CHAPTER - 3

INVENTORY MODELS FOR STOCK DEPENDENT DEMAND RATE
(Different Approach)

INTRODUCTION
EOQ models derived in the previous chapter use cost minimization criteria considering the stock dependent consumption rate for different functional relationships. The expression for assumed variable demand rate has been substituted in the total cost per unit time derived under the assumption of constant demand. Naturally this could not take care of stock-dependent demand rate except where the demand rate is depending on replenishment size.

Mandal and Phaujdar [72] suggested an EOQ model through profit maximization criteria considering the consumption rate depending upon the current stock level to yield the optimal solution, which is more realistic. They assumed a linear dependence of the demand rate on the current stock level.

In this chapter EOQ models are derived with some more plausible functional relationships existing between the demand rate and current stock level.

THE MODEL

Here the EOQ model is established in the case where the replenishment of the stock is instantaneous, shortages are not allowed and the demand rate depends upon the current stock level. The total profit per unit time during time T, obtained by Mandal and Phaujdar [72] is
\[ Z(S) = \frac{pS - [A + CS + C_1G(S)]}{F(S)} \]  

(3.1)

where \( p \) denotes unit selling price of the item, \( A \) the set up cost, \( C \) unit cost price of the item, \( C_1 \) unit holding cost per unit of the item and \( S \) is the highest stock level.

Also \[ G(S) = \int_0^S \frac{dq}{r(q)} \]  

(3.2)

and \[ F(S) = \int_0^S \frac{dq}{r(q)} = T \]  

(3.3)

The optimum value of \( S \) for total maximum profit per unit time is a solution of \( Z'(S) = 0 \), provided \( Z''(S) < 0 \) for that value of \( S \). Thus for optimal value \( S \), expression (3.1) implies,

\[ F(S) \left[ (p - C) - C_1G'(S) \right] = F(S) \left[ (p - C)S - A - C_1G(S) \right] \]  

(3.4)

where prime denotes derivative with respect to \( S \). Equation (3.4) is in general a non-linear equation which can be solved numerically by Newton-Raphson method, if the explicit form of \( r(q) \) is known. The optimal cycle length is given by \( F(S^*) \) where \( S^* \) is the optimal value of \( S \).

**ANALYSIS**

EOQ model has been established for the following functional relations—

(i) \( r(q) = \alpha + \beta q + \gamma q^2 \)

(ii) \( r(q) = \alpha + \frac{\beta}{q} \)

(iii) \( r(q) = \alpha e^{\beta q} \)

(iv) \( r(q) = \alpha e^{-\beta q} \)

(v) \( r(q) = \alpha q^{-\beta} \)
where $\alpha$, $\beta$ and $\gamma$ are non-negative constants.

Case (i) : $r(q) = \alpha + \beta q + \gamma q^2$

Putting the value of $r$ in equation (3.3), we get

$$F(S) = \int_{0}^{S} \frac{dq}{\alpha + \beta q + \gamma q^2}$$

$$= \frac{1}{N} \log \left( \frac{2S\gamma + \beta - N}{2S\gamma + \beta + N} \right) \left( \frac{\beta + N}{\beta - N} \right)$$

(3.5)

where $N^2 = \beta^2 - 4\alpha\gamma > 0$

Therefore

$$F'(S) = \frac{4\gamma}{(2S\gamma + \beta)^2 - N^2}$$

(3.6)

Also putting the value of $r$ in equation (3.2), we get

$$G(S) = \int_{0}^{S} q dq$$

$$= \frac{1}{2\gamma} \log \left( \frac{\alpha + \beta S + \gamma S^2}{\alpha} \right) - \frac{\beta}{2\gamma N} \log \left( \frac{2S\gamma + \beta - N}{2S\gamma + \beta + N} \right) \left( \frac{\beta + N}{\beta - N} \right)$$

(3.7)

So, $G'(S) = \frac{\beta + 2\gamma S}{2\gamma(\alpha + \beta S + \gamma S^2)} - \frac{2\beta}{(2\gamma S + \beta)^2 - N^2}$

(3.8)

Now substituting the values of $F(S)$, $F'(S)$, $G(S)$ and $G'(S)$ in equation (3.4) and simplifying the resulting expression, we get

$$A + \frac{C_1}{2\gamma} \log \left( \frac{\alpha + \beta S + \gamma S^2}{\alpha} \right)^2 + \frac{1}{4\gamma N} \log \left( \frac{1 + \frac{2S\gamma}{\beta - N}}{1 + \frac{2S\gamma}{\beta + N}} \right) \times$$

$$\left[ (p - C_1)\left( \gamma + 2\gamma S^2 - N^2 \right) - \frac{C_1(\beta + 2\gamma S)(\beta + 2\gamma S^2 - N^2)}{2\gamma(\alpha + \beta S + \gamma S^2)} \right] + 2C_1\beta = 0$$

(3.9)
It can be shown that if $\gamma = 0$, the above expression (3.9) reduces to that obtained by Mandal and Phaujdar [72].

Equation (3.9) is a transcendental equation which can be solved by Newton Raphson method. The solution of equation (3.9) which satisfies $Z''(S) < 0$ gives the optimal value of $S$.

Case (ii) : $r(q) = \alpha + \frac{\beta}{q}$

Substituting this value of $r$ in equation (3.3), we get

$$F(S) = \int_0^S \frac{q \, dq}{\alpha q + \beta}$$

$$= \frac{1}{\alpha} \left[ S - \frac{\beta}{\alpha} \log \left( 1 + \frac{\alpha}{\beta} S \right) \right]$$  \hspace{1cm} (3.10)

Therefore,

$$F'(S) = \frac{S}{(\alpha S + \beta)}$$  \hspace{1cm} (3.11)

Also from equation (3.2),

$$G(S) = \int_0^S \frac{q^2 \, dq}{\alpha q + \beta}$$

$$= \frac{S^2}{2\alpha} - \frac{\beta}{\alpha^2} S + \frac{\beta^2}{\alpha^3} \log \left( 1 + \frac{\alpha}{\beta} S \right)$$  \hspace{1cm} (3.12)

So,

$$G'(S) = \frac{S^2}{(\alpha S + \beta)}$$  \hspace{1cm} (3.13)

Substituting values of $F(S)$, $F'(S)$, $G(S)$ and $G'(S)$ from equations (3.10), (3.11), (3.12) and (3.13) respectively in equation (3.4) and simplifying this gives,

$$A + \frac{\beta}{\alpha} \left[ \frac{C_1}{\alpha} - \frac{(P - C)}{S} \right] \left[ \frac{1}{\alpha} \left( \alpha S + \beta \right) \log \left( 1 + \frac{\alpha}{\beta} S \right) - S \right] - \frac{C_1 S^2}{2\alpha} = 0$$  \hspace{1cm} (3.14)

If we take $\beta = 0$, then equation (3.14) reduces to
\[ A - \frac{C_1 S^2}{2\alpha} = 0 \]  

(3.15)

which gives
\[ S^* = \left( \frac{2A\alpha}{C_1} \right)^{1/2} \]  

(3.16)

This is the classical EOQ formula with uniform demand rate.

Equation (3.14) is a transcendental equation which can be solved by Newton-Raphson method. The solution of equation (3.14) which satisfies \( Z''(S) < 0 \) gives the optimum value of \( S \).

**Case (iii) :** \( r(q) = \alpha e^{\beta q} \)

For this value of \( r \), using equations (3.2) and (3.3), we obtain
\[ F(S) = \frac{1}{\alpha \beta} \left( 1 - e^{-\beta S} \right) \]  

and
\[ G(S) = -\frac{1}{\alpha \beta} S e^{-\beta S} - \frac{1}{\alpha \beta^2} e^{-\beta S} + \frac{1}{\alpha \beta^2} \]  

(3.17)

Differentiating equations (3.16) and (3.17) with respect to \( S \), we get
\[ F'(S) = \frac{1}{\alpha} e^{-\beta S} \]  

(3.18)

and
\[ G'(S) = \frac{S}{\alpha} e^{-\beta S} \]  

(3.19)

Putting these values of \( F(S) \), \( G(S) \), \( F'(S) \) and \( G'(S) \) from equations (3.16), (3.17), (3.18) and (3.19) respectively in condition (3.4) for optimality, this gives,
\[ A + \frac{C_1}{\alpha \beta^2} \left( 1 - \beta S - e^{-\beta S} \right) - (p - C) \left[ S + \frac{(1 - e^{\beta S})}{\beta} \right] = 0 \]  

(3.20)

which is a transcendental equation and can be solved using Newton-Raphson method. This solution of equation (3.20) satisfying \( Z'' < 0 \) gives the optimal value of \( S \).
**Case (iv)** : \( r(q) = \alpha e^{-\beta q} \)

With this value of \( r \), expressions for \( F(S) \), \( F'(S) \), \( G(S) \) and \( G'(S) \) are as follows:

\[
F(S) = \frac{1}{\alpha \beta} \left( e^{\beta S} - 1 \right) 
\]

(3.21)

\[
F'(S) = \frac{1}{\alpha} e^{\beta S} 
\]

(3.22)

\[
G(S) = \frac{1}{\alpha \beta} S e^{\beta S} - \frac{1}{\alpha \beta^2} e^{\beta S} + \frac{1}{\alpha \beta^2} 
\]

(3.23)

and \( G'(S) = \frac{S}{\alpha} e^{\beta S} \)

(3.24)

Substituting these values of \( F(S), F'(S), G(S) \) and \( G'(S) \) in equation (3.4) for optimality of \( S \), we obtain,

\[
A + \frac{C_1}{\alpha \beta^2} \left( 1 + \beta S - e^{\beta S} \right) - (p - C) \left[ S - \frac{(1 - e^{-\beta S})}{\beta} \right] = 0 
\]

(3.25)

The solution \( S \) of transcendental equation (3.25), which satisfies \( Z''(S) < 0 \) gives the optimum value.

**Case (v)** : \( r(q) = \alpha q^{-\beta} \)

Using this value equations (3.2) and (3.3) imply,

\[
F(S) = \frac{1}{\alpha (1 + \beta)} S^{1 + \beta} 
\]

(3.26)

and \( G(S) = \frac{1}{\alpha (2 + \beta)} S^2 + \beta \)

(3.27)

Differentiating above equations (3.26) and (3.27), we have,

\[
F'(S) = \frac{S}{\alpha} 
\]

(3.28)

and \( G'(S) = \frac{S^{1 + \beta}}{\alpha} \)

(3.29)

Optimal condition (3.4) using equations (3.26), (3.27), (3.28) and (3.29)
reduces to

\[ A - \frac{S}{(1 + \beta)} \left[ (p - C) + \frac{C_1 S^{1 + \beta}}{\alpha(2 + \beta)} \right] = 0 \] (3.30)

Which is a non-linear equation and can be solved for \( S \) by Newton-Raphson method. It can be shown that \( Z'' < 0 \) for this value of \( S \). Thus, solution of equation (3.30) gives the optimum value of \( S \).

**DISCUSSION**

Here EOQ models are derived in which production of items in the system is instantaneous and shortages do not occur. Profit maximization criteria is employed to yield the maximum profit per unit time of the system. Some functional forms of demand rate are taken in order to formulate the model.

In some cases, when conditions on demand rate are used, the model reduces to the corresponding already established inventory model for that demand rate. The model presented here can be further extended for finite rate of replenishment and (or) allowing shortages.