 we introduce the complementary tree domination number of interval graphs. Graphs considered in this chapter are all undirected, connected and simple graphs and for $S$ throughout this chapter, for the graph $G = (V, E)$ and for $S \subseteq V$, the subgraph of $G$ induced by the vertices in $S$ is denoted by $\langle S \rangle$. For any vertex $v \in V(G)$, $N(v)$ denotes the open neighborhood of $v$ and is defined as the set of all vertices adjacent to $v$ in $G$ and $N[v]$ denotes the closed neighborhood of $v$ and is defined as $N[v] = N(v) \cup \{v\}$. A vertex of degree one is called a support. A subset $S$ of the vertex set $V$ of the graph $G = (V, E)$ is a dominating set of the graph $G$ if every vertex not in $S$ is adjacent to a vertex in $S$. The domination number of the graph $G$ denoted by $\gamma(G)$ and is the minimum cardinality of a dominating set of $G$. A dominating set $S \subseteq V$ of a graph $G$ with vertex set $V(G)$ and edge set $E(G)$ is a complementary tree domination set if the induced subgraph $\langle V - S \rangle$ is a tree. Complementary tree domination number is the minimum cardinality of a complementary tree dominating set of $G$. It is denoted by $\Upsilon_{ctd}(G)$.

In chapter 3 is to study the problem of total bondage number. We consider the bondage number $b(G)$ for an interval family corresponding to an interval graph $G$. 

which is defined as the minimum number of edges whose removal results in a new graph with larger domination number. The bondage number $b(G)$ of a non-empty graph $G$ is the minimum cardinality among all sets of edges $E_i$, for which $\gamma(G - E_i) > \gamma(G)$. Thus, the bondage number of $G$ is the smallest number of edges whose removal will render every minimum dominating set in $G$ a non-dominating set in the resultant spanning sub graph. Since the domination number of every spanning sub graph of a non-empty graph $G$ is at least as great as $\gamma(G)$, the bondage number of a non-empty graph is well defined.

A subset $S$ of $V$ is called a total dominating set if every vertex in $V$ is adjacent to some vertex in $S$. The total domination number $\gamma_t(G)$ of $G$ is the minimum cardinality taken over all total dominating sets of $G$. The total bondage number $b_t(G)$ of a non-empty graph $G$ is the minimum cardinality among all sets of edges $E_i$, for which $\gamma_t(G - e) > \gamma_t(G)$.

In chapter 4 concentrates an algorithm for finding the strong dominating set and split strong dominating set of an interval graph using an algorithm. A set $D \subseteq V$ is called dominating set if every vertex in $V - D$ is adjacent to some vertex in $D$. The domination number $\gamma$ of $G$ is the minimum cardinality of a dominating set. The domination number is well-studied parameter. We can see this from the bibliography on domination. In, Sampathkumar and Pushpa Latha have introduced the concept of strong domination in graphs. Strong domination has been studied. Kulli.V.R. introduced the concept of split and non-split domination in graphs. A dominating set $D$ is called split dominating set if the induced subgraph $< V - D >$ is disconnected. The split domination number of $\gamma_s$ of $G$ is the minimum cardinality of a split dominating set. Let $G = (V, E)$ be a graph and $u, v \in V$, then $u$ strongly dominates $v$ if (i) $uv \in E$ (ii) $\deg v \leq \deg u$. A set $D_{st} \subseteq V$ is a strong dominating set of $G$ if every vertex in $V - D_{st}$ is strongly dominated by atleast one vertex in $D_{st}$. The strong domination number $\gamma_{st}(G)$ of $G$ is the minimum cardinality of a strong dominating set. Define $NI(i) = j$, if $b_i < a_j$ and there do not exist an interval $k$ such that $b_i < a_k < a_j$. If there is no such $j$, then define $NI(i) = null$. $\text{nbd}^+(i)$ is the set of all adjacent vertices to $i$ which are greater than $i$. $\text{nbd}^- (i)$ is the set of all adjacent vertices to $i$ which are less than $i$. 
\( d^+ (i) \) is the number of adjacent vertices to \( i \) which are greater than \( i \). \( d^- (i) \) is the number of adjacent vertices to \( i \) which are less than \( i \).

In chapter 5 we bring the complementary tree domination number of circular–arc graphs. A dominating set \( S \subseteq V \) of a graph \( G \) with vertex set \( V (G) \) and edge set \( E (G) \) is a complementary tree dominating set if the induced subgraph \( < V – S > \) is a tree. Complementary tree domination number is the minimum cardinality of a complementary tree dominating set of \( G \). It is denoted by \( \gamma_{ctd} (G) \). The notion of complementary tree dominating set is due to S. Muttamai et al. Some results pertaining to the bounds of Complementary tree domination number are obtained by them.