Chapter 2

Gravitational Waves

One of the remarkable predictions of the general theory of relativity is the existence of GWs. There are two types of sources mainly that generate GWs, one the astrophysical candidates such as dynamics of neutron star binaries, black hole mergers and the second is from the cosmological perturbations in the early universe [27, 28, 29]. The cosmologically generated GWs are also known as the relic GWs and were generated by the strong variable gravitational field of early universe [30]. The GWs are described by the gravitational wave equation and is the consequence of the linearized form of the Einstein field equations under suitable limiting conditions. Thus the main aim of this chapter is to present the Einstein linearized field equation, its solution and basic properties of the GWs briefly. A short discussion on the relic GWs in the expanding universe is also included. The creation of the relic GWs, during the inflationary era, through the parametric amplification mechanism also explained briefly.

2.1 Linearized Einstein Field Equations

Einstein’s general theory of relativity of gravity leads to Newtonian gravity in the suitable limit conditions, when the gravitational field is weak, static and the particles in the gravitational field move slowly compared to the velocity
of light. But consider a situation where the gravitational field is weak but not static, and there are no restrictions on the motion of particles in the gravitational field. Then the corresponding weak gravitational field can be considered as a small ‘perturbation’, \( h_{\mu\nu} \), on the flat Minkowski metric \( \eta_{\mu\nu} \),

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \mathcal{O}(h^2) + ..., \quad |h_{\mu\nu}| \ll 1. \tag{2.1}
\]

Here, we consider only first-order terms in \( h_{\mu\nu} \). In the absence of gravity, space-time is flat and is characterised by the Minkowski metric and following in the discussions, we consider it with the signature \((+,-,-,-)\). The coordinate systems involved in eq.(2.1) are called the Lorentz coordinate systems. Indices of any tensor, under the weak field approximation, can be raised or lowered using \( \eta^{\mu\nu} \) or \( \eta_{\mu\nu} \) respectively. Under a background Lorentz transformation, the perturbation transforms as a second-rank tensor:

\[
h_{\alpha\beta} = \Lambda_\alpha^\mu \Lambda_\beta^\nu h_{\mu\nu}, \tag{2.2}
\]

where \( \Lambda_\alpha^\mu \) and \( \Lambda_\beta^\nu \) are the Lorentz transformation matrices. The equations that govern \( h_{\mu\nu} \) are obtained by taking the Einstein’s field equations up to first order. The affine connection to the first order is given by

\[
\Gamma^{\lambda(1)}_{\mu\nu} = \frac{1}{2} \eta^{\lambda\rho} \left[ \partial_\mu h_{\rho\nu} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu} \right]. \tag{2.3}
\]

Therefore, the Riemann curvature tensor reduces to first order as

\[
R^{(1)}_{\mu\nu\rho\sigma} = \eta_{\mu\lambda} \partial_\rho \Gamma^{\lambda(1)}_{\sigma\nu} - \eta_{\mu\lambda} \partial_\sigma \Gamma^{\lambda(1)}_{\nu\rho}, \tag{2.4}
\]

and the Ricci tensor is obtained to the first order as

\[
R^{(1)}_{\mu\nu} = \frac{1}{2} \left[ \partial_\lambda \partial_\nu h^\lambda_{\mu} + \partial_\lambda \partial_\mu h^\lambda_{\nu} - \partial_\mu \partial_\nu h - \square h_{\mu\nu} \right], \tag{2.5}
\]

where, \( \square \equiv \eta^{\lambda\rho} \partial_\lambda \partial_\rho \) is the D’Alembertian in flat space-time. Contracting eq.(2.5) with \( \eta^{\mu\nu} \), gives the corresponding Ricci scalar as

\[
R^{(1)} = \partial_\lambda \partial_\mu h^\lambda_{\mu} - \square h. \tag{2.6}
\]

\(^1\text{Here onward superscript (1) means first order approximation.}\)
Therefore the Einstein tensor $G_{\mu \nu}$ in the limit of weak gravitational field is

$$G^{(1)}_{\mu \nu} = R^{(1)}_{\mu \nu} - \frac{1}{2} \eta_{\mu \nu} R^{(1)}$$

$$= \frac{1}{2} \left[ \partial_{\lambda} \partial_{\nu} h_{\mu}^{\lambda} + \partial_{\lambda} \partial_{\mu} h_{\nu}^{\lambda} - \partial_{\mu} \partial_{\nu} h - \eta_{\mu \nu} (\partial_{\lambda} \partial_{\sigma} h^{\lambda \sigma} - \Box h) - \Box h_{\mu \nu} \right].$$

(2.7)

Therefore the linearized Einstein’s field equations are

$$G^{(1)}_{\mu \nu} = 8\pi G T_{\mu \nu}. \quad (2.8)$$

Note that while deriving the above linearized form of the Einstein field equation the source term assumed as unperturbed. The linearized field equations (2.8) have no unique solutions as any solution to these equations will not remain invariant under a ‘gauge’ transformation. As a result, equations (2.8) can have infinitely many solutions. In other words, the decomposition eq.(2.1) of $g_{\mu \nu}$ in the weak gravitational field approximation does not completely specify the coordinate system. When a system that is invariant under a gauge transformation, then the gauge can be fixed and work in that selected coordinate system. One such coordinate system is the harmonic coordinate system and the gauge condition is given by

$$g^{\mu \nu} \Gamma_{\lambda}^{\mu \nu} = 0.$$

(2.9)

In the weak field limit, this condition reduces to

$$\partial_{\lambda} h_{\mu}^{\lambda} = \frac{1}{2} \partial_{\mu} h.$$

(2.10)

This condition is called the Lorentz gauge. In this selected gauge, the linearized Einstein’s equations simplify and reduces to

$$\Box h_{\mu \nu} - \frac{1}{2} \eta_{\mu \nu} \Box h = -16\pi G T_{\mu \nu}. \quad (2.11)$$

The ‘trace-reversed’ perturbation, $\tilde{h}_{\mu \nu}$, is defined as,

$$\tilde{h}_{\mu \nu} = h_{\mu \nu} - \frac{1}{2} \eta_{\mu \nu} h.$$  

(2.12)
Thus the harmonic gauge condition further reduces to

\[ \partial_\mu \bar{h}^\mu_\chi = 0. \quad (2.13) \]

Therefore the linearized Einstein’s equations become

\[ \Box \bar{h}_{\mu \nu} = -16\pi G T_{\mu \nu}. \quad (2.14) \]

**Plane wave solution**

The propagation of GWs in vacuum is regarded as a superposition of plane waves. The GW has two independent polarization states and their explicit form is displayed in a particular coordinate system, the transverse-traceless (TT) gauge. The linearized Einstein’s equation in vacuum, can be written as

\[ \Box \bar{h}_{\mu \nu} = 0. \quad (2.15) \]

Since the trace \( \bar{h} = -h \) satisfies the same wave equation,

\[ \Box h_{\mu \nu} = 0. \quad (2.16) \]

Consider a plane wave solution in the form of

\[ h_{\mu \nu}(x) = \epsilon_{\mu \nu} e^{\text{i} n^\alpha x^\alpha}, \quad (2.17) \]

where \( \epsilon_{\mu \nu} \) is the symmetric polarization tensor, i.e,

\[ \epsilon_{\mu \nu} = \epsilon_{\nu \mu}, \quad (2.18) \]

and \( n^\alpha \) is the 4-wavevector \( n^\alpha = (\omega, n) \). Substituting eq.(2.17) in eq.(2.16), obtain

\[ n^2 \epsilon_{\mu \nu} e^{\text{i}nx} = 0, \quad (2.19) \]

thus the wavevector is a null-vector, and

\[ n^2 = n_\alpha n^\alpha = -\omega^2 + n^2 = 0. \quad (2.20) \]
GWs propagate at the same speed $\omega/|\mathbf{n}| = c = 1$ as electromagnetic waves. Furthermore, since the wave eq.(2.16) valid only in the coordinates satisfying the Lorentz gauge condition eq.(2.10) and the polarization tensor is transverse:

$$n^\mu \epsilon_{\mu\nu} = 0.$$  \hspace{1cm} (2.21)

**The transverse-traceless gauge**

There is still some residual gauge freedom left: one can make further coordinate gauge transformations as long as the transverse condition eq.(2.21) is not violated. This requires that the associated gauge vector function $\chi_\mu$ be constrained by the condition:

$$\Box \chi_\mu = 0.$$  \hspace{1cm} (2.22)

Such coordinate freedom can be used to simplify the polarization tensor, one can pick $\epsilon_{\mu\nu}$ to be traceless, 

$$\epsilon_\mu^\mu = 0,$$  \hspace{1cm} (2.23)

as well as 

$$\epsilon_{\mu0} = \epsilon_{0\mu} = 0.$$  \hspace{1cm} (2.24)

This particular choice of the coordinates is called the transverse-traceless gauge, which is a subset of coordinates satisfying the Lorentz gauge condition.

The $4 \times 4$ symmetric polarization matrix $\epsilon_{\mu\nu}$ has ten independent elements. Equations (2.21), (2.23), and (2.24) which superficially represent nine conditions actually fix only eight parameters because the condition $n^\mu \epsilon_{\mu0} = 0$ is trivially satisfied by eq.(2.24). Thus $\epsilon_{\mu\nu}$ has only two independent elements and hence GW has only two independent polarization states. Consider a wave propagating in the $z$ direction $n^\alpha = (\omega, 0, 0, \omega)$, the transversality condition together with eq.(2.24) implies that $\epsilon_{3\nu} = \epsilon_{\nu3} = 0$. Together with the
conditions, eq.(2.23) and eq.(2.24), the metric perturbation has the form

$$h_{\mu\nu}(z, t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h^+ & h^\times & 0 \\ 0 & h^\times & -h^+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i\omega(z-t)}.$$  (2.25)

The two polarization modes can be taken to be

$$\epsilon_{\mu\nu}^{(+)} = h^+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \epsilon_{\mu\nu}^{(\times)} = h^\times \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$  (2.26)

with $h^+$ and $h^\times$ are the respectively known as “plus” and “cross” amplitudes.

### 2.2 Gravitational Waves in Expanding Universe

The perturbed metric for a homogeneous isotropic flat FLRW universe can be written as

$$ds^2 = S^2(\eta)(d\eta^2 - (\delta_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu),$$  (2.27)

where $S(\eta)$ is the cosmological scale factor, $\eta$ is the conformal time defined by $d\eta = dt/S$ and $\delta_{\mu\nu}$ is the Kronecker delta symbol. The $h_{\mu\nu}$ are metric perturbations field containing only the pure GWs and is transverse-traceless i.e; $\nabla_\mu h^{\mu\nu} = 0, \delta^{\mu\nu}h_{\mu\nu} = 0$.

The present study mainly deals with the amplitude and spectral energy density of the relic GWs generated by the expanding space-time. Thus the perturbed matter source is therefore not taken into account in present work. Since the relic GWs are very weak one needs consider only the linearized field equation given by

$$\nabla_\lambda \left( \sqrt{-g} \nabla^\lambda h_{\mu\nu}(x, \eta) \right) = 0.$$  (2.28)
Gravitational Waves

For a fixed wave number $n = |\mathbf{n}|$, here after $n$ is the comoving wave number unless it is mentioned, and a fixed polarization state $\mathbf{p}$ the linearized wave equation (2.28) gives [31]

$$h''_n(\eta) + 2 \frac{S'}{S} h'_n(\eta) + n^2 h_n(\eta) = 0,$$

(2.29)

where $'=d/d\eta$ means derivative with respect to the conformal time. The tensor perturbations have two independent physical degrees of freedom and are denoted as $h^+$ and $h^\times$, called polarization modes. Since each polarization state is same, here onwards we denote $h_n(\eta)$ without the polarization index.

Next, we rescale the filed $h_n(\eta)$ by taking

$$h_n(\eta) = \frac{\mu_n(\eta)}{S(\eta)},$$

(2.30)

where the mode functions $\mu_n(\eta)$ obey the minimally coupled Klein-Gordon equation

$$\mu''_n + \left( n^2 - U(\eta) \right) \mu_n = 0,$$

(2.31)

where $U(\eta) = S''/S$.

The general solution of eq.(2.31) is a linear combination of Hankel’s function, $H^{(1)}$ and $H^{(2)}$ with a generic power-law for the scale factor $S = \eta^q$, given by

$$\mu_n(\eta) = A_n \sqrt{\eta} H^{(1)}_{(q-\frac{1}{2})}(n\eta) + B_n \sqrt{\eta} H^{(2)}_{(q-\frac{1}{2})}(n\eta).$$

(2.32)

For a given model of the expansion of universe, consisting of a sequence of scale factors with different $q$, we can obtain a solution $\mu_n(\eta)$ by matching its value and derivative at successive stages.

### 2.3 Parametric Amplification of GWs

The main purpose of this section is to discuss the parametric (superadiabatic) amplification of relic GWs in flat FLRW universe [31, 32].
The equation (2.31) describes an oscillator with the varying frequency, known as parametrically excited oscillator. The external gravitational field is represented by the cosmological scale factor and it plays the role of a “pump” field supplying energy to the oscillator [33].

In the intervals of $\eta$-time such that $n^2 \gg |U(\eta)|$ the solutions of equation (2.31) have the form $\mu = e^{\pm in\eta}$, and are high-frequency waves with adiabatically changing amplitude $h = (1/S) \sin(n\eta + \varphi)$, where $\varphi$ is the phase of the wave. In an expanding universe, the amplitude decreases. The amplitudes of the waves with $n$ such that $n^2 \gg |U(\eta)|$ decrease adiabatically for all $\eta$.

If for a given $n$ there is an interval of time when $n^2 \ll |U(\eta)|$, the solutions to the second-order differential equation (2.31) are no longer oscillatory. In the case $U(\eta) = S''/S$ they are $\mu_1 = S$ and $\mu_2 = S \int S^{-2} d\eta$. The waves satisfying $n^2 \ll |U(\eta)|$ for some $\eta$ encounter the potential barrier\footnote{The terminology ‘barrier’ is adapted for ‘horizon’ from [33].} and are governed by the solutions $\mu_1$ and $\mu_2$ in the under-barrier region. The amplitude $\mu_f$ of the function $\mu(\eta)$ right after exit of the wave under the barrier depends on the initial phase $\varphi$ of the wave. The exiting amplitude $\mu_f$, can be larger or smaller than the entering amplitude $\mu_i$ defined right before the wave encounter the barrier. However, averaging $(\mu_f)^2$ over the initial phase $\varphi$ always leads to the dominant contribution from the solution $\mu_1$. This means that the adiabatic factor $1/S$ is cancelled out by $\mu_1 = S$ and the amplitude $h$ (with the factor $1/S$ taken into account) of a ‘typical’ wave can be regarded as remaining constant in the region occupied by the barrier. It stays constant instead of diminishing adiabatically, as the waves above the potential barrier do. Thus, the exiting amplitude $h_f$ of a ‘typical’ wave is equal to the entering amplitude $h_i$ and is larger than it would have been if the wave behaved adiabatically.

The amplification coefficient $R(n)$ for a given $n$ is the ratio $S(\eta_f)/S(\eta_i)$ where $S(\eta_i)$ is the value of the scale factor at the last oscillation of the wave.
before entering the under-barrier region, and \( S(\eta_f) \) is the value of the scale factor at the first oscillation of the wave after leaving the under-barrier region.

The waves with different wave numbers \( n \), stay under the potential barrier for different intervals of time. This means that, in general, the amplification coefficient depends on \( n \): \( R(n) = 1 \) for all \( n \) above the top of the potential, and \( R(n) \gg 1 \) for smaller \( n \). The initial spectrum of the waves \( h(n) = A(n)/S \), defined at some \( \eta \) well before the interaction began, transforms into the final spectrum \( h(n) = B(n)/S \), defined at some \( \eta \) well after the interaction is completed. The transformation occurs according to the rule: \( B(n) = R(n)A(n) \). This is the essence of the mechanism of the superadiabatic (parametric) amplification of GWs and, in fact, of any other fluctuations obeying similar equations [33].