Chapter II

Effect of Windows in Spectral Analysis
CHAPTER II
EFFECT OF WINDOWS IN SPECTRAL ANALYSIS

2.1 INTRODUCTION

This chapter pays much attention to spectral analysis to windowed signals and effects of different windows and their wide applicability in signal processing. The structure of a window in terms of its spectral behavior of its side-lobe levels and main-lobe width are described in this chapter. In general, a good window is one that requires a small increase in main-lobe width in order to achieve a specific side-lobe level. System considerations may require different variants of good windows which include windows that exhibit small scallop loss, and small integrated side-lobe level. Finally the MST radar spectral analysis techniques and estimation of different parameters of received are discussed.

Windowing is the process of taking a small subset of a larger data/signal set for processing and analysis. By necessity, any time or spatial signal observed, collected, or processed must have bounded support. Similarly, any time or spatial signal which approximates, design, and synthesize must also have bounded support. Support is the range or width of the independent variable (time, distance, or frequency) over which the dependent variable, say the signal, is nonzero. This finite support can be defined over multiple dimensions, extended over a line, a plane, or a volume. Windows can be continuous functions or discrete sequences defined over their appropriate finite supports [64, 65]. At the simplest level, a window can be considered a multiplicative operator that turns on the signal within the finite support and turns it off outside that same support. This operator affects the signal's Fourier Transform in a number of undesired ways; the most significant of which is undesired out-of-band side-lobe levels. The size and order of the discontinuities exhibited by the signal governs the level and rate of attenuation of these spectral side-lobes. Other negative effects include spectral smearing and in-band ripple. The design and application of windows is directed to minimize or control the undesired effects of in-
band ripple, out-of-band side-lobes, and spectral smearing. Examples of the application of windows to control finite aperture effects can be found in numerous disciplines [66]. Some of the applications are as follows:

i. *Finite duration Impulse Response (FIR) Filter Design:*

Windows applied to prototype filters, Impulse Response to control transition bandwidth, in-band and out-of-band side lobe levels.

ii. *Spectrum Analysis, Transforms of Sliding, Over Lapped, Windowed Data:*

Windows applied to observe time series to control Variance of Spectral Estimate while suppressing Spectral Leakage [54].

iii. *Power Spectra as Transform of Windowed Correlation Functions:*

Windows applied to sample correlation function to suppress segments of the sample correlation function exhibiting high bias and variance.

iv. *Non Stationary Spectra and Model Estimates:*

Windows applied to delayed and overlapped collected time series to localize time and spectral features (model parameters) of non-stationary signals.

v. *Modulation Spectral Mask Control:*

Design of modulation envelope to control spectral side-lobe behavior.

vi. *Synthetic Aperture RADAR (SAR):*

Windows applied to spatial series to control Antenna side-lobes

vii. *Phased Array Antenna Shading Function:*

Windows are applied to spatial function to control Antenna side-lobes.

viii. *Photolithography Aphorizing Function:*

Smooth Transmission Function applied to optical aperture to control diffraction pattern side - lobes. For convenience and consistency, the window is considered as being applied to a time domain signal. The window can, of
course, be applied to any function with the same intent and goal. The common theme of these applications is control of envelope smoothness in the time domain to obtain desired properties in the frequency domain.

2.2 POWER SPECTRUM ESTIMATION METHODS

The power spectrum of a signal is an average quantity which tells us about the power of various frequencies within that signal. The need for power spectrum estimation arises in a variety of contexts, including the measurement of noise spectra for the design of optimal linear filters, the detection of narrow-band signals in wide-band noise and the estimation of parameters of a linear system by using noise excitation.

Spectral estimation in general is the procedure of estimating the power spectral density of a random signal based on finite number of observations. The accuracy of the estimated spectrum is strictly related to the number of observations. The more observations we have, the more accurate will be the estimation of the spectrum. Another important consideration regarding the spectral estimation is that most often the observations of the signal are corrupted by noise.

The power spectrum of a Wide Stationery process is the Fourier Transform [67, 68] of autocorrelation of the sequence

\[ P(\omega) = \sum_{k=-\infty}^{\infty} r(k)e^{-j\omega k} \quad \ldots \quad (2.1) \]

Therefore, spectrum estimation is, in one sense, an autocorrelation estimation problem. The autocorrelation may be determined with the time-average.

\[ r(k) = \lim_{N \to \infty} \sum_{n=-N}^{N} h(n)h(n+k) \quad \ldots \quad (2.2) \]

However, if \( h(n) \) is only available for a finite interval (say, \([-N, N])\), the autocorrelation can be estimated, with a finite sum

\[ r(k) = \frac{1}{N} \sum_{n=-N}^{N} h(n)h(n+k) \quad \ldots \quad (2.3) \]
Taking DTFT (actually, DFT) of the autocorrelation estimate leads to an estimate of the power spectrum which is called as the periodogram [74]

\[ P(\omega) = \sum_{-N}^{N} r(k)e^{-j\omega k} \] ..................................(2.4)

It would be more convenient to express the periodogram in terms of the process itself rather than its autocorrelation. Let \( h_N(n) \) be the finite length \( N \) signal such that

\[ h_N(n) = \begin{cases} h(n) & -N \leq n \leq N \\ 0 & \text{otherwise} \end{cases} \] ..................................(2.5)

Therefore, \( h_N(n) \) is the product of \( x(n) \) and a rectangular window \( W_n(n) \)

\[ h_N(n) = W_n(n)x(n) \] ..................................(2.6)

Then, the estimated autocorrelation can be rewritten as

\[ r(k) = \frac{1}{N} \sum_{n=-N}^{N} h_N(n+k)h_N^*(n) = \frac{1}{N} h_N(k)h_N^*(-k) \] ..................................(2.7)

Where * is the convolution operator.

Taking the Fourier Transform and using the convolution theorem, the Power spectrum is

\[ P(\omega) = \frac{1}{N} |H_N(\omega)|^2 = \frac{1}{N} |H(\omega)|^2 \] ..................................(2.8)

Where \( H_N(\omega) = \sum_{n=-N}^{N} h_N(n) e^{-j\omega n} \) is the DTFT of the \( N \)-point data sequence \( h_N(n) \).

Therefore, the power spectrum is proportional to the squared magnitude of DFT of \( h_N(n) \).
2.3 DIFFERENT WINDOW FUNCTIONS

In signal processing a window function is a mathematical function that is zero-valued outside of some chosen interval. For instance, a function that is constant inside the interval and zero elsewhere is called a rectangular window, which describes the shape of its graphical representation. When another function or waveform/data sequence are multiplied by a window function, the product is also zero-valued outside the interval: all that is left is the part where they overlap; the "view through the window".

2.3.1 Fixed windows

Broadly windows can be classified as fixed windows and variable windows. In fixed windows the length of window only can be varied and the shape parameter is fixed [61, 68].

2.3.1.1 Rectangular window

\[ w(n) = 1 \quad \text{for} \quad -\frac{N}{2} \leq n \leq \frac{N}{2} \]  \quad ..........(2.9)

The rectangular window (sometimes known as the Boxcar or Dirichlet window) represented by equation (2.9) is the simplest window, equivalent to replacing all but N values of a data sequence by zeros, making it appear as though the waveform suddenly turns on and off. Figure 2.1 represents rectangular and its Fourier Transform. The transform of this window is seen to be the Dirichlet Kernel, which exhibits a DFT main lobe $2\pi/N$ and first sidelobe level approximately 13 dB down from the mainlobe peak. The sidelobe fall at 6 dB per octave, which is expected for a function with a discontinuity. This transform exhibit oscillatory behavior and need to be suppressed. Other windows are designed to moderate these sudden changes because discontinuities have undesirable effects on the discrete-time Fourier transform (DTFT) and/or the algorithms that produce samples of the DTFT.

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2.3.1.2 Triangular window

\[ w(n) = \begin{array}{ll}
1 & n = 0,1, \ldots, \frac{N}{2} \\
\frac{n}{N/2} & \frac{N}{2} < n \leq N \\
0 & N < n 
\end{array} \quad \ldots \ldots \quad (2.10a) \]

(or)

\[ w(n) = \begin{cases} 
\frac{n}{N/2} & n = 0,1, \ldots, \frac{N}{2} \\
0 & \frac{N}{2} < n \\
0 & n = N, \ldots, N-1 
\end{cases} \quad \ldots \ldots \quad (2.10b) \]

The triangular for finite Fourier transform is defined in equation (2.10) and is shown in Figure 2.2. This window can be seen as the convolution of two half-sized rectangular windows (for N even), giving it a main lobe width of twice the width of a regular rectangular window. The nearest lobe is -26 dB down from the main lobe. The sidelobes fall off at -12 dB per octave, reflecting discontinuity of the window. Convoluting two rectangle windows each of N/2 points will result in a triangle of N+1 points when zero ends are taken. The triangular window is the simplest window which exhibits non-negative transform.
2.3.1.3 Hann (Hanning) window

Hanning window is defined as

\[ w(n) = a - \beta \cos \left( \frac{2\pi n}{N} \right) \quad n=0,1,2,...,N-1 \quad \ldots\ldots(2.1a) \]

with \( a=0.5 \) and \( \beta=-0.5 \), or

\[ w(n) = 0.5 \left( 1 + \cos \left( \frac{2\pi n}{N} \right) \right) \quad -\frac{N}{2} \leq n \leq \frac{N}{2} \quad \ldots\ldots(2.11b) \]

The Hanning window is defined by equation (2.11) and is shown in figure 2.3. The ends of the cosine just touch zero, so the side-lobes roll off at about 18 dB per octave. The Hann and Hamming windows, both of which are in the family known as "raised cosine" or "generalized Hamming" windows, are respectively named after Julius von Hann and Richard Hamming. This window is commonly called a "Hanning Window". The DFT of Hanning window is sampled at multiples of \( 2\pi/N \), which are the locations of the zeros of the central Dirichlet Kernel.
2.1.1.4 Hamming window

The "raised cosine" with these particular coefficients was proposed by Richard W. Hamming. This can be thought of modified Hanning window. The window is optimized to minimize the maximum (nearest) side lobe, giving it a height of about one-fifth that of the Hann window, a raised cosine with simpler coefficients. So better sidelobe cancellation does result in and first sidelobe level is less than -42 dB. The Hamming window is represented by equation (2.12) and is shown in Figure 2.4.

\[ w(n) = \alpha + \beta \cos \left( \frac{2\pi n}{N} \right) \quad \frac{N}{2} \leq n \leq \frac{N}{2} \quad \ldots \ldots (2.12a) \]

with \( \alpha = 0.54 \), \( \beta = 1 - \alpha = 0.46 \), or

\[ w(n) = 0.54 - 0.46 \cos \left( \frac{2\pi n}{N} \right) \text{ for } n = 0, 1, 2, \ldots, N-1 \quad \ldots \ldots (2.12b) \]

instead of both constants being equal to 1/2 in the Hann window, the constants are approximations of values \( \alpha = 0.54 \) and \( \beta = 0.46 \), which cancel the first sidelobe of the
Hann window by placing a zero at frequency $2.5 \left( \frac{2\pi}{N} \right)$. Approximation of the constants to two decimal places substantially lowers the level of sidelobes, to a nearly equiripple condition.

![Hamming window and its Fourier transform](image)

**Figure 2.4**: Hamming window and its Fourier transform

### 2.3.2 Variable windows

In variable windows the length as well as the shape of the window can be varied so that leakage can be minimized for optimized main lobe width. Some of the important variable windows are:

- Dolph-Chebyshev window,
- Kaiser window,
- Gaussian window and
- Tukey window.

#### 2.3.2.1 Dolph-Chebyshev window

The solution to the minimum main lobe width for a given side lobe level is the Dolph-Chebyshev window. The two parameters are length of the window (N) and side lobe level (α).
\[
W(k) = (-1)^N \frac{\cos[N \cos^{-1}(\frac{\beta \cos(\frac{k}{N})}{N})]}{\cosh(N \cosh^{-1}(\beta))} \quad -\frac{N}{2} \leq k \leq \frac{N}{2} \quad \ldots \ldots (2.13)
\]

Where

\[\beta = \cosh\left(\frac{1}{N} \cosh^{-1}(10^6)\right)\]

\[\alpha = 20 \log(10^6)\]

\(\alpha\) is the side lobe level in dB

and

\[
\cos^{-1}(x) = \begin{cases} 
\frac{\pi}{2} - \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) & \text{for } |x| \leq 1.0 \\
\ln(x + \sqrt{x^2 - 1.0}) & \text{for } |x| \geq 1.0
\end{cases}
\]

More details of this window are presented in Chapter 3.

2.3.2.2 Kaiser window

The Kaiser window is a one-parameter family of window functions used for digital signal processing and optimizes time-bandwidth product [70]. It is defined by the formula.

\[
w(n) = I_0\left(\beta \sqrt{1 - \left(\frac{n}{N/2}\right)^2}\right) \quad -\frac{N}{2} \leq n \leq \frac{N}{2}
\]

where

\[
I_0(x) = \sum_{k=0}^{\infty} \frac{(x/2)^k}{k!}
\]

\[
\ldots \ldots (2.14)
\]

where

- \(I_0\) is the zeroth order modified Bessel function of the first kind.
- \(\beta\) is an arbitrary real number that determines the shape of the window.
In the frequency domain, it determines the trade-off between main-lobe width and side lobe level, which is a central decision in window design.

More details of this window are presented in Chapter-4.

2.3.2.3 Gaussian window

The frequency response of a Gaussian is also a Gaussian (it is an eigen-function of the Fourier Transform). Since the Gaussian function extends to infinity, it must either be truncated at the ends of the window, or itself windowed with another zero-ended window. Since the log of a Gaussian produces a parabola, this can be used for exact quadratic interpolation in frequency estimation

\[ w(n) = \exp \left[ -\frac{1}{2} \left( \gamma \frac{n}{N/2} \right)^2 \right] \] ............(2.15)

Where \(-\frac{N}{2} \leq n \leq \frac{N}{2}\), \(\gamma \geq 2\) and the window length is \(L=N+1\)

More details of this window are presented in Chapter-5.

2.3.2.4 Tukey window

\[ w(\alpha) = \begin{cases} 1.0, & 0 < |\alpha| < e^{-\frac{\alpha}{2}} \\ 0.5 \left[ 1.0 + \cos \left( \frac{\alpha - \frac{N}{2}}{\pi(1 - \alpha)} \right) \right], & \frac{N}{2} < |\alpha| < 2 \frac{N}{2} \end{cases} \] ............(2.16)

The Tukey window, also known as the tapered cosine window, can be regarded as a cosine lobe of width \(aN / 2\) that is convolved with a rectangle window of width \((1 - a / 2)N\). At \(a=0\) it becomes rectangular, and at \(a=1\) it becomes a Hann window.

More details of this window are discussed in Chapter-6.
2.4 EFFECT OF WINDOWS IN SPECTRUM ANALYSIS

A signal can be described in different coordinate systems and there is an engineering value in examining the signal in an basis system. One basis system is the set of complex exponentials and the attraction of this basis set is that complex exponentials are the eigen-functions and eigen-series of linear time invariant (LTI) differential and difference operators respectively. To put it in simplest form, it means that when a sine wave is applied to an LTI filter the steady state system response is a scaled version of the same sine wave. The system can only affect the complex amplitude (magnitude and phase) of the sine wave but can never change its frequency. Consequently complex sinusoids have become a standard tool to probe and describe LTI systems. The process of describing a signal as a summation of scaled sinusoids is standard Fourier transform analysis. The Fourier Transform and Fourier Series, shown on the left and right hand side of equation (2.17), permits to describe signals equally well in both the time domain and the frequency domain for continuous and discrete signals [71, 72].

\[
\begin{align*}
H(\omega) &= \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt \\
h(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{j\omega t} d\omega \\
H(\theta) &= \sum_{-\infty}^{\infty} h(n) e^{-j\theta n} \\
h(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\theta) e^{j\theta n} d\theta
\end{align*}
\]

Since the complex exponentials have infinite support, the limits of integration in the forward transform (time- to- frequency) are from minus to plus infinity. All signals of engineering interest have finite support, hence the limits of integration and summation of the Fourier transform may take finite form by defining finite support window to limit the signal to the appropriate finite support interval as given in equation (2.18)
\[ H_W(\omega) = \int_{t=-\infty}^{t=+\infty} w(t) h(t) e^{-jwt} dt \]

\[ h(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=+\infty} H_W(\omega) e^{j\omega t} d\omega \]

\[ H_W(\omega) = \sum_{n=-\infty}^{n=+\infty} w(n) h(n) e^{-jwn} \]

\[ h(n) = \frac{1}{2\pi} \int_{\omega=-\pi}^{\omega=+\pi} H_W(\omega) e^{j\omega n} d\omega \] 

\[ \text{..2.18..} \]

Here, \( w(t) \) and \( w(n) \) are window functions in CT and DT domains. When examining equation (2.18), it may arise that to what extent the transform of the signal is limited with the multiplicative window. Multiplication of two functions (or sequences) in the time (or sequence) domain is equivalent to convolution of their spectra in the frequency domain. So, the transform of the windowed signal is the convolution of the transform of the signal with the transform of the window.

Figure 2.5 shows the time and frequency representing the rectangular window of a sinusoid of infinite duration, and of a finite support sinusoid obtained as a product of the previous two signals. Equations (2.19) and (2.20) describe the same signals and their corresponding transforms.

\[ w(t) = \begin{cases} 
1 & \frac{T}{2} < t < \frac{T}{2} \\
0 & \text{Otherwise}
\end{cases} \]

\[ W(f) = T \frac{\sin(\pi ft)}{\pi ft} \]

\[ s(t) = A \sin(2\pi ft - \phi) \quad -\infty < t < +\infty \]

\[ S(f) = \frac{A}{2} e^{j2\pi f} \delta(f - f_0) + \frac{A}{2} e^{-j2\pi f} \delta(f + f_0) \]

\[ s_w(t) = A \sin(2\pi ft - \phi) \quad -\frac{T}{2} < t < +\frac{T}{2} \]

\[ S_w(f) = \frac{AT}{2} e^{j\pi \sin((f-f_0)T)} + \frac{AT}{2} e^{-j\pi \sin((f+f_0)T)} \] 

\[ \text{..(2.19)} \]
It can be seen that the transform of the windowed sinusoid, being the convolution of a pair of spectral impulses located at $f = f_0$ with $\sin(n\pi f T)/(n\pi f T)$ or $\text{sinc}(\pi f T)$, the transform of the window results in the window's transform being scaled and translated to the frequency of the impulses and can be considered as a special case of well known modulation theorem [73, 74].

The effects of the window on the spectrum of a signal can be seen in Figure 2.5. Here it is noted that the Fourier transform of the continuous sinusoidal signal has zero width. The windowed signal first effect is a smearing of transform's spectral width (from infinitesimally small to the main lobe of the $\sin(n\pi f T)/(n\pi f T)$). The second effect is spectral leakage, spreading of the singularity to the $\sin(n\pi f T)/(n\pi f T)$ side-lobes, a function occupying an infinite support with an envelope exhibiting a spectral decay rate of $1/f$. 

\[
W(\theta) = T \left\{ \begin{array}{ll}
\sin(\theta N) \\
\sin(\theta N)
\end{array} \right. \\
\frac{\sin(\theta N)}{\sin(\theta N)}
\]

\[
w(n) = \left\{ \begin{array}{ll}
1 - \frac{N}{2} < n < \frac{N}{2} \\
0: \text{Otherwise}
\end{array} \right.
\]

\[
s(n) = A \sin(\theta_0 n - \phi), \quad -\infty < n < +\infty
\]

\[
S_{\theta}(\theta) = \frac{A}{2} e^{j\theta} \delta(\theta - \theta_0) + \frac{A}{2} e^{-j\theta} \delta(\theta + \theta_0)
\]

\[
s_{w}(n) = A \sin(\theta_0 n - \phi), \quad -\frac{N}{2} < n < +\frac{N}{2}
\]

\[
S_{w}(\theta) = \frac{A}{2} e^{j\theta} \frac{\sin((\theta - \theta_0) N)}{\sin((\theta - \theta_0) \frac{N}{2})} + \frac{A}{2} e^{-j\theta} \frac{\sin((\theta + \theta_0) N)}{\sin((\theta + \theta_0) \frac{N}{2})}
\]
The side-lobe structure of the windowed transform limits the ability of the transform to detect spectral components of significantly lower amplitude in the presence of a large amplitude component while the main-lobe width of the windowed transform limits the ability of the transform to resolve or separate nearby spectral components. The first of these limitations is demonstrated in Figure 2.6 where a stylized power spectrum of two sinusoids of infinite extent and of finite extent is presented. From the Figure 2.6, it is observed that the side-lobe structure of the high level signal is greater than the main lobe level of the low-level signal, hence masks the presence of the low-level signal. If the low-level signal is to be detected in the presence of the nearby high level signal (noise), the window applied to the data must be modified.
Figure 2.6: Spectral Representation of Un-windowed and of a Rectangular Windowed Sinusoids of Significantly Different Amplitudes.

A comment is called for here on this example. Under the restricted condition that the frequencies of the two signals are harmonically related to the observation interval, (that is that the two signals each exhibit an integer number of cycles in the observation interval), the two signals would be resolvable and measurable. This is because, for the conditions described, the two signals are orthogonal, which when interpreted in the frequency domain means that the spectrum of the second signal is located on a zero crossing of the spectrum of the first signal.

The main-lobe width of the windowed transform limits the ability of the transform to resolve closely spaced spectral components of comparable amplitudes. This limitation is demonstrated in Figure 2.7.
Figure 2.7: Spectral Representation of Windowed Sinusoids of Successively Decreasing Spectral Distance Demonstrating Loss of Resolution Due to Merging of Main Lobe Responses.

For this example, the amplitude of the two signals is the same and the interaction between the phase of the main-lobes and the interaction between the main-lobes and the neighbor's side-lobes has been ignored. It is apparent that the spacing between adjacent spectral lines that can be resolved by a windowed transform is related to the main-lobe width of the window's spectrum. For the rectangle window, this main-lobe width (measured from peak to first zero crossing) is $\frac{1}{T}$, the reciprocal of the window's duration in the time domain. To obtain a desired reduction in side-lobe levels, these must be accompanied by an increase in main-lobe width which in turn reduces the spectral resolution capabilities of the window.

2.5 WINDOWS AS A SUM OF COSINES

It is not possible to build windows without side-lobes in their spectra. But one can design of windows with arbitrarily low level side-lobes. The common mechanism to control side-lobe levels is to control the smoothness of the window in the time domain. Figure 2.8 represents the time and spectral description of the rectangle
window. It represents a spectrum centered at zero frequency with main-lobe width $1/T$, and with amplitude of its first spectral side-lobe of $2/3 \pi$, or -13.5 dB below the main-lobe peak. The way to reduce the side-lobes is to destructively cancel them by the side-lobes of judicially placed pairs of scaled $\text{sinc}(\pi(f \pm f_c)T)$ functions. One popular option is to translate a pair of $\text{sinc}(\pi fT)$ functions to the first zero crossings of the $\text{sinc}(\pi fT)$ function. These zeros are located at frequency $\pm 1/T$, the first frequency orthogonal to the rectangle window of length $T$. These $\text{sinc}(\pi fT)$ functions represent a cosine with period exactly equal to the support of the rectangle window (frequency $= 1/T$). As seen in the Figure 2.8, the side-lobes contributed by the additional pair present opposing polarity side-lobes to those of the original $\text{sinc}(\pi fT)$ function. The effect of adding three $\text{sinc}(\pi fT)$ functions is now obvious: the main-lobe width is doubled and the side-lobe levels are reduced by an amount dependent on the particular values of $a_k$

![Figure 2.8: Reduced-complexity FIR filtering.](image)

The window just constructed is called a raised cosine window and is a member of a class of windows formed by a short cosine Fourier transform of the form shown in equation 2.21.
Windows with two-term Fourier transforms include the HANN and HAMMING windows. When the two term coefficients \((a_0, a_1) = (0.5, 0.5)\), the window is the HANN window, (often incorrectly called the HANNING window). For these weights, the highest side-lobe is 0.0267 or -31.47 dB below the peak main-lobe response and decays thereafter at 18 dB/octave. When the coefficients \((a_0, a_1) = (0.54, 0.46)\), then the window is the HAMMING window. For these weights, the highest side-lobe is 0.00735 or -42.76 dB below the peak main-lobe response and decays thereafter at 6dB/octave. We observe that we can realize over two orders of magnitude side-lobe level suppression by doubling the main-lobe width [45,55].

If additional side-lobe level suppression is desired, we have to increase the number of terms in the short Cosine transform. Each new term increases the main-lobe width by placing another pair of sinc\((\pi fT)\) functions in the main-lobe. As the main-lobe bandwidth increases, we use the additional degrees of freedom to realize additional side-lobe level suppression. Examples of windows formed by the short Cosine transforms. The number of terms in the Cosine Series expansion represents the main-lobe width between the spectral peak and the first zero crossing of the main-lobe.

2.6 SPECTRAL ANALYSIS AND WINDOW FIGURES OF MERIT

Windows are used in spectrum analysis to minimize additive biases caused by the artificial boundaries or discontinuities imposed on the time series being analyzed. Now the incidental effects of the windows on the spectrum analysis process are examined. In one use of a spectrum analyzer, a composite signal consisting of a sinusoid of interest is processed which is considered as the desired signal, other sinusoids, not of interest are considered as undesired interference, and additive white
noise [77]. Figure 2.9 is a representation of the spectra of this signal set containing a single undesired line component, the signal described in equation (2.22).

\[ s(nT) = A_s e^{j\theta_s} e^{j\omega_s T} + A_u e^{j\theta_u} e^{j\omega_u T} + N(nT) \quad \ldots \ldots (2.22) \]

\[ S(k) = \sum_{n=0}^{N-1} w(n)s(n)e^{-\frac{2\pi}{N}nk} \]

\[ = \sum_{n=0}^{N-1} w(n) \left[ A_s e^{j\theta_s} e^{\frac{2\pi}{N}(k+\delta_k)} + A_u e^{j\theta_u} e^{\frac{2\pi}{N}(k+\delta_u)} + N(n) \right] e^{-\frac{2\pi}{N}nk} \]

\[ \ldots \ldots (2.23) \]

Figure 2.9: Graphical Representation of Spectra Interacting With Observation Window.

The primary signal processing tool used to perform spectrum analysis is the Discrete Fourier Transform (DFT). The DFT of the composite signal described in equation (2.22) consists of three components as shown in Figure 2.9 and as presented in equation (2.23). In this expression \( S_k \) and \( U_k \) are the frequency displacements (in DFT bins) of the desired and undesired signal components from the DFT bin closest to the desired signal frequency [62]. The DFT bin centers are located at integer multiples of the fundamental frequency \( (2\pi/NT) \) radians/second defined by the support interval \( NT \), can be recalled. Thus, sampled data frequency is defined by the index \( k \) with units of cycles per interval or by the equivalent sampled data frequency of \( k(2\pi/N) \) radians/sample.
The frequency response of the window spectra centered at the kth bin and observed by the input sinusoid at frequency (k + δ_k) cycles/interval is the same as the frequency response of the window centered at DC and observed at frequency offset δ_k is noted. When the displacement, δ_k is zero, this gain defaults to the DC (or zero frequency) response of the window. This gain is called the peak amplitude gain of the window and as shown in equation (2.24) which is the sum of the window weights. A related gain term is called the peak power gain of the window which is also shown in equation (2.24).

Peak Signal Gain = \( W(0) = \sum_{n=0}^{N-1} w(n) \) ……… (2.24)

Peak Signal Power Gain = \( W^2(0) = \left( \sum_{n=0}^{N-1} w(n) \right)^2 \)

The noise component of S(k) is the DFT of the input noise. It is assumed that this noise has zero mean and white with variance \( \sigma^2 N \). Since the noise is a random variable and so is its DFT. The DFT of the noise can be described by its statistics. Two statistics of primary interest are as shown in equation (2.25) in the first and second moments, the DFT output variance due to input noise is a scaled version of the input noise variance. The scale term is the sum of square of the window weights. This gain, shown in equation (2.26) is termed as the peak noise power gain of the window.

This of course is bounded by N (for the rectangle).

\[
E[S_{\text{noise}}(k)] = E\left\{ \sum_{n=0}^{N-1} w(n)\psi(n)e^{-j2\pi nk} \right\} = \sum_{n=0}^{N-1} w(n)\psi(n)e^{-j2\pi nk} \\
E[|S_{\text{noise}}(k)|^2] = E\left\{ \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} w_1(n_1)w_2(n_2)\psi_1(n_1)\psi_2(n_2)e^{-j2\pi(n_1+n_2)k} \right\} \\
= \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} w(n_1)w(n_2).E[\psi(n_1)\psi(n_2)]e^{-j2\pi(n_1+n_2)k/N} \\
= \sum_{n=0}^{N-1} w^2(n)\sigma^2 \\
= \sigma^2 N^2 \sum_{n=0}^{N-1} w^2(n) \quad \text{……….}(2.25)
\]
Peak Noise Power Gain = NPG = $\sum_{n=0}^{N-1} w^2(n)$ ........ (2.26)

2.6.1 Figures of Merit

The use of a window leads to conflicting effects on the output of the transform. The window is applied to data to suppress out-of-band side-lobe levels and this is a desirable effect. The window controls side-lobes by smoothly discarding data near the boundaries of the observation interval. This has the effect of reducing the amplitude, hence energy of both signal and noise components are presented to the transform. Concurrently, the increased bandwidth of the window’s spectral main-lobe (required to purchase the reduced side-lobe levels) permits additional noise into the measurement.

To facilitate comparison of different windows, two performance measures are defined, related to the effects of the window on both signal and noise. The first of these is equivalent noise bandwidth (ENBW). This parameter indicates the equivalent rectangular bandwidth of a filter with the same peak gain of the filter that would result in the same-output noise power. ENBW is computed by dividing the total energy collected by the window by the peak power gain of the window. This is shown in equation (2.27) and is illustrated in Figure 2.10 [61].

$$ENBW = \frac{\sum_{n=0}^{N-1} w^2(n)}{\left(\sum_{n=0}^{N-1} w(n)\right)^2} \quad \ldots \ldots (2.27)$$

Figure 2.10: Equivalent Noise Bandwidth (ENBW): Area Under Power Gain Curve Allocated to Rectangle Of Same Amplitude
It is noted that the rectangle window has the smallest ENBW of $1/N$ while a Hann window has an ENBW of $1.5/N$. The units of ENBW are spectral bins, and larger ENBW indicates an increased variance of a spectral measurement. It is a common practice to normalize the ENBW of the particular window of length $N$ to the ENBW of the rectangle of the same length. Thus the normalized ENBW of the Hann window is 1.5 bins. A table of popular windows along with their ENBW is presented in Table 2.1.

A related Figure of merit for a windowed DFT is the Processing Gain (PG) or improvement in signal to-noise ratio obtained when using the window. This improvement is the ratio of output SNR to input SNR of a noisy sine wave. Processing Gain can be as large as $N$ (for a rectangle) and is usually on the order of $0.4N$ for windows with good side-lobe levels. As shown in equation (2.28), the processing gain is also equal to the reciprocal of the window’s ENBW.

\[
SNR_{out} = \frac{A^2 \left( \sum_{n=0}^{N-1} w(n) \right)^2}{\sigma_N^2 \sum_{n=0}^{N-1} w^2(n)}
\]

\[
SNR_{in} = \frac{A^2}{\sigma_N^2}
\]

\[
PG = \frac{SNR_{out}}{SNR_{in}} = \frac{\left( \sum_{n=0}^{N-1} w(n) \right)^2}{\sigma_N^2 \sum_{n=0}^{N-1} w^2(n)} = \frac{1}{ENBW}
\]

2.6.2 Scalloping Loss

The amplitude gain of a windowed transform applied to a sinusoid located at bin $(k+\delta_k)$ was presented in equation (2.23) and is repeated here in equation (2.29). The amplitude gain $W(\delta_k)$ is a function of the frequency offset, $\delta_k$, of the input sinusoid from the processed bin. This frequency dependent gain due to the offset from the DFT bin center represents a reduction in processing gain and is known as scalloping loss.
\[ S(k)_{\text{SIG}} = \sum_{n=0}^{N-1} w(n) A_s e^{i\phi_s} e^{2\pi n (k+\delta_k)/N} e^{-i2\pi nk/N} \]

\[ = A_s e^{i\phi_s} \sum_{n=0}^{N} w(n) e^{2\pi n \delta_k/N} \]

\[ = A_s e^{i\phi_s} W(\delta_k) \]

where \( S(k)_{\text{SIG}} \) is scalloping loss,

\( A_s \) is signal amplitude,

\( \phi_s \) is signal phase.

When a sinusoidal input frequency is located in the center of a particular DFT bin, the pair of filters bracketing this bin respond with equal amplitudes. If the center frequency of an input sinusoid is shifted from the bin center of filter \( k \) towards filter \((k+1)\), the amplitude response of filter \((k-1)\) and filter \( k \) drops and the response of the filter \((k+1)\) is increased. This drop in amplitude is scalloping loss. When the sinusoid is located at the midpoint between two filters, say at \((k+1/2)\), the two bracketing filters, \( k \) and \((k+1)\), respond with the same amplitude. This amplitude corresponds to the maximum reduction in filter response and is called the peak scallop loss. When Peak scallop loss is presented in dB, it represents the maximum reduction in signal to noise ratio of a windowed transform due to spectral position of input signals. The rectangle window, due to its very narrow main-lobe width, exhibits the maximum scallop loss of 3.92 dB.

Windows with deeper side-lobe levels have wider main-lobes and consequently exhibit reduced scalloping loss typically on the order of 1.0 dB. If scalloping loss is an important consideration, it can be significantly reduced by zero padding and extending the windowed data and performing a double length transform. The Figures of merit required to evaluate are listed in Table 2.1, for many useful windows.
### TABLE 2.1: Figures of Merit for Common Windows.

<table>
<thead>
<tr>
<th>Window</th>
<th>Max Side-lobe (dB)</th>
<th>Side-lobe slope (dB/Oct)</th>
<th>Coherent Gain</th>
<th>ENBW (bins)</th>
<th>Scallop Loss (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>RECTANGLE</td>
<td>-13.4</td>
<td>-6</td>
<td>1.000</td>
<td>1.000</td>
<td>-3.92</td>
</tr>
<tr>
<td>TRIANGLE</td>
<td>-26.5</td>
<td>-12</td>
<td>0.500</td>
<td>1.333</td>
<td>-1.83</td>
</tr>
<tr>
<td>HANN</td>
<td>-31.5</td>
<td>-18</td>
<td>0.500</td>
<td>1.500</td>
<td>-1.43</td>
</tr>
<tr>
<td>HAMMING</td>
<td>-42.7</td>
<td>-6</td>
<td>0.540</td>
<td>1.364</td>
<td>-1.75</td>
</tr>
<tr>
<td>DOLPH (a=40.0)</td>
<td>-40.0</td>
<td>0</td>
<td>0.589</td>
<td>1.304</td>
<td>-2.06</td>
</tr>
<tr>
<td>CHEBYSHEV (a=60.0)</td>
<td>-60.0</td>
<td>0</td>
<td>0.479</td>
<td>1.518</td>
<td>-1.42</td>
</tr>
<tr>
<td></td>
<td>(a=80.0)</td>
<td>0</td>
<td>0.414</td>
<td>1.743</td>
<td>-1.09</td>
</tr>
<tr>
<td>KAISER (β=5.47)</td>
<td>-40.0</td>
<td>-6</td>
<td>0.522</td>
<td>1.412</td>
<td>-1.62</td>
</tr>
<tr>
<td></td>
<td>(β=8.15)</td>
<td>-6</td>
<td>0.431</td>
<td>1.681</td>
<td>-1.16</td>
</tr>
<tr>
<td></td>
<td>(β=10.66)</td>
<td>-6</td>
<td>0.379</td>
<td>1.903</td>
<td>-0.09</td>
</tr>
<tr>
<td>GAUSSIAN (γ=2.46)</td>
<td>-40.0</td>
<td>-6</td>
<td>0.502</td>
<td>1.427</td>
<td>-1.62</td>
</tr>
<tr>
<td></td>
<td>(γ=3.15)</td>
<td>-6</td>
<td>0.397</td>
<td>1.784</td>
<td>-1.06</td>
</tr>
<tr>
<td></td>
<td>(γ=3.76)</td>
<td>-6</td>
<td>0.333</td>
<td>2.123</td>
<td>-0.75</td>
</tr>
<tr>
<td>TUKEY (α=0.25)</td>
<td>-14</td>
<td>-18</td>
<td>0.88</td>
<td>1.10</td>
<td>2.96</td>
</tr>
<tr>
<td></td>
<td>(α=0.50)</td>
<td>-18</td>
<td>0.75</td>
<td>1.22</td>
<td>2.24</td>
</tr>
<tr>
<td></td>
<td>(α=0.75)</td>
<td>-18</td>
<td>0.63</td>
<td>1.36</td>
<td>1.73</td>
</tr>
</tbody>
</table>
2.6.3 Spectral Resolution

A primary application of a window is control of the spectral artifacts generated by observing signals over finite supports. One spectral artifact, observed as additive bias, is side-lobe leakage [66]. The process of reducing the side-lobe levels was shown to be equivalent to adding additional spectral terms to the main-lobe to enable the destructive cancellation of the side-lobes in the spectrum of the rectangle window. This addition widens the main-lobe width and the design of good windows entails managing this increase while making the trade of main-lobe width for side-lobe level. The net effect of the window on a spectrum analysis task is an increase in dynamic range (due to side-lobe suppression) with a decrease in spectral resolution (due to main-lobe widening).

The most demanding test of a windowed transform is its ability to resolve and detect two closely spaced sinusoids of significantly different amplitudes. The effect of the window on the detection and measurement of two sine-waves of amplitude 1.0 and 0.01 and of frequency 11.5 and 16.0 cycles per interval respectively are demonstrated. The large signal at frequency 10.5 cycles per interval is located between two DFT bins and splatters its spectral side-lobes throughout the remaining DFT bins. A reference point to compare the performance of the windows is available in Figure 2.1la. The power spectrum of the two signals with the large signal moved to DFT bin 11(11 cycles per interval) is observed here. Since both signals are harmonically related to the observation interval there is no spectral splatter between the DFT bins and the signals are perfectly resolvable (within additive and arithmetic noise effects). In Figure 2.1lb, the power spectrum of the two signals after returning the large signal to bin 11.5 is seen. The side-lobe levels of the large signal almost covers the main-lobe of the small signal resulting in significant degradation of its detectability. The remaining Figures of this example (2.11c through 2.11h) present the power spectra of the same two signals processed with different windows. When the
window used in this demonstration had selectable side-lobe levels, they were selected to be 60 dB below peak. Here the power spectra were not normalized for peak power level but rather were normalized to peak power level of the rectangle window with bin centered signal. This permits the reduction in peak amplitude response due to application of the window. In Figure 2.11c the side-lobes of the Hamming window, now-centered on the spectrum of the large signal, are spilling into the spectrum and nearly covering the low level signal, which is clearly seen. By comparison, in Figure 2.11d the side-lobe structure of the Hann window quite rapidly and while not filling the entire spectrum, they spill into the spectral region occupied by the smaller signal.

In Figure 2.11c the main-lobe of the Dolph-Chebyshev is sufficiently narrow to permit excellent resolution of the two signal spectra, but the side-lobes of the two signals interact and cause incidental side-lobe amplitude modulation. This is one of the problems with windows (and filters) that exhibit constant level side-lobes. Figure 2.11f shows the excellent performance of the Kaiser-Bessel window and illustrates the window's ability to resolve closely spaced signals of significantly different amplitude levels. The spectral notch between the two spectral lines can be thought of as the window's ability to accommodate additional resolution stress (of reduced signal level or reduced spectral spacing). This window would have no difficulty in resolving these two signals even if they were 80 dB apart (rather that the 40 dB of this example). Similarly, Figure 2.11g illustrates the success of the Gaussian window in its ability to resolve the two spectral lines. Finally, Figure 2.11h demonstrates that the Tukey window resolves the two signals but the notch between the two spectral lines is not as deep as the former two examples. This reduction in resolution is due to the wider main-lobe width of the window causing the two spectra to overlap.
Figure 2.11a: Spectral resolution of Rectangle window.

Figure 2.11b: Spectral Resolution of Triangle window.

Figure 2.11c: Spectral Resolution of Hamming Window.

Figure 2.11d: Spectral resolution of Hanning window.
Figure 2.11e: Spectral Resolution of Dolph-Chebyshev Window

Figure 2.11f: Spectral Resolution of Kaiser window

Figure 2.11g: Spectral Resolution of Gaussian window

Figure 2.11h: Spectral Resolution of Tukey Window
2.7 WINDOWS IN MST SIGNAL PROCESSING AND PARAMETER EXTRACTION

Figure 2.12 shows the functional block diagram of various processing stages involved in the extraction and estimation of atmospheric parameters [78, 79]. They are namely Decoder, Coherent Integrator, Normalization, Windowing, Fourier Analysis, Power Spectrum, Incoherent Integration, Spectrum Cleaning, Noise Level Estimation, Moments Computation, SNR Estimation and UVW Computation.

The complex time series of the decoded and integrated signal samples are subjected to the process of FFT for the on-line computation of the Doppler power spectra for each range bin of the selected range window. The Doppler power spectra are recorded on a magnetic tape for off-line processing. There is a provision, however, to record raw data (complex time samples) directly for any application, if desired. The off-line data processing for parameterization of the Doppler spectrum. The computation involved in the various stages of operation and its advantages are given below. The different stages are briefly discussed below.

Figure 2.12: Processing steps for extraction of parameters
2.7.1 Decoder

The maximum range capability of the radar wind profiler is directly proportional to the square root of the average transmit power, which is the product of peak power and duty ratio ($P/T$, where $T$ is the inter-pulse period). Measurements from an unambiguous longest range, $R$, put the minimum limit on the $T$ ($\geq 2R/c$). Therefore the range coverage for a fixed inter-pulse period depends on the pulse length. However, the profiler’s range resolution is equal to $ct/2$. It is evident that best range resolution is obtained with short pulse length but the profiler’s height coverage will be minimum due to low average transmit power. On the other hand, using long pulse width increases the height coverage but degrades the range resolution. One would like to have best range resolution and maximum height coverage simultaneously. A technique of transmitting a long pulse and compressing the same upon reception, known as ‘pulse-compression and is used to accomplish this objective. In the time domain a pulse can be compressed via phase coding, especially binary phase coding, a technique which is particularly amenable to digital processing techniques. In this technique, a long pulse of length $\tau = n\tau_b$ is divided into $n$ number of small segments, each of length $\tau_b$, and each segment is phase coded on transmission. Upon reception, the signal is decoded with reference to the transmit code which results in compression of the pulse by a factor $n$, thus providing a range resolution corresponding to the sub-pulse length (also referred as baud length), $\tau_b$, but with power enhanced by a factor $n$. Hence a transmit pulse of power $P_t$ and duration $n\tau_b$ is in a sense compressed into a narrow pulse of duration $\tau_b$ and power corresponding to $nP_t$.

2.7.2 Coherent integration

Radars, by virtue of being coherent in nature, obtain amplitude and phase information of the echo signals. Demodulation of the received signal plus noise, linearly amplified to appropriate levels, is done in a quadrature detector followed by suitable post detection filters matched to the transmitted pulse shape. The strength of single-pulse echoes due to refractive turbulent scatter is generally very weak, as low as 50 dB down, compared to the mean noise floor of the receiver system and the mean radial velocities usually do not exceed few meters per second, since only a projection
of the horizontal velocity will be measured with radar beams pointed relatively close to the zenith direction. The expected shift and width of the Doppler spectrum will typically be much less than about 10 and 100 Hz at VHF and UHF, respectively. But the typical pulse repetition rates can be of the order of 1 kHz and 20 kHz at VHF and UHF, respectively. Thus the coherence time of the wind profiler signals is much longer than the typical inter-pulse period (IPP) and hence the time series data of the radar signals can be over-sampled by at least two orders of magnitude, provided that one sample per range gate is taken every IPP. Receiver noise is independent from one IPP to the next, whereas the signal remains coherent over the same period. This characteristic of received signal being phase coherent over many IPPs allows the integration of weak complex signal samples of several pulses to improve the 'detectability'. This process, carried out independently for each range gate, is known as 'coherent integration'. Coherent integration is a digital box-car low-pass filtering process achieved by adding the complex digital data samples for a pre-defined number (Nci) of pulses. Low-pass filtering process is equivalent to an increase in the apparent signal-to-noise ratio because it reduces the wide bandwidth of the noise, determined by a Nyquist frequency 1/(2Ti), Ti being the integration time (Nci x IPP), to a value close to the highest frequency of the echo signal. The apparent SNR is increased by the factor Nci (number of coherently integrated samples per range gate, corresponding to the number of IPPs), essentially because all but 1/Nci of the original wideband noise is filtered out [80].

2.7.3 Normalization of the Pre-Processed Data

The input data is to be normalized by applying a scaling factor corresponding to the operation done on it. This will reduce the chance of data overflowing due to any other succeeding operation. The Normalization has following components.

a. Sampling resolution of ADC
b. Scaling due to pulse compression in decoder
c. Scaling due to coherent integration
d. Scaling due to number of FFT points.
If \( \Delta v \) - ADC bit resolution \((10/4096)\),

\( w \) - Pulse width in microsecond,

\( M \) - Number of IPP integrated = Integrated time/inter pulse period,

\( N \) - Number of FFT points,

then the Normalization factor

\[
S = \frac{\Delta v}{w \times m \times f}
\]  

...........(2.30)

The complex time series \( \{I, Q\, where\, i = 0, \ldots, N-1\} \) at the output of the signal processor is scaled as

\[
\bar{I}_i = s \times I_i,
\]

\[
\bar{Q}_i = s \times Q_i,
\]

...........(2.31)

2.7.4 Windowing

It is well known that the application of FFT to a finite length data gives rise to leakage and picket fence effects. Weighting the data with suitable windows can reduce these effects. However the use of the data windows other than the rectangular window effects the bias, variance and frequency resolution of the spectral estimates. In general variance of the estimate increases with the use of a window. An estimate is said to be consistent if the bias and the variance both tend to zero as the number of observations are increased. Thus, the problem associated with the spectral estimation of a finite length data by the FFT techniques is establishing efficient data windows or data smoothing schemes. Processing supports the selection of different windows selectable before the FFT computation [81].

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2.7.5 Fourier Analysis

Spectral analysis is connected with characterizing the frequency content of a signal. A large number of spectral analysis techniques are available in the literature. This can be broadly classified into Non parametric or Fourier analysis based method and Parametric or Model based methods.

Fourier proposed that any finite duration signal, even a signal with discontinuities, can be expressed as an infinite summation of harmonically related sinusoidal components; that is

\[ x(t) = \sum_{k=0}^{\infty} (A_k \cos(\omega_k t) + B_k \sin(\omega_k t)) \] ........(2.32)

Where \( A_k \) and \( B_k \) are Fourier coefficients and \( \omega_0 \) is the fundamental angular frequency. Application of Fourier analysis to discrete series of data and its fast computation algorithm Fast Fourier transform (FFT) made this technique so popular in the spectral analysis. FFT is applied to complex time series \( \{ (I_i, Q_i) \}_{i=0,1,\ldots,N-1} \) to obtain complex frequency domain spectrum \( \{ (X_i, Y_i) \}_{i=0,1,\ldots,N-1} \)

\[ X_i + jY_i = \frac{1}{N} \sum_{k=0}^{N-1} (I_k + jQ_k) \exp(-2\pi ik/N) \] i=0, 1, 2, 3, ..., N-1

........ (2.33)

2.7.6 Power Spectrum

Power spectrum is calculated from the complex spectrum as

\[ P_i = X_i^2 + Y_i^2, \quad i=0,1,2, \ldots, N-1 \] ........ (2.34)

2.7.7 Incoherent Integration (Spectral Averaging)

Incoherent integration is the averaging of the power spectrum number of times.

\[ P_i = \frac{1}{m} \sum_{i=1}^{m} P_{i}, \quad i=0,1,2, \ldots, N-1 \] ........ (2.35)

where \( m \) is the number of spectra integrated.
The advantage of the incoherent integration is that it improves the detectability of the Doppler spectrum. The detectability is defined as

\[ D = \frac{P_s}{\sigma_{S+N}} \]  \hspace{1cm} (2.36)

Where \( P_s \) is the signal power and \( \sigma_{S+N} \) is the standard deviation of the power spectral density.

2.7.8 Power Spectrum Cleaning

Due to various reasons the radar echoes may get corrupted by ground clutter, system bias, interference, image formation etc. The data is to be cleaned from these problems before going for analysis.

Clutter / DC Removal: A strong DC value or zero mean velocity may be present in the spectrum after the FFT. This is removed by replacing the zero Doppler bins by the average of adjacent Doppler bins on either side.

The basic operation carried out here is,

\[ \frac{\tilde{I}_{N/2}}{2} = \frac{\left( P_{N/2}^{+1} + P_{N/2}^{-1} \right)}{2} \]  \hspace{1cm} (2.37)

N/2 corresponds to zero frequency

This can also be removed in time series by taking out the bias in I and Q channel and then perform the Fourier analysis.

Spikes (glitches) in the time series will generate a constant amplitude band all over the frequency bandwidth. Once Fourier analysis is done, it is difficult to identify the correct Doppler in the range bin. These points may be removed from the range bin and adjusted to noise floor or by an incoherent integration of the spectrum and replacing the value with good value from the second spectrum. However, this type of problem needs to be corrected before doing Fourier analysis to get a better result by finding out the outliers in data.
Constant frequency bands will form in the power spectrum by the interference generated in the system or due to extraneous signal. Due to this reason it is also possible for the formation of multiple bands in spectrum. This is removed by taking a range bin, which does not have echoes but only the interference. This range bin gets subtracted from all other range bins after the removal of mean noise. If the interference is not affecting the original Doppler trace then the analysis may be carried out in a window confined to the Doppler trace.

2.7.9 Noise Level Estimation

There are many methods adapted to find out the noise level estimation. Basically all methods are statistical approximations to the near values. The method implemented here is based on the variance decided by a threshold criterion [37]. The noise level threshold shall be estimated to the maximum level \( L \), such that the set of spectral points below the level \( S \), nearly satisfies the criterion,

\[
\frac{\text{Variance}(S)}{\text{mean}(S)} \leq 1 \quad \text{over number of spectra averaged}
\]

The algorithm steps are shown below:

**Step 1:**

Reorder the spectrum \( \{P_n, i = 0, \ldots, N-1\} \) in ascending order and form the sequence \( \{A_i, i = 0, \ldots, N-1\} \) and \( A_i < A_j \) for \( i < j \)

**Step 2:**

Compute

\[
P_n = \sum_{i=0}^{N} \frac{A_i}{(n+1)} \quad \ldots \ldots \text{(2.38)}
\]

\[
Q_n = \sum_{i=0}^{N} \frac{A_i^2}{n+1} - P_n^2 \quad \ldots \ldots \text{(2.39)}
\]

and if \( Q_n > 0 \), \( R_n = \frac{P_n^2}{Q_n^* M} \), for \( n = 0, 1, \ldots, N-1 \)

Where \( M \) is the number of spectra that were averaged for obtaining the data.
Step 3:

Noise level \((L) = P_k\) where \(k = \min n\) such that \(R_n > 1\)

if no \(n\) meets the above criterion

2.7.10 Moments Estimation

The extraction of zeroth, first and second moments is the key problem on doing all the signal processing and thereby finding out the various atmospheric and turbulence parameters in the region of radar sounding [77]. The basic steps involved in the estimation of moments are given below:

Step 1:

Reorder the spectrum to its correct index of frequency in the following manner.

Spectral index

<table>
<thead>
<tr>
<th>0</th>
<th>Ambiguous frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Maximum negative frequency</td>
</tr>
<tr>
<td>\ldots</td>
<td>\ldots</td>
</tr>
<tr>
<td>N/2</td>
<td>Zero frequency</td>
</tr>
<tr>
<td>\ldots</td>
<td>\ldots</td>
</tr>
<tr>
<td>N-1</td>
<td>Maximum positive frequency</td>
</tr>
</tbody>
</table>

Step 2:

Subtract noise level \(L\) from spectrum.

Step 3:

i) Find \(I\) the index of the peak value in the spectrum.

i.e. \(p_i \geq p_j\) for all \(i = 0 \ldots N-1\)
ii) Find the minimum index of positive point corresponding to the valley point of the detected peak Doppler spectrum.

i.e., \( p_i \geq 0 \) for all \( m \leq i \leq l \)

iii) Find the maximum index of positive point corresponding to the valley point of the detected peak Doppler spectrum.

i.e., \( p_i \geq 0 \) for all \( l \leq i \leq n \)

*Step 4:*

The moments computed as

i) \[ M_0 = \sum_{i=m}^{n} P_i \] \hspace{2cm} \text{(2.40)}

represents zeroth moment or total power in the Doppler spectrum.

ii) \[ M_1 = \frac{1}{M_0} \sum_{i=m}^{n} P_i f_i \] \text{ where } \( f_i = \frac{(i-N/2)}{IPP \cdot n \cdot N} \) \hspace{2cm} \text{(2.41)}

represents the first moment or mean Doppler in Hz

iii) \[ M_2 = \frac{1}{M_0} \sum_{i=m}^{n} (f_i - M_1)^2 \] \hspace{2cm} \text{(2.42)}

represents the second moment or variance, a measure of dispersion from central frequency.

iv) Doppler width (full) = \( 2 \sqrt{M_2} \) Hz \hspace{2cm} \text{(2.43)}

v) Signal to Noise Ratio (SNR) = \( 10 \log \left[ \frac{M_0}{(N \cdot L)} \right] \) dB \hspace{2cm} \text{(2.44)}

Where

IPP - is inter pulse period in \( \mu \) sec.

N - is the number of FFT points.

L - is Noise level
Calculation of spectral moments of spectrum with composite structure is done in a slightly different way from the procedure explained above. This type of spectrum normally comes in the upper atmospheric region (Ionosphere). Here the spectra is wide and shows multiple spikes, so after the removal of mean noise level the spectra may be crossing from positive values to negative values many times. Thus, the peak and valley detection described above cannot give the correct result. To overcome this problem, a running template is taken with seven Doppler points. The Doppler point to be checked is the central point of the template. This template will move from the peak to the either side of the spectrum, to find the lower and upper point of Doppler index from the maximum peak. The running average of seven points is checked against a threshold. The threshold is kept 3 dB above the mean noise level. The Doppler point is considered till the template average is above the threshold. Remaining part of the moments calculation is same as that of the calculation for the single peak Doppler spectrum. Figure 2.13 below shows a typical spectrum plot with height when MST Radar data is truncated for 512 points and its corresponding Signal to Noise ratio (SNR) with height.

![Typical wind profile plot and its SNR variation with height](image)

Figure 2.13: Typical wind profile plot and its SNR variation with height
2.7.11 UVW Computation

Calculation of radial velocity and height:

For representing the observation results in physical parameters, the Doppler frequency and range bin have to be expressed in terms of corresponding radial velocity and vertical height.

\[
\text{Height, } H = \frac{(c \cdot f_0 \cdot \cos \theta)}{2} \text{ meters} \quad \cdots \cdots \text{(2.45)}
\]

\[
\text{Velocity, } V = \frac{(c \cdot f_0 \cdot \lambda)}{2 \cdot f_t} \quad \text{or} \quad \frac{f_0 \cdot \lambda}{2} \text{ m/sec} \quad \cdots \cdots \text{(2.46)}
\]

Where \( c \) - velocity of light in free space, \( f_0 \) - Doppler frequency, \( f_c \)-carrier frequency, \( \lambda \) - carrier wavelength (here 5.86 m), \( \theta \) - Beam tilt angle, \( t_R \) - Range time delay.

2.7.12 Computation Of Absolute Wind Velocity Vectors (UVW)

After computing the radial velocity for different beam positions, the absolute velocity (UVW) can be calculated. To compute the UVW, at least three non-coplanar beam radial velocity data is required. If higher number of different beam data are available, then the computation will give an optimum result in the least square method.

Line of sight component of the wind vector \( V (V_x, V_y, V_z) \) is

\[
V_D = V \cdot i = V_x \cos \theta_x + V_y \cos \theta_y + V_z \cos \theta_z \quad \cdots \cdots \text{(2.47)}
\]

Where \( x, y \) and \( z \) directions are aligned to East-West, North-South and Zenith respectively. Applying least square method, residual

\[
\epsilon ^2 = \sum_i (V_{xi} \cos \theta_{xi} + V_{yi} \cos \theta_{yi} + V_{zi} \cos \theta_{zi} - V_{D_i})^2 \quad \text{(2.48)}
\]

where \( V_{D_i} = f_{D_i} \cdot \lambda / 2 \) and \( i \) represents the beam number.
To satisfy the minimum residual
\[
\frac{\partial^2 \epsilon}{\partial N_k^2} = 0 \quad k \text{ corresponding to } x, y \text{ and } z \text{ leads to}
\]
\[
\begin{bmatrix}
V_x \\
V_y \\
V_z
\end{bmatrix} =
\begin{bmatrix}
\sum \cos^2 \theta_n \cos \theta_m \\
\sum \cos \theta_n \cos \theta_m \\
\sum \cos^2 \theta_n \cos \theta_m
\end{bmatrix}
\begin{bmatrix}
\sum \cos \theta_n \cos \theta_m \\
\sum \cos^2 \theta_n \cos \theta_m \\
\sum \cos \theta_n \cos \theta_m
\end{bmatrix}^	op
\begin{bmatrix}
V_{Dx} \cos \theta_n \\
V_{Dy} \cos \theta_n \\
V_{Dz} \cos \theta_n
\end{bmatrix}
\]

\[\ldots\ldots (2.49)\]

Thus, on solving eq. (2.49) we can derive \(V_x, V_y\) and \(V_z\) which corresponds to

U (Zonal), V (Meridonal) and W (Vertical) components of velocity.