CHAPTER 8

AN EOQ MODEL WITH EXponential DEMAND AND SHORTAGES FOR VARIABLE DETERIORATING ITEMS

INTRODUCTION

In formulating inventory models, two facets of the problem have been of growing interest. One being the deterioration of items and the other variation in the demand rate. Among researchers considering inventory models in deteriorating items, Shah and Jaiswal [91] considered the rate of deterioration to be uniform, Covert and Philip [16] produced an EOQ model for items with variable rate of deterioration. Misra [67] used a two parameter Weibull distribution to fit the deterioration rate.

The present chapter deals with the generalisation of EOQ model with exponential demand for constantly deteriorating items derived in previous chapter. In the EOQ model presented here, demand rate is taken in exponential form and a special form of Weibull density function is chosen in order to make the problem mathematically tractable. Deterministic as well as probabilistic cases of demands are considered allowing shortages.

DERIVATION OF THE MODEL

DETERMINISTIC DEMAND CASE

It is assumed that replenishment size is constant and production is instantaneous during the prescribed time period $T$ of each cycle. Lead time is zero, shortages are permitted and completely accumulated. Demand rate is $\frac{d}{(e-1)T} e^{t/T}$ at any time $t$. A variable fraction $\theta(t)$ of the on hand inventory deteriorates per unit time and is of the form $\theta = \theta_0 t$, where $\theta_0$ is a constant with the condition $0 < \theta_0 < 1$, $t > 0$. 
Let $Q$ be the quantity produced or purchased at the beginning of each production cycle and after satisfying back orders, let an amount $S$ remains as a initial inventory. Let $d$ be the demand during the time period $T$. Inventory level gradually reduces and finally becomes zero at $t = t_1 < T$, then shortages occur during $(t_1, T)$ and are fully backlogged.

If $q(t)$ be the current stock level at any time $t$, their differential equations which the on hand inventory $q(t)$ must satisfy in two different parts of the cycle time $T$ are as following:

$$\frac{d}{dt} q(t) + \theta_0 t q(t) = -\frac{d}{(e - 1)T} e^{t/T}, \quad 0 \leq t \leq t_1 \quad (8.1)$$

$$\frac{d}{dt} q(t) = -\frac{d}{(e - 1)T} e^{t/T}, \quad t_1 < t \leq T \quad (8.2)$$

Solution to differential equation (8.1) follows as

$$q(t) e^{\theta_0 t^2/2} = -\frac{d}{(e - 1)T} \int e^{\theta_0 t^2/2} e^{t/T} dt + B,$$

where $e^{\theta_0 t^2/2}$ is integrating factor and $B$ is constant of integration.

$$= -\frac{d}{(e - 1)T} \int (e^{t/T} + \frac{\theta_0}{2} t^2 e^{t/T}) dt + B$$

neglecting higher order terms of $\theta$ as $0 < \theta < 1$

$$= -\frac{d}{(e - 1)T} \left[ t e^{t/T} + \frac{\theta_0}{2} \left[ T e^{t/T} - 2T \int t e^{t/T} dt \right] \right] + B$$

$$= -\frac{d}{(e - 1)T} \left[ T e^{t/T} + \frac{\theta_0}{2} T t^2 e^{t/T} - \theta_0 T^2 t e^{t/T} + \theta_0 T^3 e^{t/T} \right] + B$$

$$= -\frac{d}{(e - 1)T} \left[ \frac{\theta_0}{2} t^2 - \theta_0 T t + 1 + \theta_0 T^2 \right] e^{t/T} + B$$

Since at $t = 0$, $q = S$, we get

$$B = S + \frac{d}{(e - 1)T} (1 + \theta_0 T^2)$$

Putting value of $B$ above, solution of differential equation (8.1) becomes.
\[ \begin{align*}
&= t_1 \int_s q(t_1) \, dt \\
&= \left. \frac{d}{(e-1)} t e^t \left[ - \frac{\theta_0 t^3}{6} + \frac{\theta_0 t^2}{2} - \theta_0 T t + t + \theta_0 T^2 t \right] \right|_0^{t_1} \\
&\quad - \left. (-\theta_0 T t + 1 + \theta_0 T^2) \, T e^T + \theta_0 T^3 e^T \right|_0^{t_1} \\
&= \left. \frac{d}{(e-1)} t e^t \left[ - \frac{\theta_0 t^3}{6} + \frac{\theta_0 t^2}{2} - \theta_0 T t + t + \theta_0 T^2 t \right] \right|_0^{t_1} \\
&\quad - \left. (-\theta_0 T t + 1 + \theta_0 T^2) \, T e^T + \theta_0 T^3 e^T \right|_0^{t_1} \\
&= \left. \frac{d}{(e-1)} t e^t \left[ - \frac{\theta_0 t^3}{3} - \theta_0 T t^2 + (1 + 2\theta_0 T^2) t_1 - T(1 + 2\theta_0 T^2) \right] + T(1 + 2\theta_0 T^2) \right|_0^{t_1} \\
&= (8.8)
\end{align*} \]

Number of units running in shortages

\[ 
= \int_s^t q(t_1) \, dt \\
= \left. \frac{d}{(e-1)} t e^t \left[ T - T e^T \right] \right|_0^{t_1} \\
= \left. \frac{d}{(e-1)} (2T e^T t_1 - t_1 t e^T - e T) \right|_0^{t_1} \\
= (8.9)
\]

Finally, average total cost per unit time is given by

\[ K(t_1) = \frac{CD}{T} + \frac{C_1}{T} \int_0^{t_1} q(t) \, dt - \frac{C_2}{T} \int_{t_1}^T q(t) \, dt \]  

where \( C \) is the cost of each item, \( C_1 \) and \( C_2 \) are holding and shortage costs per unit per unit time respectively.
Substituting the values from equations (8.6), (8.8) and (8.9) cost equation (8.10) reduces to,

\[
K(t_1) = \frac{d C}{(e - 1) T} \left[ e^{t_1} T \theta_0 \left( t_1^2 - T t_1 + T^2 \right) - \theta_0 T^2 \right] \\
+ \frac{d C}{(e - 1) T} \left[ e^{t_1} T - \frac{\theta_0 t_1^3}{3} - \theta_0 T t_1^2 + \left( 1 + 2\theta_0 T^2 \right) t_1 - T \left( 1 + 2\theta_0 T^2 \right) \right] \\
+ \frac{d C}{(e - 1) T} \left[ 2T e^{t_1} T - t_1 e^{t_1} T - e T \right] \\
\]  

(8.11)

Now the total cost will be minimum if

\[ \frac{d}{dt_1} K(t_1) = 0 \]  

which gives,

\[
\frac{d C}{(e - 1) T} \left[ e^{t_1} T - \frac{\theta_0 t_1^3}{3} - \theta_0 T t_1^2 + \left( 1 + 2\theta_0 T^2 \right) t_1 - T \left( 1 + 2\theta_0 T^2 \right) \right] \\
+ e^{t_1} T \theta_0 t_1^2 - 2\theta_0 T t_1 + 1 + 2\theta_0 T^2] - \frac{d C}{e} \left[ 2e^{t_1} T - e^{t_1} T - \frac{t_1}{T} e^{t_1} T \right] = 0
\]

or

\[
\frac{d C}{(e - 1) T^2} \left[ e^{t_1} T + 2 \frac{d C}{(e - 1) T^2} \left( \theta_0 t_1^3 + \frac{\theta_0}{2} C t_1^2 + \left( C_1 + C_2 \right) t_1 - T C_2 \right) \right] = 0
\]

or

\[
\frac{d}{(e - 1) T^2} e^{t_1} T + \frac{d C}{(e - 1) T^2} \left[ -\frac{\theta_0}{3} C t_1^3 + \frac{\theta_0}{2} C t_1^2 + \left( C_1 + C_2 \right) t_1 - T C_2 \right] = 0
\]

or

\[
\frac{\theta_0}{3} C t_1^3 + \frac{\theta_0}{2} C t_1^2 + \left( C_1 + C_2 \right) t_1 - T C_2 = 0
\]

(8.12)

Equation (8.12) is a cubic equation in \( t_1 \) and can be solved by Cardon's method to get one positive real root \( t_1 = t_1^* \). The total cost \( K(t_1) \) will be minimum provided

\[
\frac{d^2 K(t_1)}{dt_1^2} > 0 \text{ at } t_1 = t_1^*, \text{ which is verified.}
\]
Therefore, minimum average cost is \( K(t_1^*) \) and other optimum quantities are

\[
S^* = d \left( e - 1 \right) \left[ e^{t_1^* T} \left( \frac{H_0^* T^2}{2} - \frac{H_0^* T}{2} t_1^* + 1 + \frac{H_0^* T^2}{2} \right) - (1 + \frac{H_0^* T^2}{2}) \right],
\]

and

\[
Q^* = d + \left( e - 1 \right) \left[ e^{t_1^* T} \left( \frac{t_1^* T}{2} - T t_1^* + T^2 \right) - T^2 \right]
\]

(8.13)

(8.14)

In particular, if \( H = 0 \), i.e. there is no deterioration, the optimal condition (8.12) implies

\[
t_1^* = \frac{C_0}{C_1 + C_2} T
\]

(8.15)

and other optimal quantities reduce to

\[
S^* = \frac{d}{(e - 1)} \left[ e^{t_1^* T} (e - 1) \right]
\]

(8.16)

and

\[
Q^* = d
\]

(8.17)

Expressions (8.15), (8.16) and (8.17) are the same given by EOQ models for non-deteriorating items, derived in seventh chapter.

PROBABILISTIC DEMAND CASE

Let demand during the period \((0, T)\) be a random variable \( X \) with probability density function \( f(x) \) \((0 < x < d)\) and demand follows exponential pattern with demand rate \( \frac{x}{(e - 1) T} e^{t T} \). Two cases exist

Case I: When shortage does not occur.

Let \( q_{1x}(t) \) be inventory level of the system at any time \( t \), then system can be described mathematically as,

\[
\frac{d}{dt} q_{1x}(t) + \int_0^t q_{1x}(t) = -\frac{x}{(e - 1) T} e^{t T}; \quad 0 \leq t \leq T
\]

(8.18)

With solution
\[ q_{1x}(t) = \left[ -\frac{x}{(e-1)} e^{t^2} \left( 1 + \frac{\theta_0 T^2}{2} - \theta_0 T t + \theta_0 T^2 \right) + S \right. \\
\left. + \frac{x}{(e-1)} (1 + \theta_0 T^2) e^{-\theta_0 T^2/2} \right] \] 

(8.19)

where \( S \) is the expected stock in hand in the beginning after satisfying the back orders.

Since there is no shortage, we have

\[ q_{1x}(T) \geq 0 \]

or

\[ \left[ -\frac{x e}{(e-1)} \left( 1 + \frac{\theta_0 T^2}{2} \right) + S + \frac{x}{(e-1)} (1 + \theta_0 T^2) \right] \geq 0 \]

or

\[ S \geq x \left[ 1 + \frac{\theta_0 T^2(e-2)}{2(e-1)} \right] \]

or

\[ S \geq \frac{x T}{2 \theta} \left( \theta_0 T^2(e-2) + 2(e-1) \right) \]

or

\[ x \leq \frac{S}{\left[ 1 + \frac{\theta_0 T^2(e-2)}{2(e-1)} \right]} = S_1 \text{ (say)} \] 

(8.20)

The average number of items carried in inventory per unit time

\[ H_1(x) = \frac{1}{T} \int_0^T q_{1x}(t) \, dt, \quad x \leq S_1 \]

\[ = \frac{x}{(e-1)} \left[ -\frac{2}{3} e \theta_0 T^2 + 1 + 2 \theta_0 T^2 \right] \] 

(8.21)

The average number of items deteriorated per unit time

\[ D_1(x) = \frac{1}{T} \left[ S - x - q_{1x}(T) \right] \]

\[ = \frac{\theta_0 T}{2} \left[ S - \frac{x}{(e-1)} \right] \] 

(8.22)
and the average shortage per unit time
\[ G_1(x) = 0 \] \hspace{1cm} (8.23)

**Case II :** When shortages occur.

In this case, differential equations describing the system are similar to equations (8.1) and (8.2) when \( d \) is replaced by \( x \) with their solutions as

\[
q_{2x}(t) = \left[ -\frac{x}{(e-1)} e^{1/T} \left( 1 + \frac{\theta_0 T^2}{2} - \theta_0 T t + \theta_0 T^2 \right) + S + \frac{x}{(e-1)} (1 + \theta_0 T^2) \right] \\
\times e^{-\theta_0 T/2}, \quad 0 \leq t \leq t_1
\] \hspace{1cm} (8.24)

and
\[
q_{2x}(t) = \frac{x}{(e-1)} (e^{1/T} - e^t) ; \quad t_1 < t \leq T
\] \hspace{1cm} (8.25)

Since shortage occur, we must have

\[
q_{2x}(T) < 0
\]
or
\[
x > S_1
\]

where value of \( S_1 \) is given by expression (8.20). Also at \( t = t_1 \), \( q_{2x} = 0 \), which gives,

\[-\frac{x}{(e-1)} e^{t_1/T} \left( 1 + \frac{\theta_0 t_1^2}{2} - \theta_0 T t_1 + \theta_0 T^2 \right) + \frac{x}{(e-1)} (1 + \theta_0 T^2) + S = 0 \]

or
\[
e^{t_1/T} \left( 1 + \frac{\theta_0 t_1^2}{2} - \theta_0 T t_1 + \theta_0 T^2 \right) = \frac{S(e-1)}{x} + (1 + \theta_0 T^2)
\]

Taking log on both the sides, we have

\[
\frac{t_1}{T} + \log \left[ 1 + \theta_0 \left( T^2 + \frac{t_1^2}{2} - T t_1 \right) \right] = \log \left[ \frac{S(e-1)}{x} + (1 + \theta_0 T^2) \right]
\]
or \[
\frac{t_1}{T} + \theta_0^2 T^2 + \frac{\theta_0^2}{2} - \theta_0 T t_1 = \log \left\{ 1 + \theta_0 T^2 + \frac{S (e - 1)}{x} \right\} \quad \text{as,} \quad \theta << 1
\]

or \[
\frac{\theta_0^2}{2} + \left( \frac{1}{T} - \theta_0 T \right) t_1 + \theta_0 T^2 - \log \left\{ 1 + \theta_0 T^2 + \frac{S (e - 1)}{x} \right\} = 0 \quad (8.26)
\]

which is a quadratic equation in \( t_1 \) and can be solved to get one positive root of \( t_1 \).

Now average number of items in inventory carried per unit time,

\[
H_2(x) = \frac{1}{T} \int_0^T q_{2x}(t) \, dt
\]

\[
= \frac{x}{(e - 1)} \left[ e^{t_1/T} \left( \frac{\theta_0 T^3}{3} T_0 T t_1^2 + (1 + 2 \theta_0 T^2) t_1 - T (1 + 2 \theta_0 T^2) \right) \right] + T (1 + 2 \theta_0 T^2)
\]

(8.27)

Average number of units deteriorating per unit time

\[
D_2(x) = \frac{1}{T} \left[ S - \frac{x}{(e - 1) T} \int_0^T e^{t / T} \, dt \right]
\]

\[
= \frac{1}{T} \left[ S - \frac{x}{(e - 1) (e^{t_1 / T} - 1)} \right] \quad (8.28)
\]

Also average shortage per unit time

\[
G_2(x) = \frac{1}{T} \int_{t_1}^T q_{2x}(t) \, dt, \quad x > S_i
\]

\[
= \frac{x}{(e - 1) T} \left[ 2T e^{t_1 / T} - t_1 e^{t_1 / T} - e^T \right] \quad (8.29)
\]

Therefore expected total cost per unit time of the system...
\[ K(t_1, S) = C \left[ \int_0^{S_1} D_1(x) f(x) \, dx + \int_{S_1}^\infty D_2(x) f(x) \, dx \right] + \\
C_1 \left[ \int_0^{S_1} H_1(x) f(x) \, dx + \int_{S_1}^\infty H_2(x) f(x) \, dx \right] \\
- C_2 \left[ \int_0^{S_1} G_1(x) f(x) \, dx + \int_{S_1}^\infty G_2(x) f(x) \, dx \right] \] (8.30)

where \( C \) is cost of unit item and \( C_1, C_2 \) are holding and shortage costs per unit per unit time respectively.

Substituting values of \( D_1(x), D_2(x), H_1(x), H_2(x), G_1(x) \) and \( G_2(x) \) and using value of \( t_1 \) obtained from equation (8.26) in equation (8.30), the total expected cost \( K(S) \) can be evaluated if probability density function \( f(x) \) is known.

The necessary condition for the total cost to be minimum is

\[ \frac{d}{dS} K(S) = 0 \] (8.31)

The total cost will be minimum is

\[ \frac{d^2}{dS^2} K(S) > 0 \] at \( S = S^* \) obtained from equation (8.31).

It can be verified that if we take \( \theta \to 0 \), i.e. there be no deterioration, all the expressions of this model reduce to the corresponding expressions of probabilistic demand model without deterioration derived in chapter seven.

**DISCUSSION**

In EOQ model presented in this chapter demand is a exponential function of time \( t \) and deterioration is variable, i.e. \( \theta = \theta_0 t \), where \( \theta_0 \) is a constant with \( 0 < \theta < 1, t > 0 \). Production rate is infinite during the cycle period \( T \) and shortages are incorporated. The model is also discussed in case of probabilistic demand. Total average cost and total expected cost respectively have been obtained.

The above inventory model can further be generalized by taking finite rate of replenishments.