Chapter 3

Gravitational Waves Spectrum in Thermal Vacuum State

The spectrum of relic GWs depends not only on the details of expansion during the inflationary era but also on the subsequent stages, including the current epoch of the universe. There are variety of sources that responsible are for generation of these waves including from the dynamics of early universe to massive astrophysical objects such as neutron star binaries and black hole mergers etc. Thus the waves have a wide spectrum of frequencies i.e, the frequency vary from very low to high ($\mathcal{O}(10^{-19} - 10^{10})$ Hz). It is possible to discriminate the relic waves from other sources on the observational point of view also. It is believed that the relic GWs are mainly generated during inflationary epoch. And the waves that amplified during the inflation are low frequency only. Since the higher frequency waves are outside the barrier the corresponding amplitudes decreased during the evolution of the universe. But these high frequency waves can re-enter into the horizon of the universe again as the universe expanded further enough. Thus these waves can also contribute to the amplitude of the present GWs. The standard inflationary models are energetically not in favor of generating thermal GWs but the existence of GWs with thermal features is not ruled out completely. However, the inflationary stage probably generated the lower frequency thermal GWS due to the stimulated emission process [9]. The pre-inflationary
period of the universe [10] and theories with higher dimensions [11] also predicts the existence of thermal GWs [12, 13]. There exist other alternative scenarios and candidate like the evaporation of mini black holes in the early universe, are also predict the existence of thermal GWs. In the present study, we mainly consider the thermal GWs that were probably generated in the pre-inflationary era, during the inflationary era due to stimulated emission mechanism and from the extra dimensional effects in the early universe. In this chapter, we consider the GWs in thermal vacuum states and study its contribution to the amplitude and energy density in the decelerated as well as accelerated FLRW universe. The contribution of the thermal GWs on its amplitude and spectral energy density due to lower and higher frequency is studied separately. A brief account of the plausible scenarios of generation of the thermal GWs and the properties of thermal state are discussed briefly. Computed contribution of the high frequency thermal GWs, due to the extra dimensional effect to its amplitude can be compared with the sensitivity of various missions to detect GWs.

3.1 GWs Spectrum in Expanding Universe

The approximate computation of the spectrum of the GWs is usually performed in two limiting cases depending up on the waves that are within or outside of the barrier. For the GWs outside barrier \( n^2 \gg S''/S \), short wave approximation) the corresponding amplitude decrease as \( h_n \propto 1/S(\eta) \) while for the waves inside the barrier \( n^2 \ll S''/S \), long wave approximation), \( h_n = C_n \) simply a constant. Thus these results can be used to estimate the spectrum for the present epoch of the universe.

The history of overall expansion of the universe is modeled as following sequence of successive epochs of power-law expansion [23].

The initial stage (inflationary)

\[
S(\eta) = l_0 |\eta|^{1+\beta}, \quad -\infty < \eta \leq \eta_1,
\]
where $1 + \beta < 0$, $\eta < 0$ and $l_0$ is a constant.

The z-stage or reheating stage

$$S(\eta) = S_z(\eta - \eta_p)^{1+\beta_s}, \quad \eta_1 < \eta \leq \eta_s,$$

where $1 + \beta_s > 0$. This z-stage is introduced to allow a general reheating epoch [34].

The radiation-dominated stage

$$S(\eta) = S_e(\eta - \eta_e), \quad \eta_s \leq \eta \leq \eta_2,$$

The matter-dominated stage

$$S(\eta) = S_m(\eta - \eta_m)^2, \quad \eta_2 \leq \eta \leq \eta_E,$$

where “e” and “m” subscripts are for the radiation and matter dominated stages and $\eta_E$ is the time when the dark energy density $\rho_\Lambda$ is equal to the matter energy density $\rho_m$. Before the discovery of accelerating expansion of the universe, the current expansion was taken to be as decelerating one because of the matter-domination. Thus, following the matter-dominated stage, it is reasonable to add an epoch of accelerating stage, which is probably driven by either the cosmological constant, or the quintessence, or some other kind of condensate [35]. The value of redshift $z_E$ at the time $\eta_E$ is $(1 + z_E) = S(\eta_0)/S(\eta_E)$, where $\eta_0$ is the present time. Since $\rho_\Lambda$ is constant and $\rho_m(\eta) \propto S^{-3}(\eta)$, we get

$$\frac{\rho_\Lambda}{\rho_m(\eta_E)} = \frac{\rho_\Lambda}{\rho_m(\eta_0)(1 + z_E)^3} = 1.$$ (3.5)

If the current value of $\Omega_\Lambda \sim 0.7$ and $\Omega_m \sim 0.3$, then it follows that

$$1 + z_E = \left(\frac{\Omega_\Lambda}{\Omega_m}\right)^{1/3} \sim 1.33.$$ (3.6)

The accelerating stage (up to the present)

$$S(\eta) = \ell_0 |\eta - \eta_a|^{-1}, \quad \eta_E \leq \eta \leq \eta_0.$$ (3.7)
This stage describes the accelerating expansion of the universe, which is a new feature and hence its influence on the spectrum of relic GWs is of interest to study. It is be noted that the actual scale factor function $S(\eta)$ differs from equation (3.7), since the matter component exists in the current universe. However, the dark energy is dominant, therefore eq.(3.7) is an approximation to the current expansion behaviour.

Given $S(\eta)$ for the various epochs, the derivative $S' = dS/d\eta$ and the ratio $S'/S$ follow immediately. Except for $\beta_s$ which is imposed upon as the model parameter, there are ten constants in the expressions of $S(\eta)$. By the continuity conditions of $S(\eta)$ and $S'(\eta)$ at $\eta_1, \eta_s, \eta_2,$ and $\eta_E$, one can fix only eight constants. The remaining two constants can be fixed by the overall normalization of $S$ and the observed Hubble constant as the expansion rate. Specifically, we put $|\eta_0 - \eta_a| = 1$ for the normalization of $S$, which fixes the $\eta_a$, and the constant $\ell_0$ is fixed by the following calculation,

$$\frac{1}{H} \equiv \left(\frac{S'^2}{S}\right)_{\eta_0} = \ell_0, \quad (3.8)$$

where $\ell_0$ is the Hubble radius at present.

In the expanding FLRW spacetime the physical wavelength is related to the comoving wave number as $\lambda \equiv 2\pi S(\eta)/n$, and the wave number $n_0$ corresponding to the present Hubble radius is $n_0 = 2\pi S(\eta_0)/\ell_0 = 2\pi$. And there is another wave number $n_E = 2\pi S(\eta_E)/(1/H) = n_0/(1 + z_E)$, whose corresponding wavelength at the time $\eta_E$ is the Hubble radius $1/H$.

By matching $S$ and $S'/S$ at the joint points of successive evolutionary stages of the universe, one gets [36]

$$l_0 = \ell_0 b \zeta_E (2+\beta) \zeta_2 \zeta_s \zeta_1^{\frac{\beta-1}{2}} \zeta_1^{\frac{-\beta-1}{2}}, \quad (3.9)$$

where $b \equiv |1 + \beta|^{-(2+\beta)}$,

$$\zeta_E \equiv \frac{S(\eta_0)}{S(\eta_E)}, \quad \zeta_2 \equiv \frac{S(\eta_E)}{S(\eta_2)}, \quad \zeta_s \equiv \frac{S(\eta_2)}{S(\eta_s)}.$$
and
\[ \zeta_1 \equiv \frac{S(\eta_s)}{S(\eta_1)}. \] (3.10)

With these specifications, the functions \( S(\eta) \) and \( S'(\eta)/S(\eta) \) are fully determined. In particular, \( S'(\eta)/S(\eta) \) rises up during the accelerating stage, instead of decreasing as in the matter-dominated stage. This causes the modifications to the spectrum of relic GWs.

### 3.2 Thermal Background of Relic GWs

Recently, the thermal gravitational waves received much attention in gravitational waves astronomy and cosmology. There are theories such as extra dimensional scenario, universe without inflation, evaporation of primordial black holes and Dirac hypothesis, etc; predict the existence of thermal GWs. In this section, we provide a brief account of various scenarios of thermal GWs.

#### 3.2.1 High energy scale and thermal GWs

One scenario is based on the assumption that the universe underwent a radiation dominated stage of expansion prior to the inflation-like acceleration phase [37]. Furthermore, it is assumed that, during this stage the temperature higher than \( \sim 10^{19} \) GeV, a thermal equilibrium between the various components, including gravitons, is maintained through gravitational interaction. In this scenario, as the universe cooled down further and the gravitons decoupled, a background of thermal relic gravitons would be left behind [38, 39].

Assume that the universe was radiation dominated before the inflationary period, and that all the particle species were highly relativistic. The interaction rate \( \Gamma \) for particles interacting solely through the gravitational force

\[ \Gamma \simeq T^5/m_{\text{pl}}^4, \] (3.11)
where \( T \) is the physical temperature in the universe [38, 40]. If the gravitons be in thermal equilibrium with the other particles were possible only provided the interaction rate is large in comparison with the expansion rate of the universe characterized by Hubble parameter

\[
H \simeq T^2/m_{pl}.
\]  

The gravitons would remain in thermal equilibrium with other particles, however, this equilibrium gets violated once \( \Gamma \lesssim H \). Thus the gravitons would decouple from the other particle species, leaving behind a free-streaming thermal graviton background.

During the expansion of universe, these GWs background would preserve its thermal spectrum, but redshifted to very low temperatures [39, 41]. To discuss further, it is convenient to consider history of the universe into three stages (a) initial radiation dominated stage \((i)\), (b) inflationary stage \((inf)\) and (c) post-inflationary stage \((p)\). Thus the present temperature \( T_0 \) of the GWs background is

\[
\frac{T_0}{m_{pl}} \simeq \frac{S_i}{S_{inf}} \times \frac{S_{inf}}{S_p} \times \frac{S_p}{S_0},
\]

where \( S_i, S_{inf}, S_p \) and \( S_0 \) are the values of the scale factors at the time of graviton decoupling, the beginning of the inflationary stage, the end of inflation and the present universe, respectively.

The temperature of the GWs decreases with the expansion. On the other hand, at the end of inflation the temperature of the thermal bath, containing the rest of the particles is significantly boosted by the process of reheating. Assuming that the observed CMB is the relic of this thermal bath, its temperature \( T_2 \) at the beginning of the post-inflationary stage can be related to value of the scale factor at this stage through the relation given by [38]

\[
\frac{S_p}{S_0} = (2.37K_{dec}/T_2)(3.91/106.75)^{1/3}.
\]  

Using this result, and denoting the temperature of the thermal bath at the
beginning of the inflationary stage as $T_1$, eq.(3.13) can be rewritten as

$$T_0 \simeq 8.0 \times 10^{-27} (T_1/T_2) e^{60-N} K_{elv},$$

(3.15)

where $N \equiv \log(S_p/S_{inf})$ is number of e-folds during the inflationary stage. This spectrum is peaked at the frequency\(^1\)

$$\nu \simeq 4.7 \times 10^{-16} (T_1/T_2) e^{60-N} \text{ Hz.}$$

(3.16)

The peak frequency depends on the value of the number of e-folds during inflation, and the ratio of the temperatures $T_1$ and $T_2$.

3.2.2 Extra dimensional scenario and thermal GWs

Cosmology with extra dimensions have been motivated since Kaluza and Klein (KK) showed that classical electromagnetism and general relativity could be combined in a five-dimensional framework [11]. The modern scenarios involving extra dimensions are being explored in particle physics, gravity and cosmology. Although there exist different models of extra dimensions, there are some general features and signals common to all of them.

In presence of $D$ extra spatial dimensions, the 3+D+1-dimensional action for gravity can be written as

$$S = \int d^4x \left[ \int d^Dy \sqrt{-g_D} \frac{R_D}{16\pi G_D} + \sqrt{-g} L_m \right],$$

(3.17)

where

$$G_D = G \frac{m_{pl}^2}{m_D^{2+D}},$$

(3.18)

and $g$ is the determinant of four dimensional metric, $G$ is Newton’s constant, $g_D, G_D$ and $R_D$ denote the higher dimensional counter parts of the determinant of metric, Newton’s constant, and the Ricci scalar, respectively. And $m_D$ is the fundamental scale of the extra dimension.

\(^1\)Here onwards $\nu$ means frequency except in Chapter 4.
Since the gravitational interactions are not strong enough to produce a thermal gravitons at temperatures below the Planck scale \( m_{pl} \sim 1.22 \times 10^{19} \) GeV, the standard inflationary cosmology predicted the existence of the cosmic GWs background, which are non-thermal in nature. However, if the universe contains extra dimensions which are favorable to generate thermal GWs, this can happen when energies in the universe are higher than the fundamental scale \( m_D \), the gravitational coupling strength increases significantly, as the gravitational field spreads out into the full spatial volume. Instead of freezing out at \( \sim O(m_{pl}) \), as in 3+1 dimensions, gravitational interactions freeze-out at \( \sim O(m_D) \). If the gravitational interactions become strong at an energy scale below the reheating temperature \( m_D < T_{RH} \), gravitons get the opportunity to thermalize, creating a thermal GWs background. The qualitative result, the creation of a thermal GWs background if \( m_D < T_{RH} \), is unchanged by the type of extra dimensions chosen [13].

Thus, if extra dimensions do exist, and the fundamental scale of those dimensions is below the reheat temperature, a relic thermal GWs background ought to exist today. Compared to the relic thermal photon background, a thermal GWs background would have the same statistics, and high degree of isotropy and homogeneity.

The energy density \( \rho_g \) and fractional energy density \( \Omega_g \) of a thermal GWs background are

\[
\rho_g = \frac{\pi^2}{15} \left(\frac{3.91}{g_*}\right)^{4/3} T_{CMB}^4, \quad (3.19)
\]

\[
\Omega_g = \frac{\rho_g}{\rho_c} \simeq 3.1 \times 10^{-4} (g_*)^{4/3}, \quad (3.20)
\]

where \( \rho_c \) is the critical energy density of the universe, \( T_{CMB} \) is the present temperature of the CMB, and \( g_* \) is the number of relativistic degrees of freedom at the scale of \( m_D \). The \( g_* \) is dependent on the particle content of the universe, i.e., whether (and at what scale) the universe is supersymmetric, has a KK tower, etc. Other quantities, such as the temperature \( T \), peak
frequency ($\nu$), number density ($n$), and entropy density ($s$) of the thermal GWs background can be derived from the CMB if $g_*$ is known, as

$$n_g = n_{CMB} \left( \frac{3.91}{g_*} \right), \quad s_g = s_{CMB} \left( \frac{3.91}{g_*} \right),$$

(3.21)

$$T_g = T_{CMB} \left( \frac{3.91}{g_*} \right)^{1/3}, \quad \nu_g = \nu_{CMB} \left( \frac{3.91}{g_*} \right)^{1/3}. \quad (3.22)$$

These quantities are not dependent on the number of extra dimensions. But if $m_D$ is just barely above the scale of the standard model, then $g_* = 106.75$. Then the thermal GWs background has a temperature of 0.905 Kelvin, with a peak frequency of 19 GHz [13].

### 3.2.3 Alternative possibilities of thermal GWs

Currently, there are alternative scenarios that would also create a thermal cosmic GWs background and we here discuss some of them briefly.

It is believed that mini black holes existed in the early universe known as primordial black holes. The primordial black holes with masses less than $10^{15}$ g would have decayed by today, producing thermal photons, gravitons, and other forms of radiation. In order to create a large mass fraction of low-mass primordial black holes but not high-mass ones, the spectral index $n'$ of the density fluctuations in the early universe is less than or equal to $2/3$ [42]. But the observed scale-invariant spectrum suggest that $n' \simeq 1$ for the density fluctuations [43]. Therefore the primordial black holes as a reasonable candidate for creating a thermal cosmic GWs background may be ruled out.

The other alternative scenario is due to the Dirac hypothesis. According to this hypothesis the difference in magnitude between the gravitational and electromagnetic coupling strengths arises due their time dependent nature [44]. Thus, the gravitational coupling would have been stronger in the early universe and hence created a thermal GWs at that epoch. But the cosmological models based on these hypothesis are difficult to reconcile [45].
and also constrained with the geophysical and astronomical observations [46].
The generation of thermal GWs subsequent to the end of inflation is difficult
because allowed variation of the gravitational coupling constant is very small.

The difficulties faced by the alternative scenarios probably points towards
the extra dimensions as the leading candidate for the existence of thermal
GWs.

3.3 Thermal Vacuum State

An effective approach to deal with thermal vacuum states is the thermo-field
dynamics (TFD)[47]. In this approach the expectation value of mixed state at
non-zero temperature is obtained by an equivalent computation with a pure
state. This is made by introducing a fictitious field which is the identical
image of the original real field. Thus a temperature dependent vacuum for
the expanded field can be obtained from the absolute vacuum by a Bogoliubov
type of transformation. Therefore, in TFD a fictitious space called the tilde
space also introduced besides the Hilbert space, and the direct product space
is made up of above two spaces. Every operator and state in the Hilbert space
has corresponding operator and state in the tilde space [47].

Based on the physical and tilde system, thermofield dynamics creates
an operator called the thermal operator $T(\theta)$ and is invariant under the
tilde conjugation i.e., $\tilde{T}(\theta) = T(\theta)$ [47]. According to the tilde conjugation
$\tilde{C}\tilde{O} = C^*\tilde{O}$, where $C$ is any coefficients appears in the expressions of
quantities for a physical system, $O$ any operator, the superscript * means
complex conjugation, and $\tilde{O}$ represents the corresponding operator for the
tilde system.

Therefore a thermal vacuum state $(Tr)$ is defined as [48]

$$[Tr] = T(\theta_n)|0\tilde{0}\rangle,$$  \hspace{1cm} (3.23)
where $|0 \tilde{0} \rangle$ is the two mode vacuum state at zero temperature, and

$$\mathcal{T}(\theta_n) = \exp[-\theta_n(a_n\tilde{a}_n - a_n^\dagger\tilde{a}_n^\dagger)],$$

(3.24)

is the thermal operator. Where $\theta_n$ is related to the average number of the thermal particle,

$$\bar{n}_n = \sinh^2\theta_n.$$  

(3.25)

For a given temperature $T$, $\bar{n}_n$ is provided by the Bose-Einstein distribution,

$$\bar{n}_n = [\exp(h\omega_n/k_BT) - 1]^{-1},$$

(3.26)

where $\omega_n$ is the resonance frequency of the field. The $a_n$, $a_n^{\dagger}$ and $\tilde{a}_n$, $\tilde{a}_n^{\dagger}$, are the annihilation and creation operators respectively in Hilbert and tilde space. These operators obey the usual commutation relations,

$$[a_n, a_{n'}^{\dagger}] = [\tilde{a}_n, \tilde{a}_{n'}^{\dagger}] = \delta^3(n - n'),$$

(3.27)

and all the other relations are zero. By the appropriate action of the thermal operators on $a_n, a_n^{\dagger}, \tilde{a}_n, \tilde{a}_n^{\dagger}$, we get [48]

$$\mathcal{T}^\dagger a_n \mathcal{T} = a_n \cosh \theta_n + \tilde{a}_n^{\dagger} \sinh \theta_n,$$

(3.28)

$$\mathcal{T}^\dagger a_n^{\dagger} \mathcal{T} = a_n^{\dagger} \cosh \theta_n + \tilde{a}_n \sinh \theta_n.$$  

(3.29)

Thus the occupation number in the thermal vacuum state can be written as

$$\langle a_n^{\dagger}a_{n'} \rangle = \left(\frac{1}{e^{\bar{n}/T} - 1}\right)\delta^3(n - n').$$

(3.30)

This important result in the thermal vacuum state is very useful for the further study.

### 3.4 Gravitational Waves Spectrum

In quantum theory of GWs, the field $h_{\mu\nu}$ is a field operator, which is written as a sum of the plane wave Fourier modes. The tensor perturbations have
two independent physical degrees of freedom and are denoted as $h^+$ and $h^\times$. Thus to compute the spectrum of GWs, we express $h^+$ and $h^\times$ in terms of the creation ($a^\dagger$) and annihilation ($a^\dagger$) operators,

$$h_{\mu\nu}(x,\eta) = \frac{\sqrt{16\pi l_{pl}}}{(2\pi)^{3/2}} \sum_p \int d^3n \epsilon^p_{\mu\nu}(n) \times \frac{1}{\sqrt{2n}} \left[ a^p_n h^p_n(\eta) e^{in\cdot x} + a^\dagger_n^p h^*_{n^p}(\eta) e^{-in\cdot x} \right], \quad (3.31)$$

where $l_{pl}$ is the Planck length.

The polarization tensor $\epsilon^p_{\mu\nu}(n)$ with $p = +, \times$, is symmetric and transverse-traceless $n^\mu \epsilon^p_{\mu\nu}(n) = 0$, $\delta^{\mu\nu} \epsilon^p_{\mu\nu}(n) = 0$ and satisfy the conditions $\epsilon^{\mu\nu\rho}(n)\epsilon^p_{\mu\nu}(n) = 2\delta^{pp'}$ and $\epsilon^p_{\mu\nu}(-n) = \epsilon^p_{\mu\nu}(n)$, the creation and annihilation operators satisfy $[a^p_n, a^\dagger_{n'}^p] = \delta^{pp'} \delta^3(n - n')$ and the initial vacuum state is defined as

$$a^p_n|0\rangle = 0, \quad (3.32)$$

for each $n$ and $p$. The energy density of the GWs in vacuum state is $t_{00} = \frac{1}{32\pi l_{pl}^2} \frac{\partial h_{\mu\nu}}{\partial x_0} \frac{\partial h^{\mu\nu}}{\partial x_0}$.

The power spectrum of GWs is defined as

$$\int_0^\infty h^2(n,\eta) \frac{dn}{n} = \langle 0|h_{\mu\nu}(x,\eta) h_{\mu\nu}(x,\eta)|0\rangle, \quad (3.33)$$

where right hand side is the vacuum expectation value of the operator $h^{\mu\nu}h_{\mu\nu}$. Substituting equation (3.31) in eq.(3.33) and taking the contribution from each polarization is same, we get

$$h(n,\eta) = \frac{4l_{pl}}{\sqrt{\pi}} n \left. | h(\eta) | \right. . \quad (3.34)$$

Thus once the mode function $h(\eta)$ is known, the spectrum $h(n,\eta)$ follows.

The spectrum at the present time $h(n,\eta_0)$ can be obtained, provided the initial spectrum is specified. The initial condition is taken to be the inflationary stage. Thus the initial amplitude of the spectrum is given by

$$h(n,\eta_i) = A \left( \frac{n}{n_0} \right)^{2+\beta}, \quad (3.35)$$
where $A = 8\sqrt{\frac{\hbar c}{m}}$ is a constant. The power spectrum for the primordial perturbation of energy density is $P(n) \propto |h(n, \eta_0)|^2$ and in terms of initial spectral index $n'$, it is defined as $P(n) \propto n'^{-1}$. Thus the scale invariant spectral index $n' = 1$ for the pure de-Sitter expansion can be obtained with the relation $n' = 2\beta + 5$ for $\beta = -2$. Here onwards $\beta$ is treated as a number.

### 3.5 Low Frequency Contribution to the Spectrum

In this section, we consider the contribution of low frequency GWs to its spectrum in thermal vacuum state. Using eqs.(3.23), (3.28) and (3.29) in eq.(3.33) the power spectrum in thermal vacuum state is obtained as

$$h^2_T(n, \eta) = \frac{16l^2}{\pi} n^2 |h(\eta)|^2 \coth \frac{n}{2T}, \quad (3.36)$$

Thus in comparison with eq.(3.35), the spectrum in thermal vacuum state is

$$h_T(n, \eta_i) = A \left(\frac{n}{n_0}\right)^{2+\beta} \coth^{1/2} \left[\frac{n}{2T}\right]. \quad (3.37)$$

The last term becomes significant when the ratio $n/(2T)$ is less than unity. The wave number $n$ and temperature $T$ are comoving quantities which are related to the physical parameters at the time of inflation [10, 13]. Thus it is expected an enhancement of the spectrum by a factor $\coth^{1/2}[n/2T] = \coth^{1/2}[HS_i/2T_i]$, here $i$ means inflationary [10].

It is convenient to consider the amplitude of waves in different range of wave numbers. Thus the amplitude of the spectrum in thermal vacuum state for different ranges are given by:

(i) when $n \leq n_E$, the corresponding wavelength is greater than the present Hubble radius. Thus the amplitude remain as the initial one and can be written as

$$h_T(n, \eta_0) = A \left(\frac{n}{n_0}\right)^{2+\beta} \coth^{1/2} \left[\frac{n}{2T}\right], \quad (3.38)$$
(ii) the amplitude remains approximately same as long as the wave is inside the barrier but begin to decrease when it leaves the barrier by a factor $1/S(\eta)$, depending on the value of scale factor at that time. This process continues until the barrier becomes higher than $n$ at a time $\eta$ earlier than $\eta_0$, so the amplitude has decreased by the ratio of the scale factor at the time of leaving the barrier $S_b$ to its value at $\eta, S(\eta)$. This is in the range $n_E \leq n \leq n_0$.

$$h_T(n, \eta_0) = A \left( \frac{n}{n_0} \right)^{\beta-1} \coth^{1/2} \left( \frac{n}{2T} \right) \frac{1}{(1 + z_E)^3}. \quad (3.39)$$

Note that this range is a new feature on account of the current acceleration of the universe which is absent in the decelerating model as pointed out in [23]. The amplitude of the waves that left the barrier at $S_b$ with wave numbers $n > n_0$ has decreased up to the present time by a factor $S_b/S(\eta_0)$. This affects the amplitude of the present spectrum and is obtained as

$$h_T(n, \eta_0) = A \left( \frac{n}{n_0} \right)^{2+\beta} \coth^{1/2} \left( \frac{n}{2T} \right) \frac{S_b}{S(\eta_0)}. \quad (3.40)$$

This result can be used to obtain the spectrum of the GWs in the remaining range of wave numbers.

(iii) the wave number that does not hit the barrier in the range $n_0 \leq n \leq n_2$ gives the amplitude as follows

$$h_T(n, \eta_0) = A \left( \frac{n}{n_0} \right)^{\beta} \coth^{1/2} \left( \frac{n}{2T} \right) \frac{1}{(1 + z_E)^3}, \quad (3.41)$$

the spectrum in this interval is different from that of the matter dominated case by a factor $1/(1 + z_E)^3$. The wave lengths of the spectrum in the range are long but smaller than the present Hubble radius.

(iv) in the range of wave number $n_2 \leq n \leq n_s$ the amplitude is

$$h(n, \eta_0) = A \left( \frac{n}{n_0} \right)^{1+\beta} \left( \frac{n_0}{n_2} \right) \frac{1}{(1 + z_E)^3}. \quad (3.42)$$

This is the interesting range on the observational point of view of Adv.LIGO, ET and LISA missions [2]-[4].
(v) for the wave number range \( n_s \leq n \leq n_1 \) which is in the high frequency case and gives the corresponding amplitude as

\[
h(n, \eta_0) = A \left( \frac{n}{n_0} \right)^{1+\beta_s} \left( \frac{n_s}{n_0} \right)^{\beta_s} \left( \frac{n_0}{n_2} \right) \frac{1}{(1 + z_E)^3}.
\]  

(3.43)

Note that the temperature dependent factor in the high frequency range is negligible, hence the term is dropped in the expression (iv) and (v) and the effect of temperature dependent factor is discussed in section.(3.7).

3.5.1 Normalization of the spectrum

The overall multiplication factor \( A \) in all the spectra is determined in absence of the temperature dependent term with the CMB data of WMAP [23]. This is based on the assumption that the contribution from GWs and the density perturbations are the same order of magnitude at low multipole moments, \( l \). Therefore it is possible to write \( \Delta T/T \simeq h(n, \eta_0) \). The observed CMB anisotropies [49] at lower multipoles is \( \Delta T/T \simeq 0.37 \times 10^{-5} \) at \( l \sim 2 \) which corresponds to the largest scale anisotropies that have been observed so far. Thus taking this to be the perturbations at the Hubble radius gives

\[
h(n_0, \eta_0) = A \frac{1}{(1 + z_E)^3} = 0.37 \times 10^{-5} \times r^{1/2};
\]  

(3.44)

where \( r \) is the tensor-to-scalar ratio and it is taken as unity for normalizing the spectrum in the present work. However, there is a subtlety here in the interpretation of \( \Delta T/T \) at low multipoles, whose corresponding scale is very large \( \sim \ell_0 \). At present the Hubble radius is \( \ell_0 \), and the Hubble diameter is \( 2\ell_0 \). On the other hand, the smallest characteristic wave number is \( n_E \), whose corresponding physical wave length at present is \( 2\pi S(\eta_0)/n_E = \ell_0(1 + z_E) \simeq 1.32\ell_0 \), which is within the Hubble diameter \( 2\ell_0 \), and is theoretically observable. So, instead of eq.(3.44), if \( \Delta T/T \simeq 0.37 \times 10^{-5} \) at \( l \sim 2 \) were taken as the amplitude of the spectrum at frequency \( \nu_E \), one would have \( h_T(n_E, \eta_0) = A/(1 + z_E)^{2+\beta} = 0.37 \times 10^{-5} \), yielding a smaller \( A \) than that in eq.(3.44) by a factor \( (1 + zE)^{1-\beta} \sim 2.3 \) [23].
3.5.2 Allowed range of $\beta$ and the spectrum

Next objective is to check the allowed range of $\beta$. During the inflationary expansion, when the $n$-mode wave enters the barrier with $\lambda_i = 1/H(\eta_i)$, it follows that $\lambda_i = \frac{b}{l_0} \left( \frac{\nu_i}{\nu_0} \right)^{2+\beta}$. For the classical treatment of the background gravitational field to be valid, this wavelength should be greater than the Planck length, $\lambda_i > l_{pl}$, so

$$\left( \frac{\nu}{\nu_0} \right)^{2+\beta} < \frac{8\sqrt{\pi}}{A}. \quad (3.45)$$

At the highest frequency $\nu = \nu_1$, this gives the following constraint

$$\beta < -2 + \ln \left( \frac{8\sqrt{\pi}}{A} \right) / \ln \left( \frac{\nu_1}{\nu_0} \right), \quad (3.46)$$

which depends on $A$. Thus, for given $A$ in (3.44), one obtains the upper limit $\beta < -1.78$. Plugging $b/l_0$ given by eq.(3.9) into $A$, using $\nu_2/\nu_0 = 58.8$ and $\ell_0/l_{pl} = 1.238 \times 10^6$, get [23]

$$1.484 \times 10^{58} \times \frac{A}{(1 + z_E)^3} = \left( \frac{\nu_1}{\nu_0} \right)^{-\beta} \left( \frac{\nu_1}{\nu_s} \right)^{\beta_s}. \quad (3.47)$$

For, given $A$ in (3.44), and $\beta = -1.9$, then $\beta_s = -0.552$.

Next, we obtain the spectrum in the thermal vacuum state with the following parameters. By taking $n = 2\pi \nu$, $\nu_E = 1.5 \times 10^{-18}$ Hz, $\nu_0 = 2 \times 10^{-18}$ Hz, $\nu_2 = 117 \times 10^{-18}$ Hz, $\nu_s = 10^8$ Hz, $\nu_1 = 3 \times 10^{10}$ Hz, the value of $\nu_1$ is taken such a way that spectral energy density does not exceed the level of $10^{-6}$, as required by the nucleosynthesis bound. The range of frequency is chosen in accordance with generation of GWs that vary from early universe to various astrophysical sources. The range of frequency is matching with the interest of CMB, Adv.LIGO, ET and LISA operations for detection of the GWs. The spectrum is computed in the thermal vacuum state with the chosen values of the parameters for the accelerated as well as decelerated FLRW universe with comoving temperatures $T = 0.001 \text{ Mpc}^{-1}$ and $T = 0.01 \text{ Mpc}^{-1}$. The selected comoving temperatures are considered
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in the context of tensor mode study of CMB [10]. Since we use the natural unit, the wave number and temperature that appear in the temperature dependent term of the spectrum is computed numerically in the Mpc\(^{-1}\) unit. And henceforth the unit of comoving temperature is taken as Mpc\(^{-1}\).

The obtained GWs spectra in thermal state are normalized with WMAP 7-year data. The amplitude of GWs spectrum in the thermal state is found enhanced in comparison to its zero temperature case (vacuum case). It is observed that the spectrum, for \(T = 0.001\) Mpc\(^{-1}\) get maximum enhancement \(\sim 1.51\) times than the vacuum case, at \(l=2\) and \(\nu = \nu_E\), and it is \(\sim 4.6\) times for \(T = 0.01\) Mpc\(^{-1}\). The plots for the amplitude of spectrum \(h_T(n, \eta_0)\) versus the frequency \(\nu\) for \(\beta = -1.9\) and \(\beta_\ast = -0.552\) are given in Fig.[3.1]. The amplitude of the spectrum get enhanced in the frequency range, \(10^{-19} \text{ Hz} \leq \nu \leq 1.49 \times 10^{-17} \text{ Hz}\), due to the thermal effect of GWs but not for the frequency range \(1.49 \times 10^{-17} \text{ Hz} \leq \nu \leq 3 \times 10^{10} \text{ Hz}\) because there is a suppression in the high frequency range due to the \(\text{coth}^{1/2}[n/2T]\) term. For comparison, the amplitude of the spectra are plotted for the decelerated and accelerated FLRW universe, panel (b), Fig.[3.1] for the frequency range \(\nu_\ast \leq \nu \leq \nu_2\) where \(\nu_\ast = 10^{-19} \text{ Hz}\). The results of study show that the amount of enhancement of the amplitude of the spectrum is independent of the model of expansion of the universe viz accelerated or decelerated and hence it is due to the thermal nature of the GWs. The obtained amplitude of the spectrum with enhancement in thermal vacuum state for both models of expansion of the universe is less than the upper bound of the WMAP 7-year data (panel (b), Fig.[3.1] the pink line is indicating the upper bound).

Further it is observed that position of the peak remains at \(\nu_E\) for the accelerated universe (as pointed out in [23] for the vacuum case) but get enhanced due to the thermal GWs. This enhancement is the new feature of the spectrum of the relic GWs in the lower frequency range \(\nu_\ast \leq \nu \leq \nu_2\) (panel(b), Fig.[3.1]).
Figure 3.1: The amplitude of the GWs for the accelerated (solid lines) and decelerated (dashed lines) universe.
3.6 Spectral Energy Density in Thermal Vacuum State

In this section, we compute and study the spectral energy density of the GWs in thermal state for the flat FLRW universe. The spectral energy density parameter $\Omega_g(\nu)$ of the GWs is defined through the relation

$$\frac{\rho_g}{\rho_c} = \int \Omega_g(\nu) \frac{d\nu}{\nu},$$

(3.48)

where $\rho_g$ is the energy density of the GWs and $\rho_c$ is the critical energy density of the universe. Thus

$$\Omega_g(\nu) = \frac{\pi^2}{3} h^2_T(k, \eta_0) \left(\frac{\nu}{\nu_0}\right)^2.$$

(3.49)

There is one more consistency condition to be satisfied. Since the space-time is assumed to be spatially flat $K = 0$ with $\Omega = 1$, the fraction density of relic GWs should be less than unity, i.e; $\rho_g/\rho_c < 1$. After normalizing the obtained spectrum with WMAP 7 year data, we integrate $\int \Omega_g(\nu) d\nu/\nu$ from the $\nu_s = 10^{-19}$ Hz up to the frequency $\nu_1 = 3 \times 10^{10}$ Hz, with $\beta = -1.9$ and $\beta_s = -0.552$, to get the total spectral energy density of GWs.

The spectral energy density is evaluated, by splitting the full range of frequency into five, for the thermal and zero temperature cases and the obtained results, in the accelerated flat FLRW universe, are:

(a) $\nu_s \leq \nu \leq \nu_E$:

$$\frac{\rho_g}{\rho_c} = 5.8 \times 10^{-11}, \quad T = 0,$$

$$\frac{\rho_g}{\rho_c} = 8.8 \times 10^{-11}, \quad T = 0.001 \text{ Mpc}^{-1},$$

$$\frac{\rho_g}{\rho_c} = 2.6 \times 10^{-10}, \quad T = 0.01 \text{ Mpc}^{-1},$$
(b) $\nu_E \leq \nu \leq \nu_H$,
\[
\frac{\rho_g}{\rho_c} = 2.3 \times 10^{-11}, \quad T = 0, \\
\frac{\rho_g}{\rho_c} = 3.5 \times 10^{-11}, \quad T = 0.001 \text{ Mpc}^{-1}, \\
\frac{\rho_g}{\rho_c} = 1.1 \times 10^{-10}, \quad T = 0.01 \text{ Mpc}^{-1},
\]

(c) $\nu_H \leq \nu \leq \nu_2$,
\[
\frac{\rho_g}{\rho_c} = 2.4 \times 10^{-11}, \quad T = 0, \\
\frac{\rho_g}{\rho_c} = 3.7 \times 10^{-11}, \quad T = 0.001 \text{ Mpc}^{-1}, \\
\frac{\rho_g}{\rho_c} = 1.2 \times 10^{-10}, \quad T = 0.01 \text{ Mpc}^{-1},
\]

(d) $\nu_2 \leq \nu \leq \nu_s$,
\[
\frac{\rho_g}{\rho_c} = 8.97 \times 10^{-9}, \quad T = 0,
\]

(e) $\nu_s \leq \nu \leq \nu_1$,
\[
\frac{\rho_g}{\rho_c} = 2.7 \times 10^{-6}, \quad T = 0.
\]

It is to be noted that in the frequency range of (d) and (e) the thermal cases are not shown because the thermal contribution in the high frequency range is negligible, due to the temperature dependent term. The combined results are plotted in Fig.[3.2]. Further, the contribution to $\rho_g/\rho_c$ from the low frequency range is $O(10^{-11} - 10^{-10})$ while from the higher frequency range it is $O(10^{-6})$. Since the order of contribution to the total spectral energy density $\rho_g/\rho_c$ from the lower frequency side is very small in contrast with the higher frequency side, we get
\[
\frac{\rho_g}{\rho_c} = 2.7 \times 10^{-6}, \quad \nu_s \leq \nu \leq \nu_1, \quad (3.50)
\]
and is the same as that of the non-zero temperature case of high frequency. However $\rho_g/\rho_c$ of the GWs with $T \neq 0$ is higher than the zero temperature
Figure 3.2: The spectral energy density of the GWs for the accelerated (solid lines) and decelerated (dashed lines) universe.
case at lower frequency range $\nu_s \leq \nu \leq \nu_2$. Therefore it is expected an enhancement for the spectral energy density in the thermal vacuum state in the frequency range $\nu_s \leq \nu \leq \nu_2$ only (panel (b), Fig.[3.2]). And it is the range of interest on the observational point of view of the relic GWs. Hence to observe the thermal effect on the relic GWs, in this range, the spectrum of GWS must be analysed separately otherwise the signature of the thermal effect gets suppressed due to the contributions from the higher frequency range.

### 3.7 High Frequency Contribution to the Spectrum

In this section, we consider contribution of very high frequency thermal GWs ($\nu_s \leq \nu \leq \nu_1$) to its spectrum and spectral energy density for the decelerated as well as accelerated flat FLRW universe. The origin of these very high frequency thermal GWs is assumed to be due to the extra dimensional effects in the early universe[13]. As studied in section 3.5, the computation of GWs spectrum again consider in the same five frequency ranges. Since this section mainly deals with the contribution of very high frequency range, the amplitude of GWs spectrum in the other frequency ranges, i.e; (i) up to (v) are not computing again which are obtained in section 3.5. The spectrum GWs in the higher frequency range is given in (v) section 3.5 and hence using eq.(3.43) the amplitude of the high thermal GWs can be expressed as

$$h_T(n, \eta_0) = A \left( \frac{n}{n_0} \right)^{1+\beta-\beta_s} \left( \frac{n_s}{n_0} \right)^{\beta_s} \left( \frac{n_0}{n_2} \right) \coth^{1/2} \left[ \frac{n}{2T} \right] \frac{1}{(1+z_E)^3}. \tag{3.51}$$

As discussed in section 3.5 the temperature dependent term can be neglected, however the inclusion of high frequency thermal GWs the term becomes significant once again in the range $\nu_s \leq \nu \leq \nu_1$.

It is already observed that the contribution of thermal GWs in the ranges $n_2 \leq n \leq n_s$ and $n_s \leq n \leq n_1$ is insignificant. But taking into account of the
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extra dimensional effect, the spectrum of GWs is peaked with a temperature $T_\ast = 1.19 \times 10^{25} \text{ Mpc}^{-1}$ [13]. Therefore it is expected an enhancement for the amplitude of spectrum (orange lines, Figs.[3.3] and [3.6]) in the range $n_s \leq n \leq n_1$ compared to $T = 0$ case, for the accelerated as well as decelerated universe. But at the same time, ignoring the thermal contribution to the amplitude of spectrum in the range $n_2 \leq n \leq n_s$ leads to a discontinuity at $n_s$, see Fig.[3.3]. This problem is evaded by fitting a new line in the range $n_2 \leq n \leq n_s$ for the amplitude $h$ of eq.(3.42) as follows:

Let the amplitude of GWs in the range $n_0 \leq n \leq n_2$ is given by eq.(3.41) as

$$h_{1T}(n, \eta_0) = A \left( \frac{n}{n_0} \right)^\beta \coth^{1/2} \left[ \frac{n}{2T} \right] \frac{1}{(1 + z_E)^3},$$

(3.52)

and the amplitude in the $n_s \leq n \leq n_1$ by eq.(3.51) as

$$h_{2T}(n, \eta_0) = A \left( \frac{n}{n_0} \right)^{1+\beta} \left( \frac{n_s}{n_0} \right)^{\beta_s} \left( \frac{n_s}{n_2} \right) \coth^{1/2} \left[ \frac{n}{2T_s} \right] \frac{1}{(1 + z_E)^3}.$$  

(3.53)

Thus the new slope for eq.(3.42), in the range $n_2 \leq n \leq n_s$, is obtained by taking $y \equiv \log_{10}(h)$ and $x \equiv \log_{10}(n)$, then

$$\log_{10}(h) - \log_{10}(h)_i = \frac{\log_{10}(h)_f - \log_{10}(h)_i}{\log_{10}(n_f) - \log_{10}(n_i)} (\log_{10}(n) - \log_{10}(n_i)),$$

(3.54)

where the subscripts $i$ and $f$ are respectively indicating the first and last points of the straight line. By putting $n_i \equiv n_2$ from eq.(3.52) and $n_f \equiv n_s$ from eq.(3.53) in eq.(3.54), we get \(^2\)

$$h = (h_{1T})_{n_2} g(n),$$

(3.55)

where

$$g(n) = \left( \frac{n}{n_2} \right)^\gamma,$$

(3.56)

and

$$\gamma = \frac{\log_{10}(h_{2T})_{n_s} - \log_{10}(h_{1T})_{n_2}}{\log_{10}(n_s) - \log_{10}(n_2)} = \frac{\log_{10} \left( \left( \frac{n_s}{n_2} \right)^{1+\beta} \coth^{1/2} \left[ \frac{n_s}{2T_s} \right] \right)}{\log_{10} \left( \frac{n_s}{n_2} \right)}.$$  

(3.57)

\(^2\)here, $\coth^{1/2} \left[ \frac{n_2}{2T} \right] = 1.$
is the slope of the line and thus we find the amplitude,

\[ h(n, \eta_0) = A \left( \frac{n}{n_0} \right)^{\beta} \frac{1}{(1 + z_E)^3} \left( \frac{n}{n_2} \right)^{\gamma}, \]

and for the convenience we designate the obtained amplitude with the thermal GWs as ‘modified amplitude’. It can be seen that, when \( T_* \) becomes zero eq.(3.57) leads to \( \gamma = 1 + \beta \), and hence eq.(3.42) is recovered from eq.(3.58) in the range \( n_2 \leq n \leq n_s \).

The overall multiplication factor \( A \) in all the spectra is determined as described in the section 3.5. The allowed values of \( \beta \), are \( \beta_s \) also obtained and are respectively given by \( \beta = -1.9 \), and \( \beta_s = -0.552 \) [23].

Next, we obtain the spectrum in the thermal vacuum state with the following parameters. By taking \( n = 2\pi \nu, \nu_E = 1.5 \times 10^{-18} \text{ Hz}, \nu_0 = 2 \times 10^{-18} \text{ Hz}, \nu_2 = 117 \times 10^{-18} \text{ Hz}, \nu_s = 10^8 \text{ Hz}, \nu_1 = 3 \times 10^{10} \text{ Hz} \). The value of \( \nu_1 \) is again taken such a way that the spectral energy density does not exceed the level of \( 10^{-6} \), as required by the rate of primordial nucleosynthesis calculation. The range of frequency is chosen in accordance with generation of GWs that vary from early universe to various astrophysical sources. The range is matching with the interest of CMB, Adv.LIGO, ET and LISA operations for detection of the GWs. The spectrum is computed in the thermal vacuum state with the chosen values of the parameters for the accelerated as well as decelerated models with \( T = 0.001 \text{ Mpc}^{-1} \) in the low frequency range \( n < n_2 \) (and similar behavior for \( T = 0.01 \text{ Mpc}^{-1} \) with this range, see subsection.(3.5.2)). This temperature is considered in the context of \( B \) mode angular spectrum of CMB spectrum in thermal state [10]. And \( T_* = 1.19 \times 10^{25} \text{ Mpc}^{-1} \) \((\dagger) \) \(^3\) the high range \( n_s \leq n \leq n_1 \) which is from the extra dimensional scenario [13]. The obtained spectra are normalized with the CMB anisotropy spectrum of WMAP 7-year data. The amplitude of the spectrum of the thermal GWs is enhanced compared to its zero temperature case (vacuum case). It is observed that the spectrum for \( T = 0.001 \text{ Mpc}^{-1} \)

\(^3(\dagger) \) here, \( T_* = 0.905 \text{ K}_{\text{cde}} = 1.19 \times 10^{25} \text{ Mpc}^{-1} \).
Figure 3.3: The amplitude of the GWs for the accelerated (solid lines) and decelerated (dashed lines) universe.

get maximum enhancement $\sim 1.51$ times than the vacuum case, at $\nu = \nu_E$, and it is $\sim 20$ times for $T_s=1.19 \times 10^{25}$ Mpc$^{-1}$ at $\nu = \nu_s$.

The plots for the amplitude of spectrum $h_T(n, \eta_0)$ versus the frequency $\nu$ for $\beta = -1.9$ and $\beta_s = -0.552$ are given in Fig.[3.3]. The amplitude of the spectrum is found enhanced in the frequency ranges, $10^{-19}$ Hz $\leq \nu < 1.49 \times 10^{-17}$ Hz, and $\nu_s \leq \nu \leq \nu_1$, the lower value of this range is selected such way that the spectral density does not exceed the upper bound of nucleosynthesis. But there is a suppression of the contribution of thermal GWs in the range $1.49 \times 10^{-17}$ Hz $\leq \nu < \nu_s$ due to the $\coth^{1/2}[n/2T]$ term. For comparison, amplitude of the spectra are plotted for the decelerated and accelerated FLRW universe in Fig.[3.3].
3.7.1 Comparison with the sensitivity of Adv.LIGO, ET and LISA

The new enhancement of the amplitude of GWs spectrum due to the high frequency GWs from extra dimensional effect (the modified amplitude, Fig.[3.3], green lines) can be compared with the sensitivity of Adv.LIGO [2], ET [3] and LISA [4]. For Adv.LIGO and ET cases, consider the root mean square amplitude per root Hz which equal to

\[ \frac{h(\nu)}{\sqrt{\nu}}. \]  \hspace{1cm} (3.59)

The comparison of the sensitivity (10 Hz - 10^4 Hz) curve of the ground based interferometer Adv.LIGO and ET with the GW spectra of \( \beta = -1.9 \) for the accelerated and decelerated flat FLRW universe are given in Fig.[3.4]. This shows that the Adv.LIGO is unlikely to detect the enhancement of the spectrum due to the extra dimensional effect with it current stands but be possible with the sensitivity of ET.

Next, we compare the enhancement of the spectrum with the sensitivity (10^{-7} Hz–10^0 Hz) of space based detector LISA. It is assumed that LISA has one year observation time which corresponds to frequency bin \( \Delta \nu = 3 \times 10^{-8} \text{Hz} \) (one cycle/year) around each frequency. Hence to make a comparison with the sensitivity curve, a rescaling of the spectrum \( h(\nu) \) is required in eq.(3.34) into the root mean square spectrum \( h(\nu, \Delta \nu) \) in the band \( \Delta \nu \), given by

\[ h(\nu, \Delta \nu) = h(\nu) \sqrt{\frac{\Delta \nu}{\nu}}. \]  \hspace{1cm} (3.60)

The plots of the LISA sensitivity with the modified amplitude of the spectrum are given in Fig.[3.5] for the accelerated and decelerated flat FLRW universe. This show that the LISA is unlikely to detect the spectrum with the new enhancement feature of the GWs of the present study.
Figure 3.4: Comparison of the modified amplitude of the spectrum for the accelerated (solid black and green lines) and decelerated (dashed black and green lines) universe with the sensitivity curves of Adv.LIGO [2] and ET [3].
Figure 3.5: Comparison of the modified amplitude of the spectrum for the accelerated (solid black and green lines) and decelerated (dashed black and green lines) universe with the LISA [4] sensitivity curve.
3.7.2 Spectral energy density

Since we obtained the contribution of the very high frequency thermal GWs, due to the extra dimensional effect, to the amplitude of GWs it is appropriate to study the corresponding spectral energy also. The computation of the spectral energy density by including the very high thermal GWs can be achieved through the spectral energy density parameter $\Omega_g(\nu)$ of GWs is defined in section 3.6. Thus, after the normalization of the obtained spectrum of thermal GWs, we integrate $\int \Omega_g(\nu) d\nu/\nu$ from $\nu_\ast = 10^{-19}$ Hz up to $\nu_1 = 3 \times 10^{10}$ Hz, with $\beta = -1.9$ and $\beta_s = -0.552$. The spectral energy density of GWs is recomputed for the thermal and zero temperature cases again by splitting the total frequency into five ranges, and the obtained results for the accelerated FLRW universe are:

(a) $\nu_\ast \leq \nu \leq \nu_E$,
\[
\frac{\rho_g}{\rho_c} = 5.8 \times 10^{-11}, \quad T = 0,
\frac{\dot{\rho}_g}{\dot{\rho}_c} = 8.8 \times 10^{-11}, \quad T = 0.001 \text{ Mpc}^{-1},
\]

(b) $\nu_E \leq \nu \leq \nu_H$,
\[
\frac{\rho_g}{\rho_c} = 2.3 \times 10^{-11}, \quad T = 0,
\frac{\dot{\rho}_g}{\dot{\rho}_c} = 3.5 \times 10^{-11}, \quad T = 0.001 \text{ Mpc}^{-1},
\]

(c) $\nu_H \leq \nu \leq \nu_2$,
\[
\frac{\rho_g}{\rho_c} = 2.4 \times 10^{-11}, \quad T = 0,
\frac{\dot{\rho}_g}{\dot{\rho}_c} = 3.7 \times 10^{-11}, \quad T = 0.001 \text{ Mpc}^{-1},
\]

(d) $\nu_2 \leq \nu \leq \nu_s$,
\[
\frac{\rho_g}{\rho_c} = 8.97 \times 10^{-9}, \quad T = 0,
\]
Figure 3.6: The spectral energy density of the GWs for the accelerated (solid lines) and decelerated (dashed lines) universe.

\((e)\) \(\nu_s \leq \nu \leq \nu_1\),

\[
\frac{\rho_g}{\rho_c} = 2.7 \times 10^{-6}, \quad T = 0,
\]

\[
\frac{\rho_g}{\rho_c} = 6.67 \times 10^{-6}, \quad T_* = 1.19 \times 10^{25} \text{ Mpc}^{-1}.
\]

It is to be noted that in the frequency range of (d) the thermal case is not shown because the thermal contribution in this frequency range is still negligible due to the temperature dependent term. However by taking into account the extra dimensional effect, the upper limit of temperature of the relic waves is \(T_* = 1.19 \times 10^{25} \text{ Mpc}^{-1}\). Thus it is expected an enhancement of the spectral energy density in range \(\nu_s \leq \nu \leq \nu_1\) compared to \(T = 0\) case for the accelerated as well as decelerated universe. But at the same time ignoring the thermal contribution to the spectral density in the range \(\nu_2 \leq \nu \leq \nu_s\) leads to a discontinuity at \(\nu_s\), see Fig.[3.6]. This problem is solved by fitting a new line as discussed in the context of estimation of the
amplitude of the spectrum and hence recomputed the spectral density in the range $\nu_2 \leq \nu \leq \nu_s$ which gives the new value for $\rho_g/\rho_c = 8.21 \times 10^{-7}$. This changes the slope indicating enhancement of the spectral energy density of the GWs in the range $\nu_2 \leq \nu \leq \nu_s$, green lines, Fig.[3.6].

The enhancement of spectral energy density $\Omega_g(\nu)$ in the frequency range (d) can be compared with the estimated upper bound of various studies and are given in Tab.[3.1]. Thus $\Omega_g^{(\text{dec})}$ and $\Omega_g^{(\text{acc})}(\nu)$ are less than the upper bound of the estimated values of the respective frequency range.

Further see that the contribution to $\rho_g/\rho_c$ from the low frequency range is $\mathcal{O}(10^{-11} - 10^{-10})$ while from the higher frequency range it is $\mathcal{O}(10^{-6})$. Since the order of contribution to the total $\rho_g/\rho_c$ from the lower frequency side is very small in contrast with higher frequency side, we get for the accelerated universe as

$$\frac{\rho_g}{\rho_c} \simeq 6.67 \times 10^{-6} \quad \nu_s \leq \nu \leq \nu_1, \quad (3.61)$$

and is of the same order as that of the zero temperature case. However $\rho_g/\rho_c$ of the GWs with $T \neq 0$ is higher than the zero temperature case at lower frequency range $\nu_s \leq \nu \leq \nu_2$. Therefore it is expected an enhancement for the spectral energy density in the thermal vacuum state in the frequency range $\nu_s \leq \nu \leq \nu_2$. 

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**Table 3.1:** Comparison of the estimated upper bound of spectral energy density of various studies with the present work. Here $\Omega_g^{(\text{dec})}$ and $\Omega_g^{(\text{acc})}$ are respectively the spectral energy density of the relic GWs in the decelerated and accelerated universe of the present study and $\Omega_g^{(\text{est})}$ is the estimated upper bound of various studies.

<table>
<thead>
<tr>
<th>Frequency$(\nu)$ Hz</th>
<th>$\Omega_g^{(\text{dec})}(\nu)$</th>
<th>$\Omega_g^{(\text{acc})}(\nu)$</th>
<th>$\Omega_g^{(\text{est})}(\nu)$</th>
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</thead>
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<tr>
<td>$10^{-9} - 10^{-7}$</td>
<td>$4.98 \times 10^{-9}$</td>
<td>$1.03 \times 10^{-9}$</td>
<td>$2 \times 10^{-8}$ [50]</td>
</tr>
<tr>
<td>69 – 156</td>
<td>$34.84 \times 10^{-8}$</td>
<td>$7.2 \times 10^{-8}$</td>
<td>$8.4 \times 10^{-4}$ [51]</td>
</tr>
<tr>
<td>41.5 – 169.25</td>
<td>$4.93 \times 10^{-7}$</td>
<td>$1.02 \times 10^{-7}$</td>
<td>$6.9 \times 10^{-6}$ [52]</td>
</tr>
</tbody>
</table>
3.8 Discussion and Conclusion

In this chapter, we considered thermal GWs with frequency range $10^{-19}$ Hz to $10^{10}$ Hz and obtained its spectrum and spectral energy density for the accelerated as well as decelerated flat FLRW universe. It is found that the spectrum gets enhanced due to the thermal effects on the GWs. This enhancement is the new feature of the spectrum if the relic GWs existed in thermal vacuum states. It is observed that the inclusion of the very high relic thermal GWs leads to a discontinuity in the amplitude of the spectrum at $\nu_s$. This is due to the fact that the temperature dependent term is insignificant in the higher frequency side $\nu_2 \leq \nu \leq \nu_s$. To evade this problem a new equation of line is derived and thus the amplitude get enhanced in the range $\nu_2 \leq \nu \leq \nu_s$. This is the new feature of the spectrum and we designates it as the ‘modified amplitude’ of the spectrum. The modified amplitude of the spectrum is compared with the sensitivity of the Adv.LIGO, ET and LISA missions. The comparison of the Adv.LIGO and LISA sensitivity shows that the modified amplitude is unlikely to be detected with its current stands but be possible with the sensitivity of ET.

The spectral energy density of the GWs is estimated in thermal vacuum state for the accelerated and decelerated flat FLRW universe. It is observed that the total spectral energy density get enhanced in the lower frequency range $\mathcal{O}(10^{-11} - 10^{-10})$ and from the higher frequency range it is $\mathcal{O}(10^{-6})$. A comparison of the estimated upper bound of spectral energy density of various studies with the present work is done. It shows that $\Omega_{g}^{(dec)}$ and $\Omega_{g}^{(acc)}$ are less than the estimated upper bound of various studies. The total estimated value of $\rho_g/\rho_c$ by including the very high frequency thermal relic GWs does not alter the upper bound of the nucleosynthesis rate. Thus the relic thermal GWs with very high frequency range are not ruled out and is testable with the upcoming data of various missions especially with ET for detecting GWs.