In the previous chapters 3, 4, and 5, it has been emphasized that one can create ‘$N$’ number of converging units, correspondingly ‘$N$’ Gaussian modes by inserting ‘$N$’ lenses in a cavity. The aim of achieving the accumulated Gouy phase variation of $N\pi$ is existing repeatedly.

Note again the overall conclusions of the analysis of SCURC, 2CURC and 3CURC in the following.

(a) SCURC for which $N = 1$, shows the Gouy phase variation of $0 - \pi$ or $\pi - 0$.

(b) The case of $N = 2$ corresponds to 2CURC, which can behave as SCURC($N = 1$) under the conditions of $L$, one can have the accu-
mulated Gouy phase variation either $0 - \pi$ or $\pi - 0$ in the limit of SCURC and $\pi - 2\pi$ or $2\pi - \pi$ or $\pi - 2\pi - \pi$ where two modes are present in 2CURC.

(c) The 3CURC for which $N = 3$, can be converted to SCURC and 2CURC, shows the accumulated Gouy phase variation discussed in the above two cases (a) and (b), in addition it also shows the accumulated Gouy phase variation of $2\pi - 3\pi$ or $3\pi - 2\pi$ or $2\pi - 3\pi - 2\pi$ when all the three modes are present in 3CURC.

From the analysis of 3CURC, it is noticed that there exist four disjointed stable regions, for $L > L_M = 4f_1 + 4f_2 + 4F$ and realized that each of them corresponds to a different phase variation. As a result, one can not be able to achieve an accumulation of the variable Gouy phase up to $3\pi$, even though different phase regions are present. As length $L$ varies from $L_M$ to lower values of $L$, it is noticed that the merging of these distinct stable regions starts taking place. In such a case the amount of accumulated Gouy phase variation is turning out to be $\pi$ and even less than $\pi$ for some cases.

One of the reasons for the existence of disjointed regions could be the creation of an odd number of modes in between the three lenses in the cavity. Therefore as a final attempt here in this chapter we consider 4CURC, to create an even number of modes i.e., four converging gaussian modes and also symmetry between the optical elements of the cavity, to see whether it is possible to get an accumulation of Gouy phase up to $N\pi$ or not. We describe the configuration of 4CURC in the following.

### 6.1 4 Converging Unit Ring Cavity (4CURC)

A schematic of the 4CURC is shown in the Fig. (6.1). This ring configuration consists four lenses $L_1, L_2, L_3$ and $L_4$ of focal lengths $f_1, f_2, f_3$.
and \( f_4 \) which are separated from one another by a distances of \( d_1, d_2, d_3 \) and \( L - d_1 - d_2 - d_3 \) respectively, where \( L \) is the length of the cavity. In addition to these four lenses, it also contains four 100\% plane mirrors \( M_1, M_2, M_3 \) and \( M_4 \).

For this configuration there exist four modes, one in the arm of \( d_1 \) in between the lenses \( L_1 \) and \( L_2 \), the second mode in the arm of length \( d_2 \), in between the lenses \( L_2 \) and \( L_3 \), the third mode in the arm of length \( d_3 \) in between the lenses \( L_3 \) and the lens \( L_4 \) and the fourth mode in the arm of length \( (L - d_1 - d_2 - d_3) \) in between the lenses \( L_4 \) and \( L_1 \). The Gaussian beams in the respective sections constitute a part of stable configuration of radiation-travel in the ring, with half Rayleigh ranges \( z_{0x} \) in the \( d_1 \) arm, \( z_{0y} \) in the \( d_2 \) arm, \( z_{0d} \) in the \( d_3 \) arm and \( z_{0b} \) in the \( L - d_1 - d_2 - d_3 \) arm; with
waist locations $x_{0s}$, $x_{0y}$, $x_{0d}$ and $x_{0b}$ respectively.

Here also each Gaussian mode contributes its Gouy phase to the round trip Gouy phase in the ring. One can fix the gain medium in any one of the three separation distances $d_1$, $d_2$, $d_3$ and vary the remaining two distances, to vary the effective focal length of the combinations, to vary $\phi_G(RT)$ which is the sum of the Gouy phases of the four parts of the 4CURC.

We define here the coordinate $x_p$ in the clockwise sense in the ring with $L_1$ at $x_p = 0$, $L_2$ at $x_p = d_1$, $L_3$ at $x_p = d_1 + d_2$ and $L_4$ at $x_p = d_1 + d_2 + d_3$. For the four Gaussian modes with half Rayleigh ranges $z_{0s}$, $z_{0y}$, $z_{0d}$ and $z_{0b}$, the points $x_{0s}(x_p = d_1 - x_{0s})$, $x_{0y}(x_p = d_2 - x_{0y})$, $x_{0d}(x_p = d_3 - x_{0d})$, and $x_{0b}(x_p = L - d_1 - d_2 - d_3 - x_{0b})$ determine the positions of the beam waists $\omega_{0s}$, $\omega_{0y}$, $\omega_{0d}$ and $\omega_{0b}$ respectively. Next we study and analyze the stability of the 4CURC in detail in the following section.

### 6.2 Stability Analysis:

For consideration of the stability let the radiation just enter before the lens $L_1$ of focal length $f_1$ and refracts off it, in the clockwise direction travel through a distance $d_1$ from the lens $L_1$, as it hits another lens $L_2$ of focal length $f_2$, travel through a distance $d_2$ from the lens $L_2$, strikes the lens $L_3$ of focal length $f_3$, travel through a distance $d_3$ from the lens $L_3$, hits the lens $L_4$ of focal length $f_4$, it refracts off it, reflecting off the mirrors $M_2$, $M_3$, $M_4$ and $M_1$ in the order, travel through the distance $(L - d_1 - d_2 - d_3)$, complete the round trip just before the lens $L_1$ where it starts its journey. The round trip $ABCD$-matrix at the lens $L_1$ in the ring is

\[
\begin{pmatrix}
A & B \\
C & D \\
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 1 \\
L - d_1 - d_2 - d_3 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 \\
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 & 1 \\
-\frac{1}{f_4} & 1 & 0 & 1 \\
0 & 1 & 0 & 1 \\
-\frac{1}{f_3} & 1 & 0 & 1 \\
-\frac{1}{f_2} & 1 & 0 & 1 \\
-\frac{1}{f_1} & 1 & 0 & 1 \\
\end{pmatrix}
\]

(6.1)
The complete round trip matrix elements can be calculated using matrix multiplication. The self consistency requires that
\[(A + D)^2 - 4 \leq 0 \quad (6.2)\]

The two alternatives for the marginal stability case of (6.2)

\[
A + D + 2 = \frac{1}{f_1f_2f_3f_4}[-d_3^2(f_1f_2 - f_1f_3 - f_2f_3) + d_2^2(f_1 + f_2)(d_3 - f_3 - f_4) + 4f_1f_2f_3f_4 + d_1^2(-f_2f_3 + d_3(f_2 + f_3) - (f_2 + f_3)f_4 + d_2(-d_3 + f_3 + f_4))
\]
\[
d_3f_1f_2L + d_3f_1f_3L + d_3f_2f_3L - f_1f_2f_3L - f_1f_2f_4L - f_1f_3f_4L - f_2f_3f_4L
\]
\[
+ d_2(f_1 + f_3) (d_3^2 + (f_3 + f_4)L - d_3(2f_3 + L)) + d_1d_3^2(f_2 + f_3)
\]
\[
+ d_1d_3^2(-d_3 + f_3 + f_4) + d_1(f_2f_3 + (f_2 + f_3)f_4)L - d_4d_3(2f_2f_3 + (f_2 + f_3)L)
\]
\[
- d_1d_2 (d_3^2 + (f_3 + f_4)(2f_2 + L) - d_3(2f_2 + f_3) + L)] \quad (6.3)
\]

\[
A + D - 2 = \frac{1}{f_1f_2f_3f_4}[-d_3^2(f_2f_3 + f_1(f_2 + f_3)) + d_2^2(f_1 + f_2)(d_3 - f_3 - f_4)
\]
\[
+ d_1^2(-f_2f_3 + d_3(f_2 + f_3) - (f_2 + f_3)f_4 + d_2(-d_3 + f_3 + f_4))
\]
\[
+ (d_3f_1f_2 + d_3f_1f_3 + d_3f_2f_3 - f_1f_2f_3 - (f_2f_3 + f_1(f_2 + f_3))f_4)L
\]
\[
+ d_2(f_1 + f_3) (d_3^2 + (f_3 + f_4)L - d_3(2f_3 + L)) + d_1d_3^2(f_2 + f_3)
\]
\[
+ d_1(-d_3^2(-d_3 + f_3 + f_4) + (f_2f_3 + (f_2 + f_3)f_4)L - 2d_1d_3f_2f_3
\]
\[
+ d_1d_3(f_2 + f_3)L - d_1d_2 (d_3^2 + (f_3 + f_4)(2f_2 + L) - d_3(2f_2 + f_3) + L)]
\]
\]
\[
(6.4)
\]

For this case the marginal stability of the condition (6.2) becomes
\[(A + D)^2 - 4 = -4 + \frac{1}{f_1f_2f_3f_4}[d_3^2(f_1f_2 + f_1f_3 + f_2f_3) - d_2^2(f_1 + f_2)(d_3 - f_3 - f_4)
- 2f_1f_2f_3f_4 + d_1^2(f_2f_3 - d_3(f_2 + f_3) + d_2(d_3 - f_3 - f_4) + (f_2 + f_3)f_4)
- d_3f_1f_2L - d_3f_1f_3L - d_3f_2f_3L + f_1f_2f_3L + f_1f_2f_4L + f_1f_3f_4L
+ f_2f_3f_4L - d_2(f_1 + f_3) (d_3^2 + (f_3 + f_4)L - d_3(2f_3 + L))
+ d_1d_3(f_2 + f_3)L - d_1(f_2f_3 + (f_2 + f_3)f_4)L
+ d_1d_2 (d_3^2 + (f_3 + f_4)(2f_2 + L) - d_3(2f_2 + f_3) + L)]^2 \quad (6.5)
\]
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The configuration of 4CURC in the Fig. (6.1) can behave equivalent to 3CURC, 2CURC and as well as SCURC, by making the separation distances go to zero as required. Before going to the analysis of the 4CURC, let us look at various possibilities, where these conversions are being taking place. First, we start with the conversion from 4CURC to 3CURC in the following section.

6.2.1 Conversion from 4CURC to 3CURC

There are four different possibilities where the configuration of the 4CURC can be converted to 3CURC as shown in the Fig. (6.2).

1(a). Consider the case when \( d_1 = 0 \), then (6.5) becomes (5.10), with the effective focal length \( \left( \frac{f_1 f_2}{f_1 + f_2} \right) \) of the combination of lenses \( L_1 \) and \( L_2 \) now behaves as a lens \( L_1 \) of 3CURC as shown in the first configuration of the Fig. (6.2).

For this case of 3CURC, we denote the value of \( L_M \) of 3CURC by \( L_{M(1,2),3,4} \), and is given by

\[
L_{M(1,2),3,4} = 4 \left( \frac{f_1 f_2}{f_1 + f_2} \right) + 4f_3 + 4f_4 \quad (6.6)
\]

1(b). Consider the case when \( d_2 = 0 \), then (6.5) becomes (5.10), with the effective focal length \( \left( \frac{f_2 f_3}{f_2 + f_3} \right) \) of the combination of lenses \( L_2 \) and \( L_3 \) now behaves as a lens \( L_2 \) of 3CURC as shown in the second configuration of the Fig. (6.2).

The value of \( L_{M_1,(2,3),4} \) is given by

\[
L_{M_1,(2,3),4} = 4f_1 + 4 \left( \frac{f_2 f_3}{f_2 + f_3} \right) + 4f_4 \quad (6.7)
\]
1(c). Consider the case when \( d_3 = 0 \), then (6.5) becomes (5.10), with the effective focal length \( \frac{f_3 f_4}{f_3 + f_4} \) of the combination of lenses \( L_3 \) and \( L_4 \) now behaves as a lens \( L_3 \) of 3CURC as shown in the third configuration of the Fig. (6.2). For this case

\[
L_{M_{1,2,(3,4)}} = 4f_1 + 4f_2 + 4 \left( \frac{f_3 f_4}{f_3 + f_4} \right) \quad (6.8)
\]

1(d). Consider the case when \( L - d_1 - d_2 - d_3 = 0 \), then (6.5) becomes (5.10), with the effective focal length \( \frac{f_4 f_1}{f_4 + f_1} \) of the combination of
lenses $L_4$ and $L_1$ now behaves as a lens $L_1$ of 3CURC as shown in the fourth configuration of the Fig. (6.2).

The value of $L_{M_{1,2,3,4}}$ is given by

$$L_{M_{(4,1),2,3}} = 4\left(\frac{f_4 f_1}{f_4 + f_1}\right) + 4f_2 + 4f_3 \quad (6.9)$$

For the case of $f_1 < f_2 < f_3 < f_4$, the value of $L_{M_{(1,2,3,4)}} (6.6)$ is the largest length among the four values. In the following we look for the possibilities of converting 4CURC to 2CURC.

### 6.2.2 Conversion from 4CURC to 2CURC

There are two possible configurations for which the 4CURC can be converted to 2CURC as shown in the Fig. (6.3). To do this, we make any two of the separation distances go to zero.

Figure 6.3: Conversion from 4CURC to 2CURC
1(a). Consider the case $d_1 = 0$ and $d_3 = 0$, for which the lenses $L_1$, $L_2$ and $L_3$, $L_4$ are combined to create only two modes in the cavity. Then the condition (6.5) becomes (4.12), the effective focal lengths $\left( \frac{f_1 f_2}{f_1 + f_2} \right)_{4\text{CURC}}$ of the combination of lenses $L_1$ and $L_2$ of 4CURC now behaves as a lens $L_1$ of 2CURC and $\left( \frac{f_3 f_4}{f_3 + f_4} \right)_{4\text{CURC}}$ of the combination of lenses $L_3$ and $L_4$ behaves as a lens $L_2$ of 2CURC as shown in the first configuration of the Fig. (6.3). For this case of 2CURC, we denote $L_M$ of 2CURC by $L_{M(1,2),(3,4)}$, which is given by

$$L_{M(1,2),(3,4)} = 4 \left( \frac{f_1 f_2}{f_1 + f_2} \right) + 4 \left( \frac{f_3 f_4}{f_3 + f_4} \right) \quad (6.10)$$

1(b). Similarly consider the case $d_2 = 0$ and $L = d_1 + d_3$, for which the lenses $L_4$, $L_1$ and $L_2$, $L_3$ are combined to create only two modes in the cavity. Then the condition (6.5) becomes (4.12), with the effective focal lengths $\left( \frac{f_4 f_1}{f_4 + f_1} \right)_{4\text{CURC}}$ of the combination of lenses $L_4$ and $L_1$ of 4CURC behaves as a lens $L_1$ of 2CURC and $\left( \frac{f_2 f_3}{f_2 + f_3} \right)_{4\text{CURC}}$ of the combination of lenses $L_2$ and $L_3$ of 4CURC behaves as a lens $L_2$ of 2CURC as shown in the second configuration of the Fig. (6.3). Here $L_{M(4,1),(2,3)}$ is given by

$$L_{M(4,1),(2,3)} = 4 \left( \frac{f_4 f_1}{f_4 + f_1} \right) + 4 \left( \frac{f_2 f_3}{f_2 + f_3} \right) \quad (6.11)$$

Finally, we discuss the possibility where the 4CURC can become SCURC in the following.

### 6.2.3 Conversion from 4CURC to SCURC

The one and only possibility where the 4CURC can behave as a SCURC is shown in the Fig. (6.4), that is when all the lenses $L_1$, $L_2$, $L_3$ and $L_4$ are
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Figure 6.4: Conversion from 4CURC to SCURC

joined together with \( d_1 = d_2 = d_3 = 0 \). In this case \( L_{\text{eff}1234} \) is given by

\[
L_{\text{eff}1234} = 4 \frac{f_1 f_2 f_3 f_4}{f_1 f_2 f_3 + f_1 f_2 f_4 + f_2 f_3 f_4 + f_3 f_4 f_1}
\]  

(6.12)

So far we have understood that the 4CURC contains all the features of 3CURC, 2CURC and SCURC. As a result, the expectations on 4CURC are high, as it is possible for the 4CURC to contain the different phase regions of 3CURC, 2CURC and SCURC and one may get rid off the forbidden zones just as it done for the 2CURC. Therefore, it is motivating further and giving assurance that one can achieve the accumulated Gouy phase variation more than \( \pi \) using 4CURC. Before going to the analysis of the Gouy phase, we study the stability analysis of 4CURC.

6.2.4 Stability of 4CURC

It is noticed that the stability function (6.5) of the considered 4CURC is quartic in \( d_1, d_2 \) and \( d_3 \). For a given \( L \), the parameters of the system which can be varied to create a variable Gouy phase are \( d_1, d_2 \) and \( d_3 \). Varying any of the distances \( d_1, d_2 \) and \( d_3 \) is equivalent to varying the effective focal length of the combined systems comprising the focal lengths \( (f_1, f_2), (f_2, f_3) \).
and \((f_3, f_4)\). First, we study (6.5) with respect to \(d_1\), keeping the rest of the system parameters fixed in the following.

### 6.2.5 Stable solutions of \(d_1\) for a given \(L, f_1, f_2, f_3, f_4, d_2\) and \(d_3\)

We look for the stable solutions of \(d_1\) in terms of the system parameters \(L, f_1, f_2, f_3, f_4, d_2\) and \(d_3\). The marginal stability of \((A + D + 2)(6.3)\) gives the stable solutions of \(d_1\) as

\[
d^{(1,2)}_{1+} = \frac{[D_0 \pm \sqrt{D_1 D_2}]}{2(f_2 f_3 - d_3 (f_2 + f_3) + d_2 (d_3 - f_3 - f_4) + (f_2 + f_3) f_4)}
\]

where \(D_0, D_1\) and \(D_2\) are given by

\[
D_0 = d_3^2 (f_2 + f_3) + d_2^2 (-d_3 + f_3 + f_4) + (f_3 f_4 + f_2 (f_3 + f_4)) L \\
- d_3^2 (f_2 f_3 + (f_2 + f_3) L) - d_2 (d_3^2 + (f_3 + f_4) (2 f_2 + L) - d_3 (2 (f_2 + f_3) + L)) \tag{6.14}
\]

\[
D_1 = -d_3^2 (f_2 + f_3) + d_2^2 (d_3 - f_3 - f_4) - d_3 (f_2 + f_3) (4 f_1 - L) + f_2 f_3 (4 f_1 - L) \\
+ ((f_2 + f_3) f_4) (4 f_1 - L) + d_2 (d_3^2 + d_3 (4 f_1 - 2 f_3 - L) + (f_3 + f_4) (-4 f_1 + L)) \tag{6.15}
\]

\[
D_2 = -d_3^2 (f_2 + f_3) + d_2^2 (d_3 - f_3 - f_4) + 4 f_2 f_3 f_4 + d_3 (f_2 + f_3) L \\
- (f_2 f_3 + (f_2 + f_3) f_4) L + d_2 (d_3^2 + (f_3 + f_4) L - d_3 (2 f_3 + L)) \tag{6.16}
\]

The parameters \(D_0, D_1\) and \(D_2\) are again quadratic in \(d_2\) and as well as in \(d_3\). Solving \(D_1\) for marginal stability of \(d_2\)

\[
d^{(1,2)}_{2+} = \frac{1}{2 (d_3 - f_3 - f_4)} \left[ (-d_3^2 - d_3 (4 f_1 - 2 f_3 - L) - (f_3 + f_4) (-4 f_1 + L)) \right. \\
\pm \sqrt{d_3^4 + 4 d_3 f_1 - 4 f_1 f_3 - 4 f_1 f_4 - d_3 f_3 + f_3 f_4 + d_3 L + f_3 L + f_4 L} \\
\left. \sqrt{d_3^4 + 4 d_3 f_1 - 4 f_1 f_3 - 4 f_1 f_4 - d_3 f_3 + f_3 f_4 + d_3 L + f_3 L + f_4 L} \right]
\]

\(135\)
(6.17) is again quartic in \(d_3\), first quadratic of the discriminant gives

\[
d^{(1,2)}_{3+} = \frac{1}{2} \left( L - 4f_1 \pm \sqrt{4f_1 - L\sqrt{4f_1 + 4f_3 + 4f_4 - L}} \right)
\] (6.18)

The second quadratic in \(d_3\) of the discriminant gives

\[
d^{(3,4)}_{3+} = \frac{1}{2} \left( L - 4f_1 - 4f_2 \pm \sqrt{4f_1 + 4f_2 + 4f_3 - L\sqrt{4f_1 + 4f_2 + 4f_4 - L}} \right)
\] (6.19)

Similarly solving \(D_2\) for marginal stability

\[
d^{(3,4)}_{2+} = \frac{1}{2(d_3 - f_3 - f_4)^2} \left[ (-d_3^2 - (f_3 + f_4)L + d_3(2f_3 + L)) \right.
\]
\[
\pm \sqrt{d_3^2 + 4d_3f_2 - 4f_2f_3 - 4f_2f_4 - d_3L + f_3L + f_4L}
\]
\[
\left] \right\} \sqrt{d_3^2 - 4f_3f_4 - d_3L + f_3L + f_4L} \] (6.20)

The above condition (6.20) is quartic in \(d_3\), first quadratic of the discriminant gives

\[
d^{(1,2)}_{3+} = \frac{1}{2} \left( L - 4f_2 \pm \sqrt{4f_2 - L\sqrt{4f_2 + 4f_3 + 4f_4 - L}} \right)
\] (6.21)

The second quadratic in \(d_3\) of the discriminant gives

\[
d^{(3,4)}_{3+} = \frac{1}{2} \left( L \pm \sqrt{-4f_3 + L\sqrt{-4f_3 + L}} \right)
\] (6.22)

Finally, for \(d_1\) to be real, there are four pairs of conditions in \(d_3\) which are given by the conditions (6.18), (6.19), (6.21) and (6.22) and can be written as

\[
d^{(1,2)}_{3+} = \frac{L - 4f_1}{2} \pm \frac{r_{11}}{2}
\] (6.23)

\[
d^{(3,4)}_{3+} = \frac{L - 4f_1 - 4f_2}{2} \pm \frac{k_{11}}{2}
\] (6.24)

\[
d^{(1,2)}_{3+} = \frac{L - 4f_2}{2} \pm \frac{t_{11}}{2}
\] (6.25)

\[
d^{(3,4)}_{3+} = \frac{L}{2} \pm \frac{p_{11}}{2}
\] (6.26)
where
\[ r_{11}^2 = (4f_1 - L)(4f_1 + 4f_3 + 4f_4 - L) \] (6.27)
\[ k_{11}^2 = (4f_1 + 4f_2 + 4f_3 - L)(4f_1 + 4f_2 + 4f_4 - L) \] (6.28)
\[ t_{11}^2 = (4f_2 - L)(4f_2 + 4f_3 + 4f_4 - L) \] (6.29)
\[ p_{11}^2 = (L - 4f_3)(L - 4f_4) \] (6.30)

The second alternative of the marginal stability (6.4) gives
\[ d_1^{(1,2)} = \frac{[D_1 \pm \sqrt{D_{11}D_{21}}]}{(2(f_2f_3 - d_3(f_2 + f_3) + d_2(d_3 - f_3 - f_4) + (f_2 + f_3)f_4))} \] (6.31)

where
\[
D_0 = d_3^2(f_2 + f_3) + d_2^2(-d_3 + f_3 + f_4) + (f_3f_4 + f_2(f_3 + f_4)L - 2d_3f_2f_3 \\
- (d_3(f_2 + f_3)L) - d_2 \left( d_3^2 + (f_3 + f_4)(2f_2 + L) - d_3(2(f_2 + f_3) + L) \right) \] (6.32)
\[
D_{11} = -d_3^2(f_2 + f_3) + d_2^2(d_3 - f_3 - f_4) + 4(f_1f_2f_3 + f_2f_3f_4 + f_1(f_2 + f_3)f_4) \\
- d_3(f_2 + f_3)(4f_1 - L) - (f_3f_4 + f_2(f_3 + f_4)L \\
+ d_2 \left( d_3^2 + d_3(4f_1 - 2f_3 - L) + (f_3 + f_4)(-4f_1 + L) \right) \] (6.33)
\[
D_{21} = -(f_2f_3 + (f_2 + f_3)f_4)L + d_2 \left( d_3^2 + (f_3 + f_4)L - d_3(2f_3 + L) \right) \] (6.34)

\(D_{11}\) and \(D_{21}\) are quadratic in \(d_2\) and as well as in \(d_3\). Solving \(D_{11}\) for marginal stability of \(d_2\)
\[
d_2^{(1,2)} = \frac{1}{2(d_3 - f_3 - f_4)} \left[ \left( d_3^2 - d_3(4f_1 - 2f_3 - L) - (f_3 + f_4)(-4f_1 + L) \right) \right. \]
\[\quad \pm \sqrt{d_3^2 + 4d_3f_1 + 4d_3f_2 - 4f_1f_3 - 4f_2f_3 - 4f_1f_4 - 4f_2f_4 - d_3L + f_3L + f_4L} \]
\[\left. \pm \sqrt{d_3^2 + 4d_3f_1 - 4f_1f_3 - 4f_1f_4 - 4f_3f_4 - d_3L + f_3L + f_4L} \right] \] (6.35)
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From the first quadratic of the discriminant, the marginal stability for $d_3$ gives

$$d_{3_{-}}^{(1, 2)} = \frac{1}{2} \left( L - 4f_1 - 4f_2 \pm \sqrt{4f_1 + 4f_2 - L \sqrt{4f_1 + 4f_2 + 4f_3 + 4f_4 - L}} \right)$$

(6.36)

The second quadratic in $d_3$ of the discriminant gives

$$d_{3_{-}}^{(3, 4)} = \frac{1}{2} \left( L - 4f_1 \pm \sqrt{4f_1 - L \sqrt{4f_1 + 4f_4 - L}} \right)$$

(6.37)

Similarly solving $D_{21}$ for marginal stability of $d_2$

$$d_{2_{-}}^{(3, 4)} = \frac{1}{2(2d_2 - f_3 - f_4)} \left[ (-d_2^2 + 2d_3f_3 + d_3L - f_3L - f_4L) \right.
\pm \sqrt{d_2^2 - d_3L + f_3L + f_4L} \n
\left. \sqrt{d_2^2 + 4d_3f_2 - 4f_2f_3 - 4f_2f_4 - 4f_3f_4 - d_3L + f_3L + f_4L} \right]$$

(6.38)

Again the above equation is quartic in $d_3$, first quadratic of the discriminant gives

$$d_{3_{-}}^{(1, 2)} = \frac{1}{2} \left( L \pm \sqrt{4f_3L - 4f_4L + L^2} \right)$$

(6.39)

The second quadratic in $d_3$ of the discriminant gives

$$d_{3_{-}}^{(3, 4)} = \frac{1}{2} \left( L - 4f_2 - \sqrt{4f_2 + 4f_3 - L \sqrt{4f_2 + 4f_4 - L}} \right)$$

(6.40)

Finally, for $d_1$ to be real, there are four pairs of conditions in $d_3$ which are given by the conditions (6.36), (6.37), (6.39) and (6.40) can be written as

$$d_{3_{-}}^{(1, 2)} = \frac{L - 4f_1 - 4f_2}{2} \pm \frac{s_{11}^1}{2}$$

(6.41)

$$d_{3_{-}}^{(3, 4)} = \frac{L - 4f_1}{2} \pm \frac{q_{11}^1}{2}$$

(6.42)

$$d_{3_{-}}^{(1, 2)} = \frac{L}{2} \pm \frac{m_{11}^1}{2}$$

(6.43)
where

\[ S_{11}^2 = (4f_1 + 4f_2 - L)(4f_1 + 4f_2 + 4f_3 + 4f_4 - L) \]
\[ q_{11}^2 = (4f_1 + 4f_3 - L)(4f_1 + 4f_4 - L) \]
\[ m_{11}^2 = (-4f_3L - 4f_4L + L^2) \]
\[ i_{11}^2 = (4f_2 + 4f_3 - L)(4f_2 + 4f_4 - L) \]

For \( f_1 = f_2 = 50\text{cm}, \ f_3 = f_4 = 100\text{cm}, \ L = 700\text{cm}, \ d_1 = 120\text{cm}, \ d_3 = 201\text{cm}, \)

![Figure 6.5: For \( f_1 = f_2 = 50\text{cm}, \ f_3 = f_4 = 100\text{cm}, \ L = 700\text{cm}, \ d_2 = 200\text{cm}, \ d_3 = 100\text{cm}, \) the stability functions \((A + D)^2 - 4(6.5), \ A + D + 2(6.3), \ A + D - 2(6.4)\) with respect to \( d_1 = \frac{L - d_2 - d_3}{2}\)](image)

the stability function (6.5) and the two parabolas (6.3), (6.4) with respect to \( d_1 \) are plotted in the Fig. (6.5). The stability function i.e., the product of \((A + D + 2)\) and \((A + D - 2)\) is a parabola, but not inverted as in the earlier cases. For this case it is in the direction of the individual parabolas of \( A + D \pm 2\) only, for a given \( d_2 \) and \( d_3 \). The above four solutions \( d_{1+}^{(1, 2)}, \ d_{1-}^{(3, 4)} \)
are identified in the Fig. (6.5). To know the behaviour of the function (6.5) further, we determine its extremum points by taking the partial derivative with respect to \( d_1 \) and equating it to zero, i.e.,

\[
\frac{\partial}{\partial d_1} [(A + D)^2 - 4] = \frac{2}{f_1 f_2 f_3 f_4} [2d_3 f_2 f_3 - d_2^2 (f_2 + f_3) + d_2^2 (d_3 - f_3 - f_4) \\
+ 2d_1 (f_2 f_3 - d_3 (f_2 + f_3) + d_2 (d_3 - f_3 - f_4) + (f_2 + f_3) f_4) + d_3 (f_2 + f_3) L \\
- (f_2 f_3 + (f_2 + f_3) f_4) L + d_2 \left( d_3^2 + (f_3 + f_4) (2f_2 + L) - d_3 (2(f_2 + f_3) + L) \right) \\
[d_3^2 f_1 f_2 + d_3^2 f_1 f_3 + d_3^2 f_2 f_3 - d_2^2 (f_1 + f_2) (d_3 - f_3 - f_4) - 2f_1 f_2 f_3 f_4 \\
+ d_1^2 (f_2 f_3 - d_3 (f_2 + f_3) + d_2 (d_3 - f_3 - f_4) + (f_2 + f_3) f_4) \\
- d_3 f_1 f_2 L - d_3 f_1 f_3 L - d_3 f_2 f_3 L + f_1 f_2 f_3 L + f_1 f_2 f_4 L + f_1 f_3 f_4 L + f_2 f_3 f_4 L \\
- d_2 (f_1 + f_2) \left( d_3^2 + (f_3 + f_4) L (-d_3 (2f_3 + L)) \right) \\
+ d_1^2 (d_3 f_2 f_3 - d_3^2 f_2 + f_3) + d_2^2 (d_3 - f_3 - f_4) + d_3 (f_2 + f_3) L \\
-(f_2 f_3 + (f_2 + f_3) f_4) L + d_2 \left( d_3^2 + (f_3 + f_4) (2f_2 + L) - d_3 (2(f_2 + f_3) + L) \right))] = 0 \\
(6.49)
\]

One can get three solutions of \( d_1 \), which are obtained as

\[
d_{1C} = \frac{\begin{pmatrix}
-2d_3 f_2 f_3 + d_3^2 (f_2 + f_3) - d_2^2 (d_3 - f_3 - f_4) \\
-d_3 (f_2 + f_3) L + (f_2 f_3 + (f_2 + f_3) f_4) L \\
-d_2 \left( d_3^2 + (f_3 + f_4) (2f_2 + L) - d_3 (2(f_2 + f_3) + L) \right)
\end{pmatrix}}{2(f_2 f_3 - d_3 (f_2 + f_3) + d_2 (d_3 - f_3 - f_4) + (f_2 + f_3) f_4)} \\
(6.50)
\]
$d_1^{(1,2)} = d_{1C} \pm \frac{1}{2(f_2f_3 - d_3(f_2 + f_3) + d_2(d_3 - f_3 - f_4) + (f_2 + f_3)f_4) \sqrt{(-4(f_2f_3 - d_3(f_2 + f_3) + d_2(d_3 - f_3 - f_4) + (f_2 + f_3)f_4)}}$

$(d^2_3f_1f_2 + d^2_3f_1f_3 + d^2_3f_2f_3 - d^2_2(f_1 + f_2)(d_3 - f_3 - f_4) - 2f_1f_2f_3f_4 - d_3f_1f_2L$

$- d_3f_1f_3L - d_3f_2f_3L + f_1f_2f_3L + f_1f_2f_4L + f_1f_3f_4L + f_2f_3f_4L$

$- d_2(f_1 + f_2)(d_3^2 + (f_3 + f_4)L - d_3(2f_3 + L))$

$+ (2d_3f_2f_3 - d^2_3(f_2 + f_3) + d^2_3(d_3 - f_3 - f_4) + d_3(f_2 + f_3)L$

$-(f_2f_3 + (f_2 + f_3)f_4)L + d_2(d^2_3 + (f_3 + f_4)(2f_2 + L) - d_3(2(f_2 + f_3) + L)) \right)^2}$

(6.51)

The determination of height $H_1$ from axis $d_1$ of the central extremum can be done by the substitution of $d_1$ given by (6.50) in the condition (6.5). We have

$H_1 = \frac{\left( d_2 - d_{2+}^{(1,2)} \right) \left( d_2 - d_{2+}^{(3,4)} \right) \left( d_2 - d_{2-}^{(1,2)} \right) \left( d_2 - d_{2-}^{(3,4)} \right)}{(16f_1^2f_2^2f_3^2f_4^2(f_2f_3 - d_3(f_2 + f_3) + d_2(d_3 - f_3 - f_4) + (f_2 + f_3)f_4)^2)}$

(6.52)

where $d_2^{(1,2)}$, $d_2^{(3,4)}$, $d_2^{(1,2)}$ and $d_2^{(3,4)}$ are related to the coefficients $r_{11}$, $k_{11}$, $t_{11}$, $p_{11}$, $s_{11}$, $q_{11}$, $m_{11}$ and $i_{11}$ which are given by the conditions (6.23-6.26), and (6.45-6.48). $H_1 \geq 0$ depending on the values of $d_2$ for a given $L$, $d_3$, $f_1$, $f_2$, $f_3$ and $f_4$ which also decide the stable zones of $d_1$. One can also look for the values of $L$ numerically, where $H_1$ becomes $-4$. In the next section we construct the length chart of 4CURC, in which all the possible cases of $L$ will be consolidated, for the case of $f_1 < f_2 < f_3 < f_4$.

### 6.3 The Length Chart of 4CURC

The parameters $r_{11}$, $k_{11}$, $t_{11}$, $p_{11}$, $s_{11}$, $q_{11}$, $m_{11}$ and $i_{11}$ which are given in the Fig. (6.6) show that the maximum length $L_M$ for the 4CURC is now extended up to $4f_1 + 4f_2 + 4f_3 + 4f_4$ and also additional new marginal
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lengths are generated in between, compare to the case of 3CURC. It can be generalized that adding an additional lens in the cavity is allowing one to extend the stable length of the corresponding cavity and also generating new additional lengths as it has been realized earlier for the cases 2CURC, 3CURC and now for 4CURC. If we consider the focal lengths \( f_1, f_2, f_3 \) and \( f_4 \) to be prime numbers then twenty three distinct and finite values of length \( L \) are possible for 4CURC as shown in the Fig. (6.7). But for the case of non primed focal lengths, the coincidence of marginal stable lengths \( L \) takes place.

Therefore for a fixed \( d_1 \), at the marginal stable \( L \) we study the behavior of the stability between the separation distances \( d_2 \) and \( d_3 \).

To analyze the stability condition (6.5), we construct 3D plots for different cases of \( L \) of 4CURC, when all the four modes present, where (6.5)i.e., \((A + D)^2 - 4\) is along z axis, while \( d_2 \) and \( d_3 \) are respectively along x and y axes. We also construct plots for the cases of \( L \), where 4CURC can get converted to 3CURC, 2CURC and SCURC, to show the behavior of stability.

<table>
<thead>
<tr>
<th>Formula 1</th>
<th>Formula 2</th>
<th>Formula 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L_{(4,1)} = 4 \frac{f_1 f_2 f_3 f_4}{f_1 f_2 + f_1 f_3 + f_1 f_4 + f_2 f_3 + f_2 f_4 + f_3 f_4} )</td>
<td>( L_{(4,1,2)} = 4 \left( \frac{f_1 f_2}{f_1 + f_2} + 4 f_3 + 4 f_4 \right) )</td>
<td>( L_{(4,1,2,3)} = 4 \left( \frac{f_1 f_2}{f_1 + f_2} + 4 f_3 + 4 f_4 \right) )</td>
</tr>
<tr>
<td>( L_{(4,2)} = 4 \left( \frac{f_1 f_2}{f_1 + f_2} + 4 f_3 + 4 f_4 \right) )</td>
<td>( L_{(4,2,3)} = 4 \left( \frac{f_1 f_2}{f_1 + f_2} + 4 f_3 + 4 f_4 \right) )</td>
<td>( L_{(4,2,3,4)} = 4 \left( \frac{f_1 f_2}{f_1 + f_2} + 4 f_3 + 4 f_4 \right) )</td>
</tr>
</tbody>
</table>

Figure 6.6: Description of length parameters in terms of focal lengths of 4CURC

\[ \begin{align*}
L_1 &= (4 f_1 - L)(4 f_2 + 4 f_3 + 4 f_4 - L) \\
L_2 &= (4 f_2 + 4 f_3 + 4 f_4 - L)(4 f_2 + 4 f_3 + 4 f_4 - L) \\
L_3 &= (4 f_2 - L)(4 f_2 + 4 f_3 + 4 f_4 - L) \\
L_4 &= (L - 4 f_2)(L - 4 f_2) \\
L_5 &= (4 f_1 + 4 f_2 - L)(4 f_1 + 4 f_2 - L) \\
L_6 &= (4 f_1 + 4 f_2 - L)(4 f_1 + 4 f_2 - L) \\
L_7 &= (L - 4 f_2)(L - 4 f_2) \\
L_8 &= (4 f_2 + 4 f_3 - L)(4 f_2 + 4 f_3 - L)
\end{align*} \]
Figure 6.7: Length chart for 4CURC for the case of $f_1 < f_2 < f_3 < f_4$
function with the corresponding variation of the length of the arms which also noted in Fig. (6.7).

For the considered case of \( f_1 < f_2 < f_3 < f_4 \), we choose \( f_1 = 50\text{cm}, f_2 = 75\text{cm}, f_3 = 100\text{cm}, f_4 = 125\text{cm} \) and \( d_1 = 160\text{cm} \).

(a). \( L > L_M \). For a fixed \( L = L_M = 1500\text{cm} \) the stable contours of \( d_2 \) and \( d_3 \) are plotted in the Fig. (6.8). There exist four stable zones of different Gouy phase variations of \( 2\pi - 3\pi, 3\pi - 2\pi \) and \( 3\pi - 4\pi \). Out of these four the central contour is completely stable inside and surrounded outside by an unstable region, where as the other three contours are surrounded inside and outside by an unstable region.

(b). \( L = L_M \). For \( L = L_M = 1400\text{cm} \), one can see that the central contour with the Gouy phase variation \( 3\pi - 4\pi \) starts decreasing as shown in the Fig. (6.9). But the remaining three stable contours are still existing in the same manner as in the previous case \( L > L_M \).

(c). \( L = L_M + L_{M3} + L_{M4} \). For the case of \( L = L_M + L_{M3} + L_{M4} = 1200\text{cm} \),
Figure 6.9: Stability diagram as a function of $d_2$ and $d_3$ for fixed $f_1 = 50\,\text{cm}$, $f_2 = 75\,\text{cm}$, $f_3 = 100\,\text{cm}$, $f_4 = 125\,\text{cm}$, $d_1 = 160\,\text{cm}$ and $L = L_M = 1400\,\text{cm}$

Figure 6.10: Stability diagram as a function of $d_2$ and $d_3$ for fixed $f_1 = 50\,\text{cm}$, $f_2 = 75\,\text{cm}$, $f_3 = 100\,\text{cm}$, $f_4 = 125\,\text{cm}$, $d_1 = 160\,\text{cm}$ and $L = L_{M2} + L_{M3} + L_{M4} = 1200\,\text{cm}$

...the central contour is completely vanished and the other three stable contours are giving an impression that, they are coming closely to merge which is shown in the Fig. (6.10).

(d). $L = L_{M1} + L_{M3} + L_{M4}$. Interestingly, one can see here in this Fig. (6.11) that for $L = L_{M1} + L_{M3} + L_{M4} = 1100\,\text{cm}$, all the three stable contours are merged.
Figure 6.11: **Stability diagram as a function of** \(d_2\) **and** \(d_3\) **for fixed** 
\(f_1 = 50\,\text{cm}, \quad f_2 = 75\,\text{cm}, \quad f_3 = 100\,\text{cm}, \quad f_4 = 125\,\text{cm}, \quad d_1 = 160\,\text{cm} \) **and** 
\(L = L_{M1} + L_{M3} + L_{M4} = 1100\,\text{cm}\)

The length region of 4CURC where the 4CURC can act as a 3CURC will come into the length chart from here onwards.

(e). \(L = L_{M(1,2),3,4}\). For \(L = L_{M(1,2),3,4} = 1020\,\text{cm}\), the merging is taking place and it is wider compare to the previous case even though the instabilities are existing as shown in the Fig. (6.12).
(f) \( L = L_{M1} + L_{M2} + L_{M4} \). As this \( L = L_{M1} + L_{M2} + L_{M4} = 1000\text{cm} \)

\[ L = L_{M1} + L_{M2} + L_{M3} = 1000\text{cm} \]

Figure 6.13: **Stability diagram as a function of** \( d_2 \) and \( d_3 \) for fixed \( f_1 = 50\text{cm}, f_2 = 75\text{cm}, f_3 = 100\text{cm}, f_4 = 125\text{cm}, d_1 = 160\text{cm} \) and \( L = L_{M1} + L_{M2} + L_{M4} = 1000\text{cm} \)

is close to the previous \( L \), there is no much difference in the stability compare to Fig. (6.12) which is shown in the Fig. (6.13).

(g) \( L = L_{M1} + L_{M2} + L_{M3} \). The stability for this case of \( L = L_{M1} + L_{M2} + L_{M3} = 900\text{cm} \)

\[ L_{M3} = 900\text{cm} \] is plotted in the Fig. (6.14). It is observed that the un

\[ L_{M3} = 900\text{cm} \]
stable region of \( d_2 \) around 400cm to 600cm for smaller \( d_3 \) is eliminated and the unstable region of \( d_3 \) around 300cm to 500cm for smaller \( d_2 \) is reduced for smaller \( d_2 \).

(h). \( L = L_{M_1,(2,3),4} \). As we change \( L \) further to \( L = L_{M_1,(2,3),4} = 871.4285 \text{cm} \),

\[ \text{Figure 6.15: Stability diagram as a function of } d_2 \text{ and } d_3 \text{ for fixed } f_1 = 50 \text{cm, } f_2 = 75 \text{cm, } f_3 = 100 \text{cm, } f_4 = 125 \text{cm, } d_1 = 160 \text{cm and } \]
\[ L = L_{M_1,(2,3),4} = 871.4285 \text{cm} \]

the unstable region of \( d_3 \) for smaller \( d_2 \) is completely removed as shown in the Fig. (6.15).

Here onwards the length region of 4CURC where it can behave as 2CURC will start.

(i). \( L = L_{M_2,(4,1),2,3} \). For the case of \( L = L_{M_2,(4,1),2,3} = 842.8571 \text{cm} \), the stable contour is started becoming narrower with respect to \( d_3 \) as shown in the Fig. (6.16).

(j). \( L = L_M + L_{M_4} \). For this case of \( L = L_M + L_{M_4} = 800 \text{cm} \), the stable contours of \( d_2 \) and \( d_3 \) are plotted in the Fig. (6.17). It is noted that just as in the case of 2CURC, there exist two disconnected stable contours.

(k). \( L = L_{M_1,(2,3,4)} \). For this very \( L = L_{M_1,(2,3,4)} = 722.22 \text{cm} \), it is shown in
Figure 6.16: Stability diagram as a function of $d_2$ and $d_3$ for fixed $f_1 = 50\, \text{cm}$, $f_2 = 75\, \text{cm}$, $f_3 = 100\, \text{cm}$, $f_4 = 125\, \text{cm}$, $d_1 = 160\, \text{cm}$ and $L = L_{M_1(2,3),4} = 842.8571\, \text{cm}$

Figure 6.17: Stability diagram as a function of $d_2$ and $d_3$ for fixed $f_1 = 50\, \text{cm}$, $f_2 = 75\, \text{cm}$, $f_3 = 100\, \text{cm}$, $f_4 = 125\, \text{cm}$, $d_1 = 160\, \text{cm}$ and $L = L_{M_2 + L_4} = 800\, \text{cm}$

the Fig. (6.18) that the stable zone is reduced further even though it is merged again.

(l). $L = L_{M_1 + L_4} = L_{M_2 + L_3}$. For this $L = L_{M_1 + L_4} = L_{M_2 + L_3} = 700\, \text{cm}$, again there exist two un connected stable contours as shown in the Fig. (6.19).

(m). $L = L_{M_1 + L_3}$. 

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Figure 6.18: Stability diagram as a function of $d_2$ and $d_3$ for fixed $f_1 = 50\text{cm}$, $f_2 = 75\text{cm}$, $f_3 = 100\text{cm}$, $f_4 = 125\text{cm}$, $d_1 = 160\text{cm}$ and $L = L_{M_{1,2},(3,4)} = 722.22\text{cm}$

Figure 6.19: Stability diagram as a function of $d_2$ and $d_3$ for fixed $f_1 = 50\text{cm}$, $f_2 = 75\text{cm}$, $f_3 = 100\text{cm}$, $f_4 = 125\text{cm}$, $d_1 = 160\text{cm}$ and $L = L_{M_1} + L_{M_4} = L_{M_2} + L_{M_3} = 700\text{cm}$

(n). $L = L_{M_4}$. For the cases (m) and (n) there is very small stable contours as shown in the Fig. (6.20) and Fig. (6.21).

(o). $L = L_{M_3}$.

(p). $L = L_{M_{(1,2),(3,4)}}$.

(q). $L = L_{M_{(4,1),(2,3)}}$. 

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Figure 6.20: Stability diagram as a function of $d_2$ and $d_3$ for fixed $f_1 = 50\,\text{cm}$, $f_2 = 75\,\text{cm}$, $f_3 = 100\,\text{cm}$, $f_4 = 125\,\text{cm}$, $d_1 = 160\,\text{cm}$ and $L = L_{M1} + L_{M3} = 600\,\text{cm}$

Figure 6.21: Stability diagram as a function of $d_2$ and $d_3$ for fixed $f_1 = 50\,\text{cm}$, $f_2 = 75\,\text{cm}$, $f_3 = 100\,\text{cm}$, $f_4 = 125\,\text{cm}$, $d_1 = 160\,\text{cm}$ and $L = L_{M4} = 500\,\text{cm}$

(r). $L = L_{M2}$. For the cases (o), (p), (q) and (r) there are stability contours but they are linear with the Gouy phase variation less than $\pi$. See Fig. (6.22)-Fig. (6.25).

(s). $L = L_{M1}$. For the considered $d_1 = 160\,\text{cm}$, the minimum length of $L$ which can be used is up to $L = L_{M1}$, even for this there is no stable region of $d_2$ and $d_3$ exist as shown in the Fig. (6.26).
Figure 6.22: Stability diagram as a function of $d_2$ and $d_3$ for fixed $f_1 = 50\text{cm}$, $f_2 = 75\text{cm}$, $f_3 = 100\text{cm}$, $f_4 = 125\text{cm}$, $d_1 = 160\text{cm}$ and $L = L_{M3} = 400\text{cm}$

Figure 6.23: Stability diagram as a function of $d_2$ and $d_3$ for fixed $f_1 = 50\text{cm}$, $f_2 = 75\text{cm}$, $f_3 = 100\text{cm}$, $f_4 = 125\text{cm}$, $d_1 = 160\text{cm}$ and $L = L_{M(1,2),(3,4)} = 342.222\text{cm}$

$L = L_{eff1234}$. For the considered $f_1$, $f_2$, $f_3$ and $f_4$, the range of $L$, from 0 to $L_{eff1234}$ as shown in the Fig. (6.27) is available for which the 4CURC behaves as a SCURC for which the Gouy phase variation is from 0 to $\pi$.

From the stability of 4CURC it is analyzed that for distinct $f_1 < f_2 < f_3 < f_4$, the unstable regions are still present. In fact, it is continuing to have
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Figure 6.24: Stability diagram as a function of $d_2$ and $d_3$ for fixed $f_1 = 50\text{cm}$, $f_2 = 75\text{cm}$, $f_3 = 100\text{cm}$, $f_4 = 125\text{cm}$, $d_1 = 160\text{cm}$ and $L = L_{M_{(4,1),(2,3)}} = 314.286\text{cm}$

Figure 6.25: Stability diagram as a function of $d_2$ and $d_3$ for fixed $f_1 = 50\text{cm}$, $f_2 = 75\text{cm}$, $f_3 = 100\text{cm}$, $f_4 = 125\text{cm}$, $d_1 = 160\text{cm}$ and $L = L_{M_2} = 300\text{cm}$

the accumulated Gouy phase equivalent to $\pi$, but not more than $\pi$. As a final attempt, we check the stability of 4CURC for symmetric $f$. We will not present the analytical solutions but we discuss some of the numerical results in the following section.
6.3.1 Stable solutions of $d_1$ for a given $L$, $d_2$, $d_3$ and for symmetric $f$

For a given $f = 100\text{cm}$, $d_1 = 100\text{cm}$, at $L = 6f\text{cm}$ onwards the merging of the stable contours starts taking place. For $L = 600$, the stability is
plotted between \(d_2\) and \(d_3\) is shown in the Fig. (6.28). It is realized that even with the symmetric \(f\), the unstable zones are occurring in between the stable zones. The corresponding Gouy phase regions are identified for this and for the following cases too. For the cases of \(L = 5f\text{cm}\) and \(L = 4f\text{cm}\)

![Figure 6.28: Stability diagram as a function of \(d_2\) and \(d_3\) for fixed \(f = 100\text{cm}\), \(d_1 = 100\text{cm}\) and \(L = 6f\text{cm}\)](image)

the stability is plotted in Fig. (6.29) and Fig. (6.30). It is again made it clear that it is not possible to achieve the accumulated Gouy phase variation beyond \(4\pi\). We calculate the half rayleigh ranges and the positions of the beam waists in each of the arms of the 4CURC by using

\[
z_{0i} = \pm \frac{1}{C} \sqrt{1 - \left(\frac{A + D}{2}\right)^2} \quad (6.53)
\]

\[
x_{0i} = \frac{A - D}{2C} \quad (6.54)
\]

Here the index \(i = s\) corresponds to the arm \(d_1\), \(i = y\) corresponds to the arm \(d_2\), \(i = d\) corresponds to the arm \(d_3\) and \(i = b\) corresponds to the arm \(L - d_1 - d_2 - d_3\) of the cavity. Having calculated the half Rayleigh ranges and
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Figure 6.29: Stability diagram as a function of $d_2$ and $d_3$ for fixed $f = 100\text{cm}$, $d_1 = 100\text{cm}$ and $L = 5f\text{cm}$

Figure 6.30: Stability diagram as a function of $d_2$ and $d_3$ for fixed $f = 100\text{cm}$, $d_1 = 100\text{cm}$ and $L = 4f\text{cm}$

their positions, one can proceed to calculate the accumulated Gouy phase
\[ \phi_G(RT) = \arctan \left( \frac{x_{0s}}{z_{0s}} \right) - \arctan \left( \frac{-d_1 + x_{0s}}{z_{0s}} \right) + \arctan \left( \frac{x_{0y}}{z_{0y}} \right) - \arctan \left( \frac{-d_2 + x_{0y}}{z_{0y}} \right) + \arctan \left( \frac{x_{0d}}{z_{0d}} \right) - \arctan \left( \frac{-d_3 + x_{0d}}{z_{0d}} \right) + \arctan \left( \frac{x_{0b}}{z_{0b}} \right) - \arctan \left( \frac{-L + d_1 + d_2 + d_3 + x_{0b}}{z_{0b}} \right) \]  

(6.55)

We discuss the Gouy phase variation for some peculiar cases and other mode properties as well in the following section.

### 6.4 Variable Gouy phase of 4CURC

In this section we analyze different cases of 4CURC to study the Gouy phase variation.

#### 6.4.1 Case(i): For a given \( f_1 = f_2 = f, f_3, f_4, d_1 \) and \( L \)

First we start with the case, when the focal lengths \( f_1 \) and \( f_2 \) of the lenses \( L_1 \) and \( L_2 \) are symmetric. For a given \( d_1, d_3 \) and \( L \), we vary the distance \( d_2 \). For a given \( f_1 = f_2 = 50\text{cm}, f_3 = 75\text{cm}, f_4 = 100\text{cm} \), and for fixed \( d_1 = 200\text{cm}, d_3 = 150\text{cm}, L = 650\text{cm} \), the stable range of \( d_2 \), which can be varied to create variable Gouy phase is from 105.104cm-344.896cm. For this range of \( d_2 \), we plot all the mode properties in the following Fig. (6.31)-(6.34).

The half Rayleigh ranges \( z_{0s}, z_{0y}, z_{0d} \) and \( z_{0b} \) are plotted in the Fig. (6.31). It is observed that \( z_{0y} \) and \( z_{0b} \) are linear through out the stable range of \( d_2 \) where as \( z_{0s} \) is showing a rapid increase at the edges of stable \( d_2 \). The Half
Rayleigh range between the lenses $L_3$ and $L_4$, i.e., $z_{0d}$ reached a maximum value of 340cm around $d_2 = 130$cm. The corresponding positions of the beam waists $x_{0x}$, $x_{0y}$, $x_{0d}$ and $x_{0b}$ are plotted with $d_2$ in the Fig. (6.32). The spot sizes before and after the lenses $L_1$, $L_2$, $L_3$ and $L_4$ are equal and given by $\omega_{1x1}$, $\omega_{2x1}$, $\omega_{3x1}$ and $\omega_{4x1}$ are plotted with $d_2$ in the Fig. (6.33). At the edges they are increasing rapidly but varying linearly through out the stable range of $d_2$. The variable Gouy phases for full round trip in each of the arms of lengths $d_1$, $d_2$, $d_3$ and $L - d_1 - d_2 - d_3$ are correspondingly $\phi_{G1}(RT)$, $\phi_{G2}(RT)$, $\phi_{G3}(RT)$, $\phi_{G4}(RT)$ and the accumulated Gouy phase $\phi_G(RT)$ as a function of $d_2$ is plotted in the Fig. (6.34). The Gouy phase variation for this case amounts to be $\frac{2\pi}{3}$ which is less than $\pi$. In the next section we consider the second and third lenses to be symmetric and analyze the mode properties in the same manner.
Figure 6.32: The positions of the beam spot sizes $x_{0x}, x_{0y}, x_{0d}$ and $x_{0b}$ as a function of $d_2$ for $f_1 = f_2 = f = 50\text{cm}$, $f_3 = 75\text{cm}$, $f_4 = 100\text{cm}$, $d_1 = 200\text{cm}$, $d_3 = 150\text{cm}$, $L = 650\text{cm}$

Figure 6.33: The spot sizes on the lenses $L_1$, $L_2$, $L_3$ and $L_4$ as a function of $d_2$ for $f_1 = f_2 = 50\text{cm}$, $f_3 = 75\text{cm}$, $f_4 = 100\text{cm}$, $d_1 = 200\text{cm}$, $d_3 = 150\text{cm}$, $L = 650\text{cm}$
Figure 6.34: Variable Gouy phases $\phi_{G1}(RT)$, $\phi_{G2}(RT)$, $\phi_{G3}(RT)$, $\phi_{G4}(RT)$ and $\phi_{G}(RT)$ as a function of $d_2$ for $f_1 = f_2 = 50$ cm, $f_3 = 75$ cm, $f_4 = 100$ cm, $d_1 = 200$ cm, $d_3 = 150$ cm, $L = 650$ cm.

6.4.2 Case(ii): For a given $f_1$, $f_2 = f_3 = f$, $f_4$, $d_1$ and $L$

Consider the focal lengths $f_2$, $f_3$ of the lenses $L_2$ and $L_3$ to be symmetric.

For $f_1 = 50$ cm, $f_2 = f_3 = 75$ cm, $f_4 = 100$ cm and $d_1 = 200$ cm, $d_3 = 100$ cm, $L = 700$ cm, the stable range of $d_2$ which is available is from $163.397$ cm-336.603 cm. For this range of $d_2$, the half Rayleigh ranges $z_{0s}$, $z_{0y}$, $z_{0d}$ and $z_{0b}$ are plotted in the Fig. (6.35). The corresponding positions of the beam waists $x_{0s}$, $x_{0y}$, $x_{0d}$ and $x_{0b}$ are plotted with $d_2$ in the Fig. (6.32). The spot sizes before and after the lenses $L_1$, $L_2$, $L_3$ and $L_4$ are equal and given by $\omega_{1x}$, $\omega_{2x}$, $\omega_{3x}$ and $\omega_{4x}$ are plotted with $d_2$ in the Fig. (6.37). The variable Gouy phases for full round trip in each of the arms of lengths $d_1$, $d_2$, $d_3$ and $L - d_1 - d_2 - d_3$ are correspondingly $\phi_{G1}(RT)$, $\phi_{G2}(RT)$, $\phi_{G3}(RT)$, $\phi_{G4}(RT)$ and the accumulated Gouy phase $\phi_G(RT)$ as a function of $d_2$ is plotted in the Fig. (6.38). The Gouy phase variation observed for this case is $\frac{2\pi}{3}$. 

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Figure 6.35: **Half Rayleigh ranges** $z_{0s}$, $z_{0y}$, $z_{0d}$ and $z_{0b}$ as a function of $d_2$ for $f_1 = 50\text{cm}$, $f_2 = f_3 = 75\text{cm}$, $f_4 = 100\text{cm}$, $d_1 = 200\text{cm}$, $d_3 = 100\text{cm}$, $L = 700\text{cm}$

Figure 6.36: **The positions of the beam spot sizes** $x_{0s}$, $x_{0y}$, $x_{0d}$ and $x_{0b}$ as a function of $d_2$ for $f_1 = 50\text{cm}$, $f_2 = f_3 = 75\text{cm}$, $f_4 = 100\text{cm}$, $d_1 = 200\text{cm}$, $d_3 = 100\text{cm}$, $L = 700\text{cm}$
Figure 6.37: The spot sizes on the lenses $L_1$, $L_2$, $L_3$ and $L_4$ as a function of $d_2$ for $f_1 = 50\text{cm}$, $f_2 = f_3 = 75\text{cm}$, $f_4 = 100\text{cm}$, $d_1 = 200\text{cm}$, $d_3 = 100\text{cm}$, $L = 700\text{cm}$

6.4.3 Case(iii): For a given $f_1$, $f_2$, $f_3 = f_4 = f$, $d_1$ and $L$

Consider the focal lengths $f_3$, $f_4$ of the lenses $L_3$ and $L_4$ to be symmetric. For $f_1 = 50\text{cm}$, $f_2 = 75\text{cm}$, $f_3 = f_4 = 100\text{cm}$ and $d_1 = 200\text{cm}$, $d_3 = 180\text{cm}$, $L = 800\text{cm}$, the stable range of $d_2$ which is available is from 41.1508cm-445.516cm. For this long range of $d_2$, the half Rayleigh ranges $z_{0x}$, $z_{0y}$, $z_{0d}$ and $z_{0b}$ are plotted in the Fig. (6.39). The corresponding positions of the beam waists $x_{0x}$, $x_{0y}$, $x_{0d}$ and $x_{0b}$ are plotted with $d_2$ in the Fig. (6.40). The variable Gouy phases for full round trip in each of the arms of lengths $d_1$, $d_2$, $d_3$ and $L - d_1 - d_2 - d_3$ are correspondingly $\phi_{G1}(RT)$, $\phi_{G2}(RT)$, $\phi_{G3}(RT)$, $\phi_{G4}(RT)$ and the accumulated Gouy phase $\phi_G(RT)$ as a function of $d_2$ is plotted in the Fig. (6.42). For this case the Gouy phase variation is turning out to be $\frac{2\pi}{3}$. It is realized that even if we make any two of the lenses of the 4CURC to be symmetric the accumulated Gouy phase variation

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which one can get is of the order of $\frac{2\pi}{3}$. Therefore, what happens to the Gouy phase variation when all of the lenses are symmetric? In the following section we study different cases leading to different Gouy phase variations with the symmetric focal lengths.

### 6.5 For a symmetric $f_1 = f_2 = f_3 = f_4 = f$

Consider all the focal lengths of the 4CURC to be symmetric i.e., $f_1 = f_2 = f_3 = f_4 = f$. At different values of $L$, $d_1$, $d_2$ and $d_3$ we check the Gouy phase variation as in the following.

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**Figure 6.38:** Variable Gouy phases $\phi_{G_1}(RT)$, $\phi_{G_2}(RT)$, $\phi_{G_3}(RT)$, $\phi_{G_4}(RT)$ and $\phi_G(RT)$ as a function of $d_2$ for $f_1 = 50\text{cm}$, $f_2 = f_3 = 75\text{cm}$, $f_4 = 100\text{cm}$, $d_1 = 200\text{cm}$, $d_3 = 100\text{cm}$, $L = 700\text{cm}$
Figure 6.39: Half Rayleigh ranges $z_{0s}$, $z_{0y}$, $z_{0d}$ and $z_{0b}$ as a function of $d_2$ for $f_1 = 50 \text{cm}$, $f_2 = 75 \text{cm}$, $f_3 = f_4 = 100 \text{cm}$, $d_1 = 200 \text{cm}$, $d_3 = 180 \text{cm}$, $L = 800 \text{cm}$

6.5.1 Case(i): $3\pi - 4\pi - 3\pi$

For the given values of $f = 30 \text{cm}$, $L = 480 \text{cm}$, $d_2 = 120 \text{cm}$, $d_3 = 120 \text{cm}$, the stable range of $d_1$ which satisfies the stability function is from 85.359cm to 154.641cm. Therefore for this range of $d_1$, we plot all the beam properties which are shown in the Figs. (6.43)-(6.46). The Fig. (6.43) shows the behaviour of half Rayleigh ranges $z_{0s}$, $z_{0y}$, $z_{0d}$ and $z_{0b}$ in between the lens combinations $(L_1, L_2)$, $(L_2, L_3)$, $(L_3, L_4)$ and $(L_4, L_1)$ respectively. It is noticed that all of them are becoming zero at $d_1 = 120 \text{cm}$. The corresponding positions of the beam waists are plotted in the Fig. (6.44). The spot sizes on the lenses $\omega_{1x1}$, $\omega_{2x1}$, $\omega_{3x1}$ and $\omega_{4x1}$ are plotted in the Fig. (6.45). The Gouy phase variation in all the four sections $\phi_{G1}(RT)$, $\phi_{G2}(RT)$, $\phi_{G3}(RT)$, $\phi_{G4}(RT)$ and the accumulated Gouy phase $\phi_{G}(RT)$ is plotted in the Fig. (6.46). It is noticed that the accumulated Gouy phase shows the variation
Figure 6.40: The positions of the beam spot sizes $x_{0s}$, $x_{0y}$, $x_{0d}$ and $x_{0b}$ as a function of $d_2$ for $f_1 = 50\text{cm}$, $f_2 = 75\text{cm}$, $f_3 = f_4 = 100\text{cm}$, $d_1 = 200\text{cm}$, $d_3 = 180\text{cm}$, $L = 800\text{cm}$

The spot sizes before and after the lenses $L_1$, $L_2$, $L_3$ and $L_4$ are equal and given by $\omega_{1x1}$, $\omega_{2x1}$, $\omega_{3x1}$ and $\omega_{4x1}$ are plotted with $d_2$ in the Fig. (6.41).

$3\pi - 4\pi - 3\pi$.

6.5.2 Case(ii): $3\pi - 2\pi$, $2\pi - 3\pi$

For the chosen $f = 30\text{cm}$, $L = 470\text{cm}$, $d_2 = 120\text{cm}$ and $d_3 = 274\text{cm}$, there are two stable regions of $d_1$ available, in between 24cm to 75cm. First region of $d_1$ is from 27cm to 41cm and the second region of $d_1$ is from 46cm to 75cm which are separated by an unstable region. For this case the half Rayleigh ranges $z_{os}$, $z_{0y}$, $z_{0d}$ and $z_{ob}$ are plotted in the Fig. (6.47). The corresponding positions of the beam waists are plotted in the Fig. (6.48). The corresponding spot sizes on the lenses $\omega_{1x1}$, $\omega_{2x1}$, $\omega_{3x1}$ and $\omega_{4x1}$ are plotted in the Fig. (6.49). The Gouy phase variation in all the four sections $\phi_{G1}(RT)$, $\phi_{G2}(RT)$, $\phi_{G3}(RT)$, $\phi_{G4}(RT)$ and the accumulated Gouy phase
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Figure 6.41: The spot sizes on the lenses $L_1$, $L_2$, $L_3$ and $L_4$ as a function of $d_2$ for $f_1 = 50\text{cm}$, $f_2 = 75\text{cm}$, $f_3 = f_4 = 100\text{cm}$, $d_1 = 200\text{cm}$, $d_3 = 180\text{cm}$, $L = 800\text{cm}$

$\phi_G(RT)$ is plotted for two regions and is shown in Fig. (6.50). In the first region of $d_1$ the accumulated Gouy phase is varying from $3\pi - 2\pi$ whereas in the second region of $d_1$ it is varying from $2\pi - 3\pi$.

6.5.3 Case(iii): $\pi - 2\pi$, $2\pi - \pi$

As it is discussed, the Gouy phase variations $\pi - 2\pi$ and $2\pi - \pi$ correspond to the length region of 2CURC. For this case we make $d_2 = d_3 = 0\text{cm}$. For the chosen $f = 30\text{cm}$, $L = 470\text{cm}$ there exist two small stable regions of $d_1$. First stable region of $d_1$ is from 41.0284cm to 44.1284cm and the second stable region is from 425.8535cm to 428.9535cm. For this set of parameters the half Rayleigh ranges are plotted for the two stable regions separately which are shown in the Figs. (6.51a) and (6.51b).

In the first region of $d_1$, the half Rayleigh ranges $z_{0x}$, $z_{0y}$ and $z_{0d}$ are
Figure 6.42: Variable Gouy phases $\phi_{G1}(RT)$, $\phi_{G2}(RT)$, $\phi_{G3}(RT)$, $\phi_{G4}(RT)$ and $\phi_{G}(RT)$ as a function of $d_2$ for $f_1 = 50\text{cm}$, $f_2 = 75\text{cm}$, $f_3 = f_4 = 100\text{cm}$, $d_1 = 200\text{cm}$, $d_3 = 180\text{cm}$, $L = 800\text{cm}$

Figure 6.43: Half Rayleigh ranges $z_{0x}$, $z_{0y}$, $z_{0d}$ and $z_{0b}$ as a function of $d_1$ for $f = 30\text{cm}$, $d_2 = 120\text{cm}$, $d_3 = 120\text{cm}$, $L = 480\text{cm}$
Figure 6.44: The positions of the beam spot sizes $x_{0s}$, $x_{0y}$, $x_{0d}$ and $x_{0b}$ as a function of $d_1$ for $f = 30\text{cm}$, $d_2 = 120\text{cm}$, $d_3 = 120\text{cm}$, $L = 480\text{cm}$

Figure 6.45: The spot sizes on the lenses $L_1$, $L_2$, $L_3$ and $L_4$ as a function of $d_1$ for $f = 30\text{cm}$, $d_2 = 120\text{cm}$, $d_3 = 120\text{cm}$, $L = 480\text{cm}$

very small compare to $z_{0b}$. For the second region, $z_{0s}$ reaches higher values, where as the other three half Rayleigh ranges $z_{oy}$, $z_{0d}$ and $z_{ob}$ are very much confined and they are small in the size.
Figure 6.46: Variable Gouy phases $\phi_{G1}(RT)$, $\phi_{G2}(RT)$, $\phi_{G3}(RT)$, $\phi_{G4}(RT)$ and $\phi_G(RT)$ as a function of $d_1$ for $f = 30\, \text{cm}$, $d_2 = 120\, \text{cm}$, $d_3 = 120\, \text{cm}$, $L = 480\, \text{cm}$

Figure 6.47: Half Rayleigh ranges $z_{0\alpha}$, $z_{0\beta}$, $z_{0\delta}$, $z_{0\varepsilon}$ as a function of $d_1$ for fixed $f = 30\, \text{cm}$, $d_2 = 120\, \text{cm}$, $d_3 = 120\, \text{cm}$, $L = 470\, \text{cm}$
Figure 6.48: The positions of the beam spot sizes $x_{0x}$, $x_{0y}$, $x_{0d}$ and $x_{0b}$ as a function of $d_1$ for fixed $f = 30\text{cm}$, $d_2 = 120\text{cm}$, $d_3 = 120\text{cm}$, $L = 470\text{cm}$

Figure 6.49: The spot sizes on the lenses $L_1$, $L_2$, $L_3$ and $L_4$ as a function of $d_1$ for fixed $f = 30\text{cm}$, $d_2 = 120\text{cm}$, $d_3 = 120\text{cm}$, $L = 470\text{cm}$
Figure 6.50: Variable Gouy phases \( \phi_{G1} \), \( \phi_{G2} \), \( \phi_{G3} \), \( \phi_{G4} \) and \( \phi_{G(RT)} \) as a function of \( d_1 \) for fixed \( f = 30 \text{cm} \), \( d_2 = 120 \text{cm} \), \( d_3 = 120 \text{cm} \) and \( L = 470 \text{cm} \)

Figure 6.51: Half Rayleigh ranges \( z_{0s} \), \( z_{0y} \), \( z_{0d} \), \( z_{0b} \) as a function of \( d_1 \) for fixed \( f = 30 \text{cm} \), \( d_2 = d_3 = 0 \text{cm} \), \( L = 470 \text{cm} \)
The corresponding positions $x_{0s}, x_{0y}, x_{0d}$ and $x_{0b}$ of the beam waists are plotted in the Fig. (6.52a) for the first stable region of $d_1$ and in Fig. (6.52b) for the second stable region of $d_1$.

The corresponding spot sizes on the lenses $\omega_{1x1}, \omega_{2x1}, \omega_{3x1}$ and $\omega_{4x1}$ are plotted in the Figs. (6.53a) and (6.53b). The Gouy phase variation in all the four sections $\phi_{G1}(RT)$, $\phi_{G2}(RT)$, $\phi_{G3}(RT)$, $\phi_{G4}(RT)$ and the accumulated Gouy phase $\phi_G(RT)$ is plotted for two regions in Figs. (6.54b). In the first stable region of $d_1$ the accumulated Gouy phase is varying from $\pi - 2\pi$ where as in the second stable region of $d_1$ it is varying from $2\pi - \pi$.

To summarize, we have analyzed the Gouy phase variation for different cases of 4CURC. By the insertion of ‘$N$’ lenses into the cavity, one can generate $N$ regions showing different phases, but with the accumulated Gouy phase variation to be $\pi$. 

Figure 6.52: The positions of the beam spot sizes $x_{0s}, x_{0y}, x_{0d}$ and $x_{0b}$ as a function of $d_1$ for fixed $f = 30\text{cm}$, $d_2 = d_3 = 0\text{cm}$, $L = 470\text{cm}$.
(a) for the first stable region

Figure 6.53: The spot sizes on the lenses $L_1$, $L_2$, $L_3$ and $L_4$ as a function of $d_1$ for fixed $f = 30\, \text{cm}$, $d_2 = d_3 = 0\, \text{cm}$, $L = 470\, \text{cm}$

(b) for the second stable region

Figure 6.54: $\phi_{G1}(RT)$, $\phi_{G2}(RT)$, $\phi_{G3}(RT)$, $\phi_{G4}(RT)$ and $\phi_G(RT)$ as a function of $d_1$ for fixed $f = 30\, \text{cm}$, $d_2 = d_3 = 0\, \text{cm}$, $L = 470\, \text{cm}$
6.6 Conclusion

The stability analysis of SCURC, 2CURC, 3CURC and 4CURC is presented in the chapters 3, 4, 5 and 6 in detail. This study suggests that it is not possible to get the accumulated Gouy phase variation more than $\pi$. But the advantage of this very amount of $\pi$ in the Ring laser gyros has been explained in the chapter 5. Chapter-7 deals with the question (b) raised at the end of Chapter-3.