CHAPTER – I

INTRODUCTION AND REVIEW OF LITERATURE

1.1 Introduction

The demand for advanced technological equipments has been increasing day by day may because of their efficient working, least cost of operation and reliable services. Now the well developed systems have taken over the traditional functioning of the old ones ensuring flawless operations. At the same time, the customers want regular and prompt delivery of services like water and power supply, scheduled transport services as well as problem free equipments such as washing machines, air conditioners, invertors, computers, refrigerators, etc. Though in case of relatively less critical systems the consumer may allow failures within certain limits but on the other hand there are systems of hazardous nature (nuclear power plants etc.) in case of which a small system failure can result in serious consequences in terms of huge economic loss, threats to human life or serious damage to the environment. Therefore, the failures in such systems cannot be allowed. This fact makes the theme of reliability theory much more essential and brought it at forefront of research.

The reliability of a system can be defined in many ways but most commonly it is characterized as the ability that the system will perform its intended function for a given period of time under stated conditions. As the system failures are inherent and inevitable so the occurrence of failures can neither be stopped nor be neglected completely but there effects can be minimized by incorporating the techniques of reliability theory i.e. either by using various redundancy methods or by employing effective and proper repair policies. Actually, the importance of reliability was born out of the demands of modern technology used in World War 2nd. In Second World War, the failures of equipments and components particularly electronic tubes were more than expected. Therefore, during the decades following the war many research laboratories and universities initiated work on failures of equipments and components. The first major committee (AGREE) was set up by the US department of defense in 1950.
The exponential distribution for life testing has been used by Epstein and Sobel (1953). This paper becomes first milestone in reliability modeling and popularized the choice of exponential distribution in life testing research. Birnbaum and Saunders (1958) have studied a statistical model for life length of material. This paper has been considered to set up the foundation for mathematical theory of reliability (Barlow and Proschan, 1965). Gaver (1963), analyzed the time to system failure and the long run availability of parallel redundant system subjected to the random failure and arbitrarily distributed repair times. Barlow and Proschan (1965) have exceptionally portrayed the early developments of mathematical theory of reliability. Mine et al. (1968) examined multiple unit redundant systems by assuming generalized repair time distribution.

However, some of them have tried to probe the real functioning of the industrial systems with focus on elements of reliability. Srinivasan and Gopalan [1973] adopted the regenerative point technique for obtaining reliability characteristics of a system. Barolini [1974, 75] has discussed some applications of regenerative stochastic processes to reliability theory for evaluating reliability and availability of 2-item redundant systems. Gopalan [1975] analyzed availability and reliability of a series-parallel system with a single repair facility. Arora [1977] obtained reliability of several standby priority redundant systems.


Kumar et al. [1995] studied probabilistically a two-unit cold standby system with instructions at need. Singh and Goel [1997] determined availability of a thermal power plant

The reliability models for a computer system with the concepts of cold standby redundancy and inspection have been studied by Anand (2012). Later on, Ashish (2013) carried out cost-benefit analysis of computer systems subject to maximum operation and repair times. Also, Malik (2013) developed reliability model of a computer system with preventive maintenance and priority subject to maximum operation and repair times. Malik and Dhankar (2013) made cost analysis of a system with replacement of the server and unit subject to inspection. Chilla and Malik (2013) introduced the concepts of priority to maintenance over repair while analyzing shock model for a parallel system. Gitanjali (2014) proposed reliability models for a parallel system considering different servers subject to maximum repair times. The
studies referred above indicate that the reliability models of cold standby systems have not been discussed much more with the concepts of preventive maintenance and repair.

To fill up this gap and strengthening the existing literature, the purpose of the present study is to carry out profit analysis of some reliability models developed for a cold standby system considering the ideas of preventive maintenance and repair. Hence the thesis “Profit Analysis of System Reliability Models with Preventive Maintenance and Repair”

1.2 Some Fundamental Concepts

1.2.1 Reliability

Reliability theory is the foundation of reliability engineering. For engineering purposes, reliability is defined as: the probability that a device will perform its intended function during a specified period of time under stated conditions.

Mathematically, this may be expressed as,

\[ R(t) = \int_{0}^{\infty} f(x)dx \]

Where

\( f(x) \) is the failure probability density function.

Reliability engineering is concerned with four key elements of this definition:

- First, reliability is a probability. This means that there is always some chance for failure. Reliability engineering is concerned with meeting the specified probability of success, at a specified statistical confidence level.
- Second, reliability is predicated on “intended function:” Generally, this is taken to mean operation without failure. However, even if no individual part of the system fails, but the system as a whole does not do what was intended, then it is still charged against the
system reliability. The system requirements specification is the criterion against which reliability is measured.

- Third, reliability applies to a specified period of time. In practical terms, this means that a system has a specified chance that it will operate without failure before time t. Reliability engineering ensures that components and materials will meet the requirements during the specified time. Units other than time may sometimes be used. The automotive industry might specify reliability in terms of miles, the military might specify reliability of a gun for a certain number of rounds fired. A piece of mechanical equipment may have a reliability rating value in terms of cycles of use.

- Fourth, reliability is restricted to operation under stated conditions. This constraint is necessary because it is impossible to design a system for unlimited conditions. A Mars Rover will have different specified conditions than the family car. The operating environment must be addressed during design and testing.

1.2.2 Reliability Engineering

Reliability engineering is the discipline of ensuring that a system (or a device in general) will perform its intended function(s) when operated in a specified manner for a specified length of time. Reliability engineering is performed throughout the entire life cycle of a system, including development, test, production and operation.

Reliability engineers rely heavily on statistics, probability theory, and reliability theory. Many engineering techniques are used in reliability engineering, such as reliability prediction, Weibull analysis, thermal management, reliability testing and accelerated life testing. Because of the large number or reliability techniques, their expense, and the varying degrees of reliability required for different situations, most projects develop a reliability program plan to specify the reliability tasks that will be performed for that specific system.

The function of reliability engineering is to develop the reliability requirements for the product, establish an adequate reliability program, and perform appropriate analyses and tasks to ensure the product will meet its requirements. These tasks are managed by a reliability
engineer, who usually holds an accredited engineering degree and has additional reliability specific education and training. Reliability engineering is closely associated with maintainability engineering and logistics engineering. Many problems from other fields, such as security engineering, can also be approached using reliability engineering techniques.

1.2.3 Reliability Requirements

Requirements are specified using reliability parameters. The most common reliability parameter is the mean-time-between-failure (MTBF), which can also be specified as the failure rate or the number of failures during a given period. These parameters are very useful for systems that are operated on a regular basis, such as most vehicles, machinery, and electronic equipment. Reliability increases as the MTBF increases. The MTBF is usually specified in hours; but can also be used with any unit of duration such as miles or cycles.

In other cases, reliability is specified as the probability of mission success. For example, reliability of a scheduled aircraft flight can be specified as a dimensionless probability or a percentage.

A special case of mission success is the single-shot device or system. These are devices or systems that remain relatively dormant and only operate once. Examples include automobile airbags, thermal batteries and missiles. Single-shot reliability is specified as a probability of success, or is subsumed into a related parameter. Single-shot missile reliability may be incorporated into a requirement for the probability of hit.

In addition to system level requirements, reliability requirements may be specified for critical subsystems. In all cases, reliability parameters are specified with appropriate statistical confidence intervals.

It is a general praxis to model the early failure rate with an exponential distribution. This less complex model for the failure distribution has only one parameter: the constant failure rate.
1.2.4 Failure Rate

Failure rate is the frequency with which an engineered system or component fails, expressed for example in failures per hour. It is often denoted by the Greek letter \( \lambda \) (Lamda) and is important in reliability theory. In practice, the reciprocal rate MTBF is more commonly expressed and used for high quality components or systems.

Failure rate is usually time dependent, and an intuitive corollary is that both rates change over time versus the expected life cycle of a system. For example, as an automobile grows older, the failure rate in its fifth year of service may be many times greater than its failure rate during its first year of service – one simply does not expect to replace an exhaust pipe, overhaul the brakes, or have major power plant-transmission problems in a new vehicle.

1.2.4.1 Failure Rate in the Discrete Sense

In words appearing in an experiment, the failure rate can be defined as “The total number of failures within an item population, divided by the total time expended by that population, during a particular measurement interval under stated conditions”.

Here failure rate \( \lambda \) (t) can be thought of as the probability that a failure occurs in a specified interval, given no failure before time \( t \). It can be defined with the aid of the reliability function or survival function \( R(t) \), the probability of no failure before time \( t \), as:

\[
\lambda(t) = \frac{R(t_1) - R(t_2)}{(t_2 - t_1)R(t_1)} = \frac{R(t) - R(t + \Delta t)}{\Delta t . R(t)}
\]

where \( t_1 \) (or \( t \)) and \( t_2 \) are respectively the beginning and ending of a specified interval of time spanning \( \Delta t \). Note that this is a conditional probability, hence the \( R(t) \) in the denominator.

1.2.4.2 Failure Rate in the Continuous Sense (Instantaneous Hazard rate)

By calculating the failure rate for smaller intervals of time \( \Delta t \), the interval becomes infinitesimally small. This results in the hazard function, which is the instantaneous failure rate at any point in time:
Continuous failure rate depends on a failure distribution, $F(t)$, which is a cumulative distribution function that describes the probability of failure prior to time $t$,

$$P(t > t) = F(t) = 1 - R(t), \ t \geq 0.$$

The failure distribution function is the integral of the failure density function, $f(x)$,

$$F(t) = \int_0^t f(x)dx.$$

The hazard function can be defined now as

$$h(t) = \frac{f(t)}{R(t)}.$$

There are many failure distributions. A common failure distribution is the exponential failure distribution.

$$F(t) = \int_0^t \lambda e^{-\lambda t}dx = 1 - e^{-\lambda t},$$

which is based on the exponential density function. This leads to a constant hazard rate. For other distributions, such as the Weibull distribution, log-normal distribution, or bathtub curve, the hazard function is not constant, which means that the failure rate varies with time.

1.2.4.3 Failure Rate Units

Failure rates can be expressed using any measure of time, but hours is the most common unit in practice. Other units, such as miles, revolutions, etc., can also be used in place of “time” units.

Failure rates are often expressed in engineering notation as failures per million, or $10^6$, especially for individual components, since their failure rates are often very low.
The Failures In Time (FIT) rate of a device is the number of failures that can be expected in one billion \((10^9)\) hours of operation. This term is used particularly by the semiconductor industry.

### 1.2.5 System Configurations

By system, we mean an arbitrary device having several units/sub systems/components assuming that their reliabilities are known which help us to predict the reliability of whole system. It is now important that the system structure be known. Various system structures have been considered as follows:

#### 1.2.5.1 Series Configuration

A system having \(n\)-units is said to have series configuration if the failure of an arbitrary unit, say \(i^{th}\) unit causes the entire system failure. The examples of the series configurations are:

- (i) The aircraft electronic system consists of a sensor sub system, a guidance subsystem, computer subsystem and the fire control subsystem. These systems can only operate successfully if all these operate simultaneously.
- (ii) Deepawali or Christmas glow bulb, where if one bulb fails the whole lead fails. The block diagram of a series system is shown in Fig. 1.2.

Let \(R_i(t)\) be the reliability of \(i^{th}\) component, then the system reliability is given by

\[
R(t) = \Pr[T > t] = \Pr[\min(T_1, T_2, T_3, \ldots, T_n) > t] = \prod_{i=1}^{n} R_i(t)
\]

Where \(T_i\) is the life time of the \(i^{th}\) unit of the system. The system hazard rate, therefore is

\[
r(t) = \sum_{i=1}^{n} r_i(t)
\]

where

- \(r_i(t)\) is the instantaneous failure rate of \(i^{th}\) unit.
1.2.5.2 Parallel Configuration

In this configuration, all the units are connected in parallel i.e. the failure of the system occurs only when all the units of system fail. For example, four engined aircraft which is still able to fly with only two engines working. Block diagram representing a parallel configuration is shown in fig. 1.3.

Suppose $R_i(t)$ and $T_i$ be the reliability of $i^{th}$ components and the life time of the $i^{th}$ unit in time $t$ respectively, then the system reliability is given by

$$R(t) = Pr [T > t]$$

$$= Pr [\text{max.}(T_1, T_2, T_3, \ldots, T_n) > t]$$

$$= 1 - Pr[T_1 \leq t, T_2 \leq t, T_3 \leq t, \ldots, T_n \leq t]$$

If the units function independently, then

$$R(t) = 1 - [1-R_1(t)][1-R_2(t)][1-R_3(t)]\ldots[1-R_n(t)]$$

$$= 1 - \prod_{i=1}^{n}[1 - R_i(t)]$$

1.2.5.3 Standby Redundant Configuration

To assure high reliability of a system, redundancy is incorporated. In redundant system more units than the required are used so that when failures occur in a system, it does not stop functioning. In standby redundant system with $n$ units, only one unit is on-line at a time. When it fails, it is replaced manually or automatically by a standby unit. This process continues until all $(n-1)$ standby units have been exhausted. For example, consider a cinema hall in a city where power supply is irregular. In order to ensure uninterrupted supply of power apart from the regular source of supply, a generator is kept as standby. The generator is switched on as and when the main supply is resumed. A block diagram of such system is shown in Fig. 1.4.

Gnedenko (1967) classified the standby units as follows:-

...
a) If the off-line unit can fail and is loaded in exactly the same way as the operating unit. It is called the hot standby unit.
b) If the off-line unit can fail and can diminish the load, it is called warm standby unit. The probability of failure for a warm standby is less than the failure of operative unit.
c) If the off-line unit cannot fail and is completely unloaded, it is called cold standby.

Reliability $R(t)$ of an $n$-unit standby system at any time instant $t$ is given as

$$R(t) = \Pr\left[\sum_{i=1}^{n} T_i > t\right]$$

where $T_i$ is the life time of $i^{th}$ unit and all the $n$ units are independent.

### 1.2.6 Transforms and Convolutions

#### 1.2.6.1 Laplace Transform

A transform is merely a mapping or function from one space to another. While it may be very difficult to solve certain equations directly for a particular function of interest, it is often easier to solve a corresponding equation in terms of a transform of the function and then invert the transform to obtain the function. One particular transform, that is, very useful for solving some types of differential equations as well as certain integral equations, is the Laplace transform (L. T.).

Let $f(t)$ be a function of positive real variable $t$. Then the Laplace transform (L. T.) of $f(t)$ is defined as

$$L[f(t)] = f^*(s) = \int_{0}^{\infty} e^{-st} f(t) \, dt.$$

For the range of value of $s$ for which the integral exists. Here $f(t)$ is called an inverse Laplace transform of $f^*(s)$ and we write $f(t) = L^{-1}f^*(s)$. The following are some important properties of Laplace transform:

i) $L\left\{\sum_{i=1}^{n} c_i f_i(t)\right\} = \sum_{i=1}^{n} c_i f_i^*(t)$
ii) \( L[t^0F(t)] = (-1)^0 \frac{d^n \ast f(s)}{ds^n} \)

iii) \( L \int_0^\infty f(u)du \Bigg| = L[F(t)] = \frac{f^*(s)}{s} \)

iv) \( \lim_{t \to 0} F(t) = \lim_{s \to \infty} sf^*(s) \) (initial value theorem)

v) \( \lim_{t \to \infty} F(t) = \lim_{s \to 0} sf^*(s) \) (final value theorem)

vi) \( \lim_{s \to \infty} sf^*(s) = 0 \)

vii) \( \lim_{s \to 0} f^*(s) = 1 \) if \( f^*(s) \) is L.T. of a p.d.f.

\textbf{1.2.6.2 Laplace Stieltjes Transforms}

Let \( X \) be a non-negative random variable with distribution function

\[ F(x) = \Pr [X \leq x] \]

then Laplace Stieltjes transform (L.S.T.) of \( F(x) \) is defined for \( s > 0 \) by

\[ F^\ast\ast(s) = \int_0^\infty e^{-sx} dF(X) \]

under certain regular conditions, we have

\[ F^\ast\ast(s) = s \int_0^\infty e^{-sx} F(X)dX = sF^\ast(s) \text{ and} \]

\[ F^\ast\ast(s) = \int_0^\infty e^{-sx} f(X)dX = f^\ast(s) \]

Where

\[ f(x) = \frac{dF(x)}{dx} \]
1.2.6.3 Convolution

Let \( f(t) \) and \( g(t) \) be two real valued non-negative continuous functions \( t \), then the integral

\[
\int_0^1 f(t-u)g(u)du = \int_0^1 g(t-u)f(u)du
\]

\[= f(t) \odot g(t)\]

\[= L^{-1} [f^*(s) g^*(s)]\]

is called Laplace convolution of the functions \( f(t) \) and \( g(t) \). If \( F(t) \) and \( G(t) \) be two real valued distribution functions defined for \( t \geq 0 \), the resulting convolution is again a distribution and integral

\[
\int_0^1 F(t-u)dG(u) = \int_0^1 G(t-u)dF(u) = F(t)(s)G(t)
\]

is known as Stieltjes convolution of \( F(t) \) and \( G(t) \).

1.2.7 Mean Sojourn Time in a State

The expected time taken by the system in a particular state before transiting to any other state is known as mean sojourn time or mean survival time in that state. If \( T_i \) be the sojourn time in state \( i \), then the mean sojourn time in state \( i \) is

\[\mu_i = \int_0^\infty \Pr(T_i > t)dt\]

1.2.8 First Passage Time

Suppose that a system starts with the state \( j \), then time taken to reach a given state \( k \) for the first time from state \( j \) is called first passage time. In general, first passage time is a measure of how long it takes to reach a given state from another state.
1.2.9 Mean Time to System Failure

It is defined as the expected time for which the system is in operation before it completely fails.

Let $f(t)$ be the probability density function of life time of the system, then we have

$$ MTTF = E(T) = \int_0^\infty f(t) \, dt = \int_0^\infty R(t) \, dt $$

Also

$$ \lim_{s \to 0} R^*(s) = \int_0^\infty R(t) \, dt $$

$$ \Rightarrow MTTF = \lim_{s \to 0} R^*(s) $$

Let $\phi_0(t)$ be the cumulative distribution function of the first passage time from initial state to a failed state, then

$$ R^*(s) = \frac{1 - \phi_0^*(s)}{s} $$

from above equations, we have

$$ MTTF = \lim_{s \to 0} R^*(s) = \lim_{s \to 0} \frac{1 - \phi_0^*(s)}{s} $$

where

$R^*(s)$ and $\phi_0^*(s)$ are respectively the Laplace transform and Laplace Stieltjes transform of $R(t)$ and $\phi_0(t)$.

1.2.10 Availability

Availability is well established in the literature of Stochastic Modeling and optimal maintenance. Barlow and Proschan [1975] define availability of a repairable system as “the probability that the system is operating at a specified time $t$”. While Blanchard [1998] gives a
qualitative definition of availability as “measure of the degree of a system which is in the operable and committable state at the start of mission when the mission is called for at an unknown random point in time.

In telecommunications and reliability theory, the term availability has the following meanings:

The degree to which a system, subsystem, or equipment is operable and in a committable state at the start of a mission, when the mission is called for at an unknown, i.e., a random, time. Simply put, availability is the proportion of time a system is in a functioning condition.

**Representation**

The most simple representation for availability is as a ratio of the expected value of the uptime of a system to the aggregate of the expected values of up and down time, or

\[
A = \frac{E[Uptime]}{E[Uptime] + E[Downtime]}
\]

If we define the status function \(X(t)\) as

\[
X(t) = \begin{cases} 
1, & \text{sys functions at time } t \\
0, & \text{otherwise}
\end{cases}
\]

therefore, the availability is represented by

\[
A(t) = \Pr[X(t) = 1].
\]

\[
A(t) = \Pr[X(t) = 1] = E[X(t)], \ t > 0.
\]

Average availability must be defined on an interval of the real line. If we consider an arbitrary constant \(c\), then average availability is represented as

\[
A_c = \frac{1}{c} \int_0^c A(t)dt \ , \ c > 0
\]
Limiting (or steady-state) availability is represented by

\[ A = \lim_{t \to \infty} A(t) \]

Limiting average availability is defined as also defined on an interval (0, c] as,

\[ A_c = \lim_{c \to \infty} A_c = \lim_{c \to \infty} \frac{1}{c} \int_0^c A(t), \quad c > 0. \]

### 1.2.10.1 Types of Availability

- **Instantaneous Availability**
  
  This is the probability that the system will be able to operate within the tolerances at a given instant of time \( t \) (say). Let this probability be denoted by \( A(t) \).

  Let \( X(t) = 1 \) if the system is operable at time \( t \) and \( X(t) = 0 \) when it is not operable. The availability \( A(t) \) of the system at time \( t \) is given by

  \[ A(t) = \Pr[X(t) = 1 | X(0) = 1] \]

- **Average (Interval) Availability**

  It is the expected fraction of a given interval of time that the system will be able to operate within tolerances.

  Suppose the given interval of time is \( (0, t] \) then interval availability \( H(0, t] \) of this interval is given by:

  \[ H(0, t] = \frac{1}{t} \int_0^t A(u)du = \frac{\mu_{up}(t)}{t} \]

  when \( \mu_{up}(t) = \) expected up time of the system during \( (0, t] \).

- **Steady-state (Limited Interval) Availability**

  The long run or steady-state availability is defined as the proportion of the time during which equipment is available for use.
Mathematically, it is the limiting value of the point wise availability when t become finitely large i.e.,

\[ A = \lim_{t \to \infty} A(t) \]

1.2.11 Maintainability

Maintainability is the probability that the system will be restored to operational effectiveness within a specified time when the maintenance action is taken in accordance with prescribed conditions. Maintenance is one of the effective ways of increasing the reliability of a system. Maintenance action can be classified in several categories: preventive maintenance, corrective maintenance and repair maintenance.

Preventive maintenance includes actions such as lubrication, replacement of a nut or screw of some part of the system, refueling, cleaning, etc. It is designed to minimize the limit that the system will spend in degraded states, it is a sort of repair that is done before a unit actually fails. Corrective maintenance deals with the system performance when it gives wrong result and it involves minor repairs that may creep up between inspections.

Repair maintenance is also concerned with increasing the system availability. In order to increase availability, failed unit upon failure is returned to operation by sending it to a repair facility if available, otherwise waits for repair. There may be two types of repair policies:

- **Repeat repair policy**
  Due to certain reasons the repair of a failed unit has to be stopped. When the repair is begun again, it is started all over again.

- **Resume repair policy**
  The repair of failed component is terminated before completion due to one reason or the other. When it begins again, it is started from the stage where it was prior to the termination of repair.
Fig. 1.2. Series configuration

Fig. 1.3. Parallel configuration
Fig. 1.4 Standby redundant configuration

Fig. 1.5. Series parallel configuration
1.2.12 Busy Period of the Server

Let $B(t)$ be the probability that a server is busy with the system in the interval $(0, t)$, then in the long run the total fraction of time for which a server is busy is given by
\[ B = \lim_{t \to \infty} B(t) \]

### 1.2.13 Expected Number of Visits by the Server

Let \( N(t) \) be a random variable representing the number of times, the server has visited the system in the interval \((0, t]\), then the expected number of visits by the server to the system in \((0, t]\), is \( E[N(t)] \) and in the long run this number per unit time is given by

\[ N = \lim_{t \to \infty} \frac{E[N(t)]}{t} \]

### 1.2.14 Profit Analysis

Today the main objective of most of the industrialists is to earn maximum by selling their products with minimum investment on production. In fact, no organization can survive for long without minimum financial returns for its investment. And, so there must be an optimal balance between the reliability aspect of a product and its cost. This can be achieved by contributing major cost to research and development, production, spares and maintenance. How the cost of these individual items varies with reliability is shown in Fig. 1.9. In order to increase the reliability of the products, we would require a correspondingly high investment in the research and development activities. The production cost also would increase with the requirement of greater reliability.

The revenue and cost functions lead to the profit function of a firm, as the profit is excess of revenue over the cost of production. The profit function in time \( t \) is given by

\[ P(t) = \text{Expected revenue in } (0, t] - \text{expected total cost in } (0, t] \]

In general, the optimal policies can more easily be derived for an infinite time span or compared to a finite time span. The profit per unit time, in infinite time span is expressed as

\[ \lim_{t \to \infty} \frac{P(t)}{t} \]
i.e. profit per unit time = total revenue per unit time – total cost per unit time. Considering the various costs, the profit equation is given as

\[ P = K_0 A_0 - K_1 B_0 - K_2 N_0 \]

where

- \( P \) = Profit per unit time incurred to the system
- \( K_1 \) = Revenue per unit up time of the system
- \( A_0 \) = Total fraction of time for which the system is up
- \( K_2 \) = Cost per unit time for which server is busy
- \( B_0 \) = Total fraction of time for which the server is busy
\( K_3 \) = Cost per unit visit by the server

\( N_0 \) = Expected number of visits per unit time for the server

### 1.2.15 Distributions

In the concept of reliability theory, the failure times, the repair times, inspection times, waiting time, etc. are random variables. A random variable is completely characterized by the distribution function which is defined in terms of probability parameters, the distribution function \( F(x) \) is defined as:

\[
F(x) = P[X < x] = \int_0^x f(t) \, dt, \quad X \geq 0
\]

Here \( f(t) \) is known as the probability density function.

#### 1.2.15.1 Exponential Distribution

In probability theory and statistics, the exponential distributions are a class of continuous probability distribution. They are often used to model the time between independent events that happen at a constant average rate.

**Characterization**

**Probability density function**

The probability density function (pdf) of an exponential distribution has the form

\[
F(x; \lambda) = \begin{cases} 
\lambda e^{-\lambda x}, & x \geq 0, \\
0, & 0 < 0.
\end{cases}
\]

Where \( \lambda > 0 \) is a parameter of the distribution, often called the rate parameter. The distribution is supported on the interval \([0, \infty)\). If a random variable \( X \) has this distribution, we write \( X \sim \text{Exponential}(\lambda) \).
Cumulative distribution function

The cumulative distribution function is given by

\[
F(x;\lambda) = \begin{cases} 
1 - e^{-\lambda x}, & x \geq 0, \\
0, & x < 0. 
\end{cases}
\]

Occurrence and applications

The exponential distribution is used to model Poisson processes, which are situations in which an object initially in state A can change to state B with constant probability per unit time \( \lambda \). The time at which the state actually changes is described by an exponential random variable with parameter \( \lambda \). Therefore, the integral from 0 to T over f is the profitability that the object is in state B at time T.

The exponential distribution may be viewed as a continuous counterpart of the geometric distribution, which describes the number of Bernoulli trials necessary for a discrete process to change state. In contrast, the exponential distribution describes the time for a continuous process to change state.

In real-world scenarios, the assumption of a constant rate (or probability per unit time) is rarely satisfied. For example, the rate of incoming phone calls differs according to the time of day. But if we focus on a time interval during which the rate is roughly constant, such as from 2 to 4 p.m. during work days, the exponential distribution can be used as a good approximate model for the time until the next phone call arrives. Similar caveats apply to the following examples which yield approximately exponentially distributed variables:

- The time until a radioactive particle decays, or the time between beeps of a Geiger counter;
- The number of dice rolls needed until you roll a six 11 times in a row;
• The time it takes before your next telephone call
• The time until default (on payment to company debt holders) in reduced form credit risk modeling
• Exponential variables can also be used to model situations where certain events occur with a constant probability per unit distance:
  • The distance between mutations on a DNA strand;
  • The distance between roadkill on a given street;

Reliability theory and reliability engineering also make extensive use of the exponential distribution. Because of the memoryless property of this distribution, it is well-suited to model the constant hazard rate portion of the bathtub curve used in reliability theory. It is also very convenient because it is so easy to add failure rates in a reliability model. The exponential distribution is however not appropriate to model the overall lifetime of organisms or technical devices, because the “failure rates” here are not constant: more failures occur for very young and for very old systems.

Properties

Mean and variance

The mean or expected value of an exponentially distributed random variable $X$ with rate parameter $\lambda$ is given by

$$E[X] = \frac{1}{\lambda}$$

In light of the examples given above, this makes sense: if you receive phone calls at an average rate of 2 per hour, then you can expect to wait half an hour for every call.

The variance of $X$ is given by

$$\text{Var}[X] = \frac{1}{\lambda^2}.$$
Memorylessness

An important property of the exponential distribution is that it is memoryless. This means that if a random variable $T$ is exponentially distributed, its conditional probability obeys

$$P(T > s + t | T > s) = P(T > t) \text{ for all } s, t \geq 0.$$ 

This says that the conditional probability that we need to wait, for example, more than another 10 seconds before the first arrival, given that the first arrival has not yet happened after 30 seconds, is no different from the initial probability that we need to wait more than 10 seconds for the first arrival. This is often misunderstood by students taking courses on probability: the fact that $P(T>40 | T>30) = P(T>10)$ does not mean that the events $T>40$ and $T>30$ are independent. To summarize: “memorylessness” of the probability distribution of the waiting time $T$ until the first arrival means

(Right) $P(T>40 | T>30) = P(T>10)$.

The exponential distributions and the geometric distributions are the only memory less probability distributions. The exponential distribution also has a constant hazard function.

Exponential distribution plays an important role in reliability theory. Besides a number of mathematical properties it has a very important memory less property. For example an electric fuse (assuming it can not melt partially) whose failure life distribution is practically unchanged as long as it has not yet failed.

1.2.15.2 Weibull Distribution

In probability theory and statistics, the Weibull distribution is a continuous probability distribution.

The probability density function of a Weibull random variable $x$ is:

$$f(x; \lambda, k) = \begin{cases} \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k} & x \geq 0, \\ 0 & x < 0, \end{cases}$$
where \( k > 0 \) is the shape parameter and \( \lambda > 0 \) is the scale parameter of the distribution.

While it’s complementary cumulative distribution function is a stretched exponential function given as

\[
F(x; k, \lambda) = \begin{cases} 
1 - e^{\left(\frac{x}{\lambda}\right)^k} & \text{for } x \geq 0 \\
0 & \text{for } x < 0
\end{cases}
\]

![Figure 1.10](image)

The Weibull distribution interpolates between the exponential distribution \((k = 1)\) and the Rayleigh distribution \((k = 2)\). If the quantity \( x \) is a "time-to-failure", the Weibull distribution gives a distribution for which the failure rate is proportional to a power of time. The shape parameter, \( k \), is that power plus one, and so this parameter can be interpreted directly as follows:

- A value of \( k<1 \) indicates that the failure rate decreases over time. This happens if there is significant "infant mortality", or defective items failing early and the failure rate decreasing over time as the defective items are weeded out of the population.
- A value of \( k=1 \) indicates that the failure rate is constant over time. This might suggest random external events are causing mortality, or failure.
A value of $k>1$ indicates that the failure rate increases with time. This happens if there is an "aging" process, or parts that are more likely to fail as time passes on.

In the field of materials science, the shape parameter $k$ of a distribution of strengths is known as the Weibull modulus.

**Application of the Weibull Distribution**

The Weibull distribution is one of the most widely used lifetime distributions in reliability engineering. It is a versatile distribution that can take on the characteristics of other types of distributions, based on the value of the shape parameter $\lambda$. The most important application of the Weibull distribution can be seen in the following areas:

- In survival analysis
- In reliability engineering and failure analysis
- In industrial engineering to represent manufacturing and delivery times
- In extreme value theory
- In weather forecasting
- To describe wind speed distributions, as the natural distribution often matches the Weibull shape

**1.2.16 Stochastic Process**

A stochastic process is a family of random variables $\{X(t) \mid t \in \mathbb{T}\}$, defined on a given probability space, indexed by the parameter $t$, where $t$ varies over an index set $\mathbb{T}$. Both the parametric set and state space can be independently either discrete or continuous.

In stochastic process $\{X(t), t \in \mathbb{T}\}$, where $X(t)$, $t$ and $\mathbb{T}$ respectively, the state space, parameter (generally taken to be time) and the index set if $\mathbb{T}$ is a countable set as $\mathbb{T}=\{0,1,2,3,\ldots\}$, then the stochastic process is said to be a discrete parameter process and if $\mathbb{T}=\{t : -\infty < t < \infty\}$, the stochastic process is said to be a continuous parametric process. The
state space is classified as discrete if it is finite or countable and continuous if it consists of an interval on the real line. In the present study, we have only dealt with discrete state space continuous time parameter stochastic processes.

1.2.16.1 Markov Process

If \( \{X(t), t \in T\} \) is a stochastic process such that given the value of \( X(s) \), the value of \( X(t) \), \( t > s \) do not depend on the values of \( X(u) \), \( u < s \) i.e. for \( t > s \), \( i \in s \).

\[
\Pr \{X(t) = i | X(u), 0 < u < s\} = \Pr \{X(t) = i | X(s)\}
\]

Then the process \( \{X(t), t \in T\} \) is a Markov process.

1.2.16.2 Markov Chain

A discrete parameter Markov process is known as a Markov Chain. The stochastic process \( \{X_n, n = 0, 1, 2, \ldots\} \) is called a Markov chain, if, for \( j, k, j_1, j_2, \ldots, j_{n-1} \in \mathbb{N} \),

\[
\Pr \{X_n = k | X_{n-1} = j, X_{n-2} = j_1, \ldots, X_0 = j_{n-1}\} = \Pr \{X_n = k | X_{n-1} = j\} = p_{jk} \text{ (say)}
\]

The conditional probability \( p_{jk} \) is called transition probability from the state \( j \) at \( n \)th trial to the state \( k \) at \( (n+1) \)th trial. If the transition probability \( p_{jk} \) is independent of \( n \), the Markov Chain is said to be homogenous; and if it is dependent of \( n \), the chain is said to be non-homogeneous.

1.2.16.3 Semi-Markov Process

In the above, assume that the process is time homogeneous, i.e.

\[
\Pr \{X_{n+1} = j, t_{n+1} - t_n \leq t | X_n = i\} = Q_{ij}(t), i, j \in s, \text{ is independent of } n, \text{ then there exist limiting transition probabilities.}
\]

\[
p_{ij} = \lim_{t \to \infty} Q_{ij}(t) = \Pr \{X_{n+1} = j | X_n = i\}
\]
then \( \{X_n, \ n = 0, 1, 2, \ldots\} \) constitute a Markov Chain with state space \( E \) and transition probability matrix (t.p.m.)

\[
P = [p_{ij}]
\]

The continuous parameter stochastic process \( Y(t) \) with state space \( E \) defined by

\[
Y(t) = X_n, \ t_n < t < t_{n+1}
\]

is called a semi-Markov process. The Markov Chain \( X_n \) is said to be an embedded Markov chain of the semi-Markov process.

In other words, we define the semi-Markov process as a process in which transition from one state to another is governed by the transition probabilities of a Markov process but the time spent in each state before a transition occurs, is a random variable depending upon the last transition made. Thus at transition instants the semi-Markov process behaves just like a Markov process. However, the times at which transitions occur are governed by a different probability mechanism.

### 1.2.16.4 Regenerative Process

Regenerative stochastic process was defined by Smith (1955) and has been crucial in the analysis of complex systems. In this, we take time points at which the system history prior to the time points is irrelevant to the system conditions. These points are called regenerative points. Let \( X(t) \) be the state of the system of epoch. If \( t_1, t_2, \ldots \) are the epochs at which the process probabilistically restarts, then these epochs are called regenerative epochs and the process \( \{X(t), \ t = t_1, t_2, \ldots\} \) is called regenerative process.

### 1.3 Thesis at a Glance

It is observed that continued operation and ageing of operable systems reduce their performance, reliability and safety. And, the method of redundancy has been proved as one of the effective techniques for improving performance and reliability of these systems without changing the reliability of individual component that forms the system. Also, the deterioration
process of operable systems can be slow down by conducting preventive maintenance after a pre-
specific time ‘t’. Cold standby systems are one of the most important structures in reliability
engineering and have been widely applied in industries. Hence in the present thesis entitled
“Profit Analysis of System Reliability Models with Preventive Maintenance and Repair”
some reliability models of a cold standby system of two identical units are developed and
analyzed stochastically in detail by considering different repair policies. The expressions for
several reliability and economic measures are derived in steady state by adopting semi-Markov
process and regenerative point technique. The graphical behavior of some important measures of
system effectiveness has been shown for particular values of various parameters and costs. The
profit comparison of the system models has also been made to know the usefulness of different
repair policies.

The research work reported in this thesis is covered into seven chapters which are
described as:

Chapter 1: Introduction and Review of Literature

The first chapter of the thesis is introductory in nature which contains origin,
development of the subject and a brief review of literature. The basic concepts like reliability,
reliability engineering, failure rate, system configurations, probability distributions, Laplace
Stieltjes Transform, Laplace Transform, convolution, stochastic process, reliability measures and
profit function are described in brief to update knowledge about the subject. Structure of the
thesis is given at the end of the chapter along with main findings of the study.

Chapter 2: Profit Analysis of a Cold Standby System Reliability Model

with Preventive Maintenance and Repair

The profit analysis of a cold standby system considering the concepts of preventive
maintenance and repair has been carried out in this chapter. There are two identical units in
the system which may fail directly from normal mode. A single server is provided immediately
to conduct different repair activities. The operative unit undergoes for preventive maintenance
after a pre-specific time ‘t’ up to which no failure occurs. However, repair of the unit is done at
its failure. The unit works as new after repair and preventive maintenance. The random variables are statistically independent. The failure time of the unit and the time by which unit undergoes for preventive maintenance are exponentially distributed while the distributions of preventive maintenance and repair times are assumed as arbitrary with different probability density functions. The expressions for some more reliability measures such as mean sojourn times, busy period of the server due to preventive maintenance and repair, expected number of visits by the server, expected number of preventive maintenances and repairs have been derived in steady state using semi-Markov process and regenerative point technique. Graphs for mean time to system failure (MTSF), availability and profit function are drawn with respect to preventive maintenance rate to know the usefulness of different repair policies.

Chapter 3: Profit Analysis of a Cold-Standby System Reliability Model

with Priority to Repair over Preventive

In this chapter, a reliability model for a two-unit cold standby system is developed considering the idea of priority in repair disciplines. The units are identical in nature which may fail directly from normal mode. There is a single server who visits the system immediately to carry out repair activities. Server conducts preventive maintenance of the unit after a maximum operation time ‘t’. However, repair of the unit is done at its complete failure. Priority is given to repair of one unit over maintenance of the other unit. The random variables associated with failure time, completion of maximum operation time, preventive maintenance and repair times are statistically independent. The failure time and time by which unit goes for preventive maintenance follow exponential distribution while the distributions for maintenance and repair times are taken as arbitrary. The reliability measures of the system model as mentioned in the previous chapter are obtained for particular values of various parameters and costs to depict their graphical behavior. Profit comparison of the present model has been made with that of the model discussed in 2nd chapter.

Chapter 4: Profit Analysis of a Cold-Standby System Reliability Model
with Priority to Preventive Maintenance over Repair

The motive of the present chapter is to analyze a cold standby system by giving priority to preventive maintenance over repair. Two identical units are taken having two modes—operative and complete failure. There is a single server who visits the system immediately for conducting maintenance and repair. Server conducts preventive maintenance of the unit after a maximum operation time ‘t’. However, repair of the unit is done at its complete failure. Priority is given to preventive maintenance of one unit over repair of the other unit. The random variables associated with failure time, completion of maximum operation time, preventive maintenance and repair times are statistically independent. The failure time of the unit and the time by which unit undergoes for preventive maintenance and repair times are taken as arbitrary. Several measures of system effectiveness such as mentioned in the previous chapter are obtained by using semi-Markov process and regenerative point technique. Graphs are drawn to depict the behavior of MTSF, availability and profit function for particular values of various parameters and costs. The profit of the present model is also compared with that of the models discussed in chapters 2nd and 3rd.

Chapter 5: Profit Analysis of a Cold Standby System Reliability Model

with Preventive Maintenance and Inspection for Feasibility of Repair

While considering the practical utility of preventive maintenance and inspection in repairable systems, here a cold standby system of two-identical units has been studied stochastically in detail. Each unit has two modes—normal and complete failure. There is a single server who visits the system immediately to perform different repair activities. The server conducts preventive maintenance of the operative unit after a pre-specific time ‘t’. The failed unit undergoes for inspection to see the feasibility of its repair. If repair of the unit is not feasible, it is replaced immediately by new-one. The random variables are statistically independent. The unit works as new after preventive maintenance and repair. The failure time
and time by which unit undergoes for preventive maintenance follow negative exponential distribution while the distributions for inspection and repair times are taken as arbitrary with different probability density functions.

The semi-Markov process and regenerative technique are adopted to derive the expressions for several reliability measures such as transition probabilities, mean sojourn times, mean time to system failure (MTSF), availability, busy period of the server due to preventive maintenance, inspection and repair, expected number of preventive maintenances, inspections and repairs of the unit and profit function. Graphs are drawn to depict the behavior of MTSF, availability and profit function for a particular case.

Chapter 6: Profit Analysis of a Cold Standby System Reliability Model

with Priority to Preventive Maintenance over Repair

Subject to Inspection

The present chapter is devoted to the profit analysis of a cold standby system of two identical units each having two modes - operative and complete failure. A single server is provided immediately to carry out different repair activities. The unit undergoes for preventive maintenance after a maximum operation time. However, inspection of the unit is done at its failure to see the feasibility of repair. If the repair of the unit is not feasible, it is replaced immediately by new one in order to avoid unnecessary expanses on repair. The unit works as new after preventive maintenance and repair. Priority is given to the preventive maintenance over repair.

The random variables are statistically independent. The failure time and time by which unit goes for preventive maintenance follow exponential distribution while the distributions for
preventive maintenance, inspection and repair times are taken as arbitrary. Several measures of system effectiveness such as mentioned in the previous chapter are obtained in steady state using semi-Markov process and regenerative point technique. Graphs are drawn to depict the behavior of MTSF, availability and profit function for particular values of various parameters and costs. The profit of the present model has also been compared with the model discussed in 5th chapter.

Chapter 7: Profit Analysis of a Cold Standby System Reliability Model with Priority to Preventive Maintenance over Inspection

In chapter 5th, reliability model of a cold standby system has been examined stochastically in detail by conducting inspection of the failed unit to see the feasibility of repair. And, the same system model has been discussed in chapter 6th by giving priority to preventive maintenance over repair subject to inspection. The findings revealed that such an idea of priority is useful in making the system more profitable. But sometimes a system has complex faults which cannot be identified through inspection in a specific time and in such a situation priority may be given to some other repair activity in order to reduce down time of the system under techno-economic constraints. Thus the study in the present chapter is concentrated on the profit analysis of a two-unit cold standby system with priority to preventive maintenance over inspection. Each unit of the system has two modes- operative and complete failure. There is a single server who conducts repair activities such as preventive maintenance, repair, replacement and inspection of the unit as and when needed. The preventive maintenance of the unit is conducted after a specific time of operation. Server inspects the unit at its failure to see the feasibility of repair. The failed unit is replaced immediately by new one in case its repair is not feasible to the system. Priority is given to preventive maintenance of one unit over inspection of the other failed unit. The repair activities and switch devices are prefect.

The random variables associated with failure time, the time by which unit undergoes for preventive maintenance, preventive maintenance and repair times are statistically
independent. The failure time and time by which unit goes for preventive maintenance follow exponential distribution while the distributions for maintenance and repair times are taken as arbitrary. Several measures of system effectiveness as mentioned in the previous chapter are obtained by adopting semi-Markov process and regenerative point technique. Graphs are drawn to depict the behavior of MTSF, availability and profit function for particular values of various parameters and costs. The profit comparison of the present model has been made with the models discussed in chapters 5th and 6th.

1.4 Main Findings of the Study

The research work reported in the thesis has been concentrated on the profit analysis of a cold standby system of two identical units under different preventive maintenance and repair policies. The expressions for several reliability measures of the system models have been obtained in steady state. The results indicate that

- The reliability measures go on increasing with the increase of preventive maintenance and repair rates.
- The reliability measures decline with the increase of failure rate and the rate by which unit undergoes for preventive maintenance.
- The effect of preventive maintenance rate on reliability measures is more than that of the repair rate.
- The system becomes more profitable by giving priority to repair over preventive maintenance. However, the concept of priority to preventive maintenance over repair is not much economically beneficial.
- The performance of the system can be improved by conducting inspection of the failed unit to see the feasibility of repair.
The concept of priority to repair over preventive maintenance is not much economically beneficial in case inspection of the failed unit is conducted to see the feasibility of repair.

The idea of priority to preventive maintenance over repair is more useful in making the system profitable as compared the idea of giving priority to preventive maintenance over inspection provided inspection of the failed unit is conducted to see the feasibility of repair.