CHAPTER IV

NEW SUBCLASSES OF THE CLASS OF CLOSE-TO-CONVEX FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS

4.1. INTRODUCTION AND DEFINITIONS

In the present chapter, we introduce some subclasses of the class of close to convex functions with respect to symmetric points, derive inclusion relations, establish integral representation formulas and determine coefficient estimates for the functions of such classes. The results are sharp.

Most of the definitions of the classes of analytic functions which are useful in the present chapter are defined in the previous chapter. We are hereby state some of the other basic definitions which are backbone of our results of this chapter.

A function \( f(z) \in \mathcal{A} \) is said to belong the class \( \mathcal{C}_s \) of close -to-convex functions with respect to symmetric points if there exists a function

\[
g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in S^*_s, \; z \in E
\]

for which

\[
Re \left\{ \frac{zf'(z)}{g(z) - g(-z)} \right\} > 0, \; z \in E.
\]
If

\[(4.1.3) \quad h(z) = z + \sum_{k=2}^{\infty} c_k z^k \in \mathcal{K}_s, \quad z \in E\]

for which

\[(4.1.4) \quad Re \left\{ \frac{zf'(z)}{h(z) - h(-z)} \right\} > 0, \quad z \in E,\]

the class of functions \(f(z) \in \mathcal{A}\) satisfying \(4.1.4\) is denoted by \(\mathcal{C}_{1(s)}\).

Evidently \(\mathcal{C}_{1(s)} \subset \mathcal{C}_s\).

Let \(\mathcal{C}'_s\) represents the class of functions \(f(z)\) in \(\mathcal{A}\) which satisfy the condition

\[(4.1.5) \quad Re \left\{ \frac{(zf'(z))'}{(g(z) - g(-z))} \right\} > 0, \quad g \in S^* \text{ and } z \in E.\]

If

\[(4.1.6) \quad Re \left\{ \frac{(zf'(z))'}{(h(z) - h(-z))} \right\} > 0, \quad h \in \mathcal{K}_s \text{ and } z \in E,\]

the corresponding class is denoted by \(\mathcal{C}'_{1(s)}\).

Mehrok et al [56] introduced the subclass \(\mathcal{C}_{s}(A, B; C, D)\) of \(\mathcal{C}_s\) and obtained its coefficient estimates.

Let \(-1 \leq D \leq B < A \leq C \leq 1\). A function \(f(z)\) in \(\mathcal{A}\) belongs to \(\mathcal{C}_{s}(A, B; C, D)\) if

\[(4.1.7) \quad \left\{ \frac{2zf'(z)}{g(z) - g(-z)} \right\} < \frac{1 + Cz}{1 + Dz}, \quad g \in S^*(A, B), \quad z \in E.\]

If

\[(4.1.8) \quad \left\{ \frac{2zf'(z)}{h(z) - h(-z)} \right\} < \frac{1 + Cz}{1 + Dz}, \quad h \in \mathcal{K}_s(A, B), \quad z \in E.\]
the corresponding class of functions satisfying (4.1.8) is denoted by 
\[ \mathcal{C}_{1(s)}(A, B; C, D) \].

\[ \mathcal{C}'(A, B; C, D) \] is the class of functions \( f(z) \) in \( \mathcal{A} \) for which

\[ \left( \frac{2zf'(z)}{g(z) - g(-z)} \right) < \frac{1 + Cz}{1 + Dz}, \quad g \in S_s^*(A, B) \text{ and } z \in E. \]

Moreover \( \mathcal{C}_{1(s)}(A, B; C, D) \) is the class of functions \( f(z) \) in \( \mathcal{A} \) such that

\[ \left( \frac{2zf'(z)}{(h(z) - h(-z))} \right) < \frac{1 + Cz}{1 + Dz}, \quad h \in \mathcal{K}_s(A, B) \text{ and } z \in E. \]

Let \( \alpha \geq 0 \) and \( \frac{f(z)f'(z)}{z} \neq 0 \). Then \( \mathcal{C}_s(\alpha) \) is the class of \( \alpha \)–close to convex functions \( f(z) \) in \( \mathcal{A} \) with respect to symmetric points if there exists \( g \in S_s^* \) such that

\[ \text{Re} \left\{ \frac{2(1 - \alpha)zf'(z)}{g(z) - g(-z)} + \frac{2\alpha zf'(z)}{(g(z) - g(-z))} \right\} > 0, \quad z \in E. \]

If

\[ \text{Re} \left\{ \frac{2(1 - \alpha)zf'(z)}{h(z) - h(-z)} + \frac{2\alpha zf'(z)}{(h(z) - h(-z))} \right\} > 0, \quad h \in \mathcal{K}_s, z \in E, \]

the class of functions satisfying (4.1.12) is denoted by \( \mathcal{C}_{1(s)}(\alpha) \).

Let \( 1 \leq D \leq B < A \leq C \leq 1, \ 0 \leq \alpha \leq 1 \) and \( z \in E \). Then \( \mathcal{C}_s(\alpha; A, B; C, D) \) and \( \mathcal{C}_{1(s)}(\alpha; A, B; C, D) \) represent the classes of functions \( f(z) \) in \( \mathcal{A} \) which satisfy, respectively, the conditions

\[ \left( \frac{2(1 - \alpha)zf'(z)}{g(z) - g(-z)} + \frac{2\alpha zf'(z)}{(g(z) - g(-z))} \right) < \frac{1 + Cz}{1 + Dz}, \quad g \in S_s^*(A, B). \]

and
(4.1.14) \[
\left\{ \begin{array}{c}
2(1 - \alpha)zf'(z) + 2\alpha(zf'(z))' \\
h(z) - h(-z) \end{array} \right\} < \frac{1 + Cz}{1 + Dz}, \ h \in \mathcal{K}_s(A, B).
\]

Throughout this chapter we assume that
\[
\left\{ \begin{array}{c}
\alpha \geq 0, \ g \in S^*(A, B), \ h \in \mathcal{K}_s(A, B), \ G(z) = \frac{g(z) - g(-z)}{2}, \\
H(z) = \frac{h(z) - h(-z)}{2}, \ P(z) = \frac{1 + Cw(z)}{1 + Dw(z)}, \ w \in \mathcal{U}, \\
-1 \leq D \leq B < A \leq C \leq 1 \text{ and } z \in E.
\end{array} \right\}
\]

### 4.2. PRELIMINARY LEMMAS

**Lemma 4.2.1** [8]. Let \( \alpha \geq 0 \) and \( \mathcal{D}(z) \) be starlike in \( E \). Let \( \mathcal{N}(z) \) be analytic in \( E \) such that \( \mathcal{N}(0) = \mathcal{D}(0) = 0 = \mathcal{N}'(0) - 1 = \mathcal{D}'(0) - 1 \), then
\[
\text{Re} \left\{ \frac{\mathcal{N}(z)}{\mathcal{D}(z)} \right\} > 0
\]
whenever
\[
\text{Re} \left\{ (1 - \alpha) \frac{\mathcal{N}(z)}{\mathcal{D}(z)} + \alpha \frac{\mathcal{N}'(z)}{\mathcal{D}'(z)} \right\} > 0.
\]

**Lemma 4.2.2** Under the same conditions of Lemma 4.2.1, we have
\[
\frac{\mathcal{N}(z)}{\mathcal{D}(z)} < \frac{1 + Cz}{1 + Dz}
\]
whenever
\[
\left\{ (1 - \alpha) \frac{\mathcal{N}(z)}{\mathcal{D}(z)} + \alpha \frac{\mathcal{N}'(z)}{\mathcal{D}'(z)} \right\} < \frac{1 + Cz}{1 + Dz}.
\]

**Proof.** By definition of Principle of subordinate, we have
\[
\left\{(1 - \alpha) \frac{\mathcal{N}(z)}{\mathcal{D}(z)} + \alpha \frac{\mathcal{N}'(z)}{\mathcal{D}'(z)}\right\} = \frac{1 + Cw(z)}{1 + Dw(z)}.
\]

This implies that
\[
\text{Re}\left\{(1 - \alpha) \frac{\mathcal{N}(z)}{\mathcal{D}(z)} + \alpha \frac{\mathcal{N}'(z)}{\mathcal{D}'(z)}\right\} = \text{Re}\left\{\frac{1 + Cw(z)}{1 + Dw(z)}\right\}
\geq \frac{1 - Cr}{1 - Dr}
\]
\[
> \frac{1 - C}{1 - D} = \beta \quad (0 \leq \beta < 1)
\]

which can be put into the form
\[
(4.2.1) \quad \frac{1}{(1 - \beta)} \text{Re}\left\{(1 - \alpha) \left(\frac{\mathcal{N}(z)}{\mathcal{D}(z)} - \beta\right) + \alpha \left(\frac{\mathcal{N}'(z)}{\mathcal{D}'(z)} - \beta\right)\right\} > 0.
\]

Setting
\[
(4.2.2) \quad \mathcal{M}(z) = \frac{\mathcal{N}(z) - \beta \mathcal{D}(z)}{(1 - \beta)}
\]

(4.2.1) takes the form
\[
\text{Re}\left\{(1 - \alpha) \frac{\mathcal{M}(z)}{\mathcal{D}(z)} + \alpha \frac{\mathcal{M}'(z)}{\mathcal{D}'(z)}\right\} > 0.
\]

By Lemma 4.2.1, we obtain
\[
\text{Re}\left\{\frac{\mathcal{M}(z)}{\mathcal{D}(z)}\right\} > 0 \text{ whenever } \text{Re}\left\{(1 - \alpha) \frac{\mathcal{M}(z)}{\mathcal{D}(z)} + \alpha \frac{\mathcal{M}'(z)}{\mathcal{D}'(z)}\right\} > 0.
\]

This means that
\[
\text{Re}\left\{\frac{\mathcal{N}(z)}{\mathcal{D}(z)}\right\} > \beta \text{ whenever } \text{Re}\left\{(1 - \alpha) \frac{\mathcal{N}(z)}{\mathcal{D}(z)} + \alpha \frac{\mathcal{N}'(z)}{\mathcal{D}'(z)}\right\} > \beta.
\]

That is
\[
\frac{\mathcal{N}(z)}{\mathcal{D}(z)} < \frac{1 + Cz}{1 + Dz} \text{ whenever } \left\{(1 - \alpha) \frac{\mathcal{N}(z)}{\mathcal{D}(z)} + \alpha \frac{\mathcal{N}'(z)}{\mathcal{D}'(z)}\right\} < \frac{1 + Cz}{1 + Dz}.
\]
4.3. **SUBCLASSES OF CLOSE-TO-CONVEX FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS**

**Theorem 4.3.1**  \( C_s(\alpha; A, B; C, D) \subset C_s(A, B; C, D) \subset C_s \).

**Proof.** Putting \( \mathcal{N}(z) = zf'(z) \) and \( \mathcal{D}(z) = G(z) \) (which is odd starlike) in Lemma 4.2.2 so that
\[
\frac{zf'(z)}{G(z)} < \frac{1 + Cz}{1 + Dz'},
\]
whenever
\[
\left\{(1 - \alpha)\frac{zf'(z)}{G(z)} + \alpha \left(\frac{zf'(z)}{G'(z)}\right)\right\} < \frac{1 + Cz}{1 + Dz}
\]
and the desired result follows.

From the above theorem it follows that all \( \alpha \)-close to convex functions with respect to symmetric points are close to convex with respect to symmetric points. Similarly we can prove

**Theorem 4.3.2**  \( C_{1(s)}(\alpha; A, B; C, D) \subset C_{1(s)}(A, B; C, D) \subset C_{1(s)} \).

**Theorem 4.3.3** Let \( f \in C_s(\alpha; A, B; C, D) \), then

(i) for \( \alpha = 0 \),
\[
f(z) = \int_0^z \frac{G(t)P(t)}{t} dt.
\]

(ii) for \( \alpha > 0 \) and \( c = \frac{1}{\alpha} - 1 \),

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\[ f(z) = (1 + c) \left\{ \int_0^z \frac{1}{t^c (G(t))^c} \left( \int_0^t (G(u))^c G'(u)P(u) du \right) dt \right\}. \]

**Proof.** We have

\[ (4.3.1) \quad (1 - \alpha) \frac{zf'(z)}{G(z)} + \alpha \frac{(zf'(z))'}{G'(z)} = P(z). \]

For \( \alpha = 0 \), there is nothing to prove.

Consider the case when \( \alpha > 0 \).

Dividing (4.3.1) by \( \alpha \) and putting \( c = \frac{1}{\alpha} - 1 \), (4.3.1) takes the form

\[ (4.3.2) \quad \frac{czf'(z)}{G(z)} + \frac{(zf'(z))'}{G'(z)} = (1 + c)P(z) \]

Multiplying (4.3.2) by \((G(z))^c G'(z)\), we get

\[ czf'(z)(G(z))^{c-1} G'(z) + (zf'(z))' (G(z))^c = (1 + c)[(G(z))^c G'(z)P(z)] \]

which reduces to

\[ (4.3.3) \quad \frac{d}{dz} \left\{ zf'(z)(G(z))^c \right\} = (1 + c)\left\{ (G(z))^c G'(z)P(z) \right\}. \]

Integrating (4.3.3) from 0 to \( z \), we obtain

\[ f'(z) = \frac{(1 + c)}{z(G(z))^c} \left\{ \int_0^z (G(u))^c G'(u)P(u) du \right\} \]

which again on integration gives the desired result.
On the same lines we can prove

**Theorem 4.3.4** Let \( f \in C_1(s) (\alpha; A, B; C, D) \), then

(i) for \( \alpha = 0 \),

\[
f(z) = \int_0^z \frac{H(t)P(t)}{t} \, dt.
\]

(ii) for \( \alpha > 0 \) and \( c = \frac{1}{\alpha} - 1 \),

\[
f(z) = (1 + c) \left\{ \int_0^z \frac{1}{t(H(t))^c} \left( \int_0^t (H(u))^c H'(u)P(u) \, du \right) \, dt \right\}.
\]

**Theorem 4.3.5** Let \( f \in \mathbb{C}'_s (A, B; C, D) \), then

(4.3.4) \[ |a_{2n}| \leq \frac{(C - D)}{(2n)^2} \left\{ 1 + (A - B) \left( \sum_{k=2}^{n} \frac{(2k - 1)}{2^{k-1}(k-1)!} \prod_{j=1}^{k-2} (A - B + 2j) \right) \right\}, \]

(4.3.5) \[ |a_{2n+1}| \leq \frac{1}{(2n + 1)^2} \left[ \frac{(2n + 1)(A - B)}{2^n n!} \prod_{j=1}^{n-1} (A - B + 2j) \right]^n + (C - D) \left\{ 1 + (A - B) \sum_{k=2}^{n} \frac{(2k - 1)}{2^{k-1}(k-1)!} \prod_{j=1}^{k-2} (A - B + 2j) \right\} \]

The estimates (4.3.4) and (4.3.5) are sharp.

**Proof.** We have

\[
(zf'(z))' = G'(z)P(z)
\]

which on expansion yields

\[
1 + 2^2 a_2 z + 3^2 a_3 z^2 + \cdots + (2n)^2 a_{2n} z^{2n-1} + (2n + 1)^2 a_{2n+1} z^{2n} + \cdots
\]

\[
= [1 + p_1 z + p_2 z^2 + \cdots + p_{2n-1} z^{2n-1} + p_{2n} z^{2n} + \cdots]
\]

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\[1 + 3b_3z^2 + 5b_5z^4 + \cdots + (2n - 1)b_{2n-1}z^{2n-2} + (2n + 1)b_{2n+1}z^{2n} + \cdots\]

Equating coefficients of \(z^{2n-1}\) and \(z^{2n}\) in the above equation, we get
\[(4.3.6) \quad (2n)^2a_{2n} = (2n - 1)p_1b_{2n-1} + (2n - 3)p_3b_{2n-3} + \cdots + 3p_{2n-3}b_3 + p_{2n-1}.
\]
\[(4.3.7) \quad (2n + 1)^2a_{2n+1} = (2n + 1)b_{2n+1} + (2n - 1)p_2b_{2n-1} + \cdots + 3p_{2n-2}b_3 + p_{2n}.
\]

Using Lemma 2.2.1 of Chapter-II and Lemma 3.2.1 of Chapter-III in (4.3.6), we obtain
\[(2n)^2|a_{2n}| \leq (C - D) \left( 1 + \sum_{k=2}^{n} (2k - 1)|b_{2k-1}| \right)
\]
\[\leq (C - D) \left[ 1 + (A - B) \left\{ \sum_{k=2}^{n} \frac{(2k - 1)}{2^{k-1}(k - 1)!} \prod_{j=1}^{k-2} (A - B + 2j) \right\} \right]
\]
from which (4.3.4) follows.

Again applying Lemma 2.2.1 of Chapter-II and Lemma 3.2.1 of Chapter-III in (4.3.7), we have
\[(2n + 1)^2|a_{2n+1}| \leq (2n + 1)|b_{2n+1}| + (C - D) \left( 1 + \sum_{k=2}^{n} (2k - 1)|b_{2k-1}| \right)
\]
\[\leq \left[ \frac{(2n + 1)(A - B)}{2^n n!} \prod_{j=1}^{n-1} (A - B + 2j) \right] + (C - D) \left\{ 1 + (A - B) \left( \sum_{k=2}^{n} \frac{(2k - 1)}{2^{k-1}(k - 1)!} \prod_{j=1}^{k-2} (A - B + 2j) \right) \right\}
\]
which gives the desired result (4.3.5).

The extremal function is obtained by choosing \(G(z) = \frac{g_0(z) - g_0(-z)}{2}\),
$g_0(z)$ is defined by (3.2.1) of Chapter III, and $P(z) = \frac{1 + Cz}{1 + Dz}$ in the
integral representation formula

$$
f(z) = \int_0^z \frac{1}{t} \left( \int_0^t G'(u)P(u)du \right) dt.
$$

Taking $A = C = 1$ and $B = D = -1$ in the theorem, we have the following

**Corollary 4.3.1** If $f \in C_s$, then

$$
|a_{2n}| \leq \frac{1}{2},
$$

$$
|a_{2n+1}| \leq \frac{1 + (2n + 1)^2}{2(2n + 1)^2}.
$$

**Theorem 4.3.6** Let $f \in C_s(\alpha; A, B; C, D)$. Then

$$
(4.3.8) \quad |a_2| \leq \frac{(C - D)}{2(1 + \alpha)},
$$

$$
(4.3.9) \quad |a_3| \leq \frac{(1 + 2\alpha)(A - B) + 2(C - D)}{6(1 + 2\alpha)},
$$

$$
(4.3.10) \quad |a_4| \leq \frac{(C - D)}{8(1 + \alpha)(1 + 3\alpha)}[(1 + 5\alpha)(A - B) + 2(1 + \alpha)],
$$

$$
(4.3.11) \quad |a_5| \leq \frac{1}{5(1 + 2\alpha)(1 + 4\alpha)} \left[ \frac{1}{8} (1 + 2\alpha)(1 + 4\alpha)(A - B)(A - B + 2) \right]^{+4(C - D)[(1 + 8\alpha)(A - B) + 2(1 + 2\alpha)]},
$$
\[(4.3.12) \ |a_6| \leq \frac{(C - D)}{48(1 + \alpha)(1 + 3\alpha)(1 + 5\alpha)} \left[ 4\alpha|\alpha - 1| \left( \frac{A - B}{4} \right)^2 \right.
\hspace{2cm} \left. + (1 + 3\alpha)(1 + 9\alpha)(A - B)(A - B + 2) \right.
\hspace{2cm} \left. + 4(1 + \alpha)(1 + 11\alpha)(A - B) \right.
\hspace{2cm} \left. + 8(1 + \alpha)(1 + 3\alpha) \right] \]

The bounds are sharp.

**Proof.** Since \( f \in C_\omega (\alpha; A, B; C, D) \), therefore
\[
(1 - \alpha)zf' (z)G'(z) + \alpha(zf' (z))' G(z) = P(z)G(z)G'(z).
\]

Expanding the series, we get
\[
(1 - \alpha)[z + 2a_2z^2 + 3a_3z^3 + 4a_4z^4 + 5a_5z^5 + 6a_6z^6 + \ldots ][1 + 3b_3z^2 + 5b_5z^4 + \ldots ]
\hspace{2cm} + \alpha[1 + 4a_2z + 9a_3z^2 + 16a_4z^3 + 25a_5z^4 + 36a_6z^5 + \ldots ][z + b_3z^3 + b_5z^5 + \ldots ]
\hspace{2cm} = [1 + p_1z + p_2z^2 + p_3z^3 + p_4z^4 + p_5z^5 + \ldots ][z + b_3z^3 + b_5z^5 + \ldots ][1 + 3b_3z^2 + 5b_5z^4 + \ldots ]
\]
which on simplification yields
\[(4.3.13) \quad 1 + 2(1 + \alpha)a_2z + \{3(1 + 2\alpha)a_3 + (3 - 2\alpha)b_3\}z^2
\hspace{2cm} + \{4(1 + 3\alpha)a_4 + (6 - 2\alpha)a_2b_3\}z^3
\hspace{2cm} + \{5(1 + 4\alpha)a_5 + (5 - 4\alpha)b_5 + 9a_3b_3\}z^4
\hspace{2cm} + \{6(1 + 5\alpha)a_6 + 4(3 + \alpha)a_4b_3 + 2(1 + \alpha)a_2b_5\}z^5 + \ldots
\hspace{2cm} = 1 + p_1z + (4b_3 + p_2)z^2 + (4p_1b_3 + p_3)z^3
\hspace{2cm} + ((6b_5 + 3b_3^2) + 4p_2b_3 + p_4)z^4
\hspace{2cm} + \{p_1 (6b_5 + 3b_3^2) + 4p_2b_3 + p_5\}z^5 + \ldots.
\]

Identifying terms in (4.3.13), we get
\[(4.3.14) \quad 2(1 + \alpha)a_2 = p_1,
\]
(4.3.15) \[3(1 + 2\alpha)a_3 = (1 + 2\alpha)b_3 + p_2,\]
(4.3.16) \[4(1 + 3\alpha)a_4 = (2\alpha - 6)a_2b_3 + 4p_1b_3 + p_3,\]
(4.3.17) \[5(1 + 4\alpha)a_5 = (1 + 4\alpha)b_5 - 9a_3b_3 + 3b_3^2 + 4p_2b_3 + p_4,\]
(4.3.18) \[6(1 + 5\alpha)a_6\]
\[= 4(3 + \alpha)a_4b_3 - 2(5 - 3\alpha)a_2b_5 + p_1(6b_5 + 3b_3^2) + 4p_3b_3 + p_5.\]

Applying Lemma 2.2.1 of Chapter-II in (4.3.14), we get (4.3.8). Using Lemma 2.2.1 of Chapter-II and Lemma 3.2.1 of Chapter-III in (4.3.15), (4.3.9) follows. (4.3.16) in conjunction with (4.3.14) leads us to (4.3.19) \[4(1 + \alpha)(1 + 3\alpha)a_4 = (1 + 5\alpha)p_1b_3 + (1 + \alpha)p_3.\]

With the aid of Lemma 2.2.1 of Chapter-II and Lemma 3.2.1 of Chapter-III in (4.3.19), (4.3.10) follows.

Eliminating \(a_3\) from (4.3.15) and (4.3.17), we arrive at
(4.3.20) \[5(1 + 2\alpha)(1 + 4\alpha)a_5\]
\[= (1 + 2\alpha)(1 + 4\alpha)b_5 + (1 + 8\alpha)p_2b_3 + (1 + 2\alpha)p_4.\]

Using Lemma 2.2.1 of Chapter-II and Lemma 3.2.1 of Chapter-III in (4.3.20), we obtain (4.3.11).

From (4.3.14), (4.3.16) and (4.3.18), we get
(4.3.21) \[6(1 + \alpha)(1 + 3\alpha)(1 + 5\alpha)a_6\]
\[= 4\alpha(\alpha - 1)p_1b_3^2 + (1 + 3\alpha)(1 + 9\alpha)p_1b_5\]
\[+ (1 + \alpha)(1 + 11\alpha)p_3b_3 + (1 + \alpha)(1 + 3\alpha)p_5.\]

With the application of Lemma 2.2.1 of Chapter-II and Lemma 3.2.1 of Chapter-III in (4.3.21), (4.3.12) follows.

The extremal function is obtained by choosing \(G(z) = \frac{g_0(z) - g_0(-z)}{2},\)
$g_0(z)$ is defined by (2.2.1) of Chapter II and $P(z) = \frac{1 + Cz}{1 + Dz}$ in the integral representation formula proved in Theorem 3.3.

Letting $A = C = 1$ and $B = D = -1$ in the theorem, we have the following

**Corollary 4.3.2** If $f \in C_s(\alpha)$, then

$$
|a_2| \leq \frac{1}{(1 + \alpha)},
$$

$$
|a_3| \leq \frac{3 + 2\alpha}{3(1 + 2\alpha)},
$$

$$
|a_4| \leq \frac{1}{(1 + \alpha)},
$$

$$
|a_5| \leq \frac{5 + 26\alpha + 8\alpha^2}{5(1 + 2\alpha)(1 + 4\alpha)},
$$

$$
|a_6| \leq \begin{cases} 
\frac{3 + 32\alpha + 37\alpha^2}{3(1 + \alpha)(1 + 3\alpha)(1 + 5\alpha)}, & 0 \leq \alpha \leq 1, \\
\frac{1}{(1 + \alpha)}, & \alpha \geq 1.
\end{cases}
$$

**Theorem 4.3.7** Let $f \in C_{1(s)}(A, B; C, D)$, then

(4.3.22) \hspace{1cm} |a_{2n}| \leq \frac{(C - D)}{2n} \left\{ 1 + \left( \sum_{k=2}^{n} \frac{(A - B)}{(2k - 1)2^{k-1}(k-1)!} \prod_{j=1}^{k-2} (A - B + 2j) \right) \right\},

(4.3.23) \hspace{1cm} |a_{2n+1}| \leq
\[
\frac{1}{2n + 1} \left[ \frac{(A - B)}{(2n + 1)2^n n!} \prod_{j=1}^{n-1} (A - B + 2j) \right] 
\]

\[
+(C - D) \left\{ \frac{1}{\prod_{j=1}^{k-2} (A - B + 2j)} \sum_{k=2}^{n} \frac{(A - B)}{(2k - 1)2^{k-1}(k - 1)!} \right\}
\]

**Proof.** Since \( f \in C_{1(s)}(A, B; C, D) \), we have
\[
z f'(z) = H(z)P(z)
\]
which on expansion gives
\[
z + 2a_2 z^2 + 3a_3 z^3 + \ldots + 2na_{2n} z^{2n} + (2n + 1)a_{2n+1} z^{2n+1} + \ldots
\]
\[
= \left[ 1 + p_1 z + p_2 z^2 + \ldots + p_{2n-1} z^{2n-1} + p_{2n} z^{2n} + \ldots \right] [z + c_3 z^3 + \ldots
\]
\[
+ c_{2n-1} z^{2n-1} + c_{2n+1} z^{2n+1} + \ldots \]

Equating coefficients of \( z^{2n} \) and \( z^{2n+1} \) in the above equation, we get
\[
2na_{2n} = p_1 c_{2n-1} + p_3 c_{2n-3} + \ldots + p_{2n-3} c_3 + p_{2n-1}. 
\]
\[
(4.3.24)
\]
\[
(2n + 1)a_{2n+1} = c_{2n+1} + p_2 c_{2n-1} + \ldots + p_{2n-2} c_3 + p_{2n}. 
\]
\[
(4.3.25)
\]

Applying triangular inequality and Lemma 2.2.1 of Chapter-II in (4.3.24), we have
\[
2n|a_{2n}| \leq (C - D) \left( 1 + \sum_{k=2}^{n} |c_{2k-1}| \right)
\]

With the aid of Lemma 3.2.2 of Chapter-III, the above inequality becomes
\[
2n|a_{2n}| \leq (C - D) \left[ 1 + \left\{ \sum_{k=2}^{n} \frac{(A - B)}{(2k - 1)2^{k-1}(k - 1)!} \prod_{j=1}^{k-2} (A - B + 2j) \right\} \right]
\]

from which (4.3.22) follows.

By the application of Lemma 2.2.1 of Chapter-II and Lemma 3.2.2 of Chapter-III in (4.3.25), we get
\[(2n + 1)|a_{2n+1}| \leq |c_{2n+1}| + (C - D) \left( 1 + \sum_{k=2}^{n} |c_{2k-1}| \right)\]
\[\leq \frac{(A - B)}{(2n + 1)2^n n!} \prod_{j=1}^{n-1} (A - B + 2j)\]
\[+ (C - D) \left[ 1 + \left\{ \sum_{k=2}^{n} \frac{(A - B)}{(2k - 1)2^{k-1}(k - 1)!} \prod_{j=1}^{k-2} (A - B + 2j) \right\} \right]\]

which gives (4.3.23).

Choosing \( H(z) = \frac{h_0(z) - h_0(-z)}{2} \), \( h_0(z) \) is defined by (3.2.2) of

Chapter III, \( P(z) = \frac{1 + Cz}{1 + Dz} \), the extremal function is

\[ f(z) = \int_{0}^{z} \frac{H(t)P(t)}{t} \, dt. \]

If we take \( A = C = 1 \) and \( B = D = -1 \), we have the following

**Corollary 4.3.3** Let \( f \in C_{1(s)} \), then

\[ |a_{2n}| \leq \frac{1}{n} \left( 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1} + \cdots \right), \]

\[ |a_{2n+1}| \leq \frac{1}{2n+1} \left( \frac{1}{2n+1} + 2 \left( 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} \right) \right). \]

**Theorem 4.3.8** Let \( f \in C'_{1(s)}(A, B; C, D) \), then

\[ (4.3.26) \quad |a_{2n}| \leq \frac{(C - D)}{(2n)^2} \left\{ 1 + \left( \sum_{k=2}^{n} \frac{(A - B)}{2^{k-1}(k - 1)!} \prod_{j=1}^{k-2} (A - B + 2j) \right) \right\}, \]
\((4.3.27)\quad |a_{2n+1}| \leq \frac{1}{(2n+1)^2} \left[ \frac{(A - B)}{2^n n!} \prod_{j=1}^{n-1} (A - B + 2j) \right] + (C - D) \left\{ 1 + \sum_{k=2}^{n} \frac{(A - B)}{2^{k-1} (k-1)!} \prod_{j=1}^{k-2} (A - B + 2j) \right\} \right].

The results are sharp.

**Proof.** We have

\((4.3.28)\quad (zf^\prime(z))^\prime = P(z)H^\prime(z).\)

Equating coefficients of \(z^{2n-1}\) and \(z^{2n}\) in \((4.3.28)\), we have

\((4.3.29)\quad (2n)^2 a_{2n} = (2n - 1)p_1 c_{2n-1} + (2n - 3)p_3 c_{2n-3} + \cdots + 3p_{2n-3} c_3 + p_{2n-1},\)

\((4.3.30)\quad (2n + 1)^2 a_{2n+1} = (2n + 1)c_{2n+1} + (2n - 1)p_2 c_{2n-1} + \cdots + 3p_{2n-2} c_3 + p_{2n}.\)

Using Lemma 2.2.1 of Chapter-II and Lemma 3.2.2 of Chapter-III in \((4.3.29)\), we get

\[(2n)^2 |a_{2n}| \leq (C - D) \left( 1 + \sum_{k=2}^{n} (2k - 1) |c_{2k-1}| \right) \leq (C - D) \left( 1 + \left\{ \sum_{k=2}^{n} \frac{(A - B)}{2^{k-1} (k-1)!} \prod_{j=1}^{k-2} (A - B + 2j) \right\} \right),\]

which is the result \((4.3.26)\).

From \((4.3.30)\), we obtain by using Lemma 2.2.1 of Chapter-II and Lemma 3.2.2 of Chapter-III,
\[(2n + 1)^2 |a_{2n+1}| \leq (2n + 1)|c_{2n+1}| + (C - D) \left( 1 + \sum_{k=2}^{n} (2k - 1)|c_{2k-1}| \right)\]

\[\leq \frac{(A - B)}{2^n n!} \prod_{j=1}^{n-1} (A - B + 2j)\]

\[+(C - D) \left( 1 + \left\{ \sum_{k=2}^{n} \frac{(A - B)}{2^{k-1} (k - 1)!} \prod_{j=1}^{k-2} (A - B + 2j) \right\} \right)\]

from which (4.3.27) follows.

Choosing \( H(z) = \frac{h_0(z) - h_0(-z)}{2} \), \( h_0(z) \) is defined by (3.2.2) of Chapter – III, \( P(z) = \frac{1 + Cz}{1 + Dz} \), the extremal function is

\[f(z) = \int_0^z \frac{1}{t} \left( \int_0^t H'(u) P(u) du \right) dt.\]

Taking \( A = C = 1 \) and \( B = D = -1 \) in the theorem, we get

**Corollary 4.3.4** If \( f \in C_{1(s)} \), then

\[|a_{2n}| \leq \frac{1}{2n} \quad \text{and} \quad |a_{2n+1}| \leq \frac{1}{2n + 1}.\]

That is

\[|a_n| \leq \frac{1}{n}.\]

Proceeding as in the Theorem 4.3.6 and using Lemma 2.2.1 of Chapter-II and Lemma 3.2.2 of Chapter-III, we can also prove that
Theorem 4.3.9 Let $f \in \mathcal{C}_{1(s)}(\alpha; A, B; C, D)$. Then

$$|a_2| \leq \frac{(C - D)}{2(1 + \alpha)},$$

$$|a_3| \leq \frac{(1 + 2\alpha)(A - B) + 6(C - D)}{18(1 + 2\alpha)},$$

$$|a_4| \leq \frac{(C - D)}{24(1 + \alpha)(1 + 3\alpha)} [(1 + 5\alpha)(A - B) + 6(1 + \alpha)],$$

$$|a_5| \leq \frac{1}{600(1 + 2\alpha)(1 + 4\alpha)} \left[ \frac{3(1 + 2\alpha)(1 + 4\alpha)(A - B)(A - B + 2)}{+20(C - D)\{(1 + 8\alpha)(A - B) + 6(1 + 2\alpha)\}} \right],$$

$$|a_6| \leq \frac{(C - D)}{2160(1 + \alpha)(1 + 3\alpha)(1 + 5\alpha)} \left[ \frac{40\alpha|\alpha - 1|(A - B)^2}{+9(1 + 3\alpha)(1 + 9\alpha)(A - B)(A - B + 2)} \right.\left. +60(1 + \alpha)(1 + 11\alpha)(A - B) \right] +360(1 + \alpha)(1 + 3\alpha) \right].$$

The extremal function is obtained by choosing $H(z) = \frac{h_0(z) - h_0(-z)}{2}$, $h_0(z)$ is defined by (3.2.2) of Chapter – III, and $P(z) = \frac{1 + Cz}{1 + Dz}$ in the integral representation formula

(i) for $\alpha = 0$,

$$f(z) = \int_0^z \frac{H(t)P(t)}{t} \, dt;$$

(ii) for $\alpha > 0$,

$$f(z) = (1 + c) \left[ \int_0^z \frac{1}{t(H(t))^c} \left\{ \int_0^t (H(u))^c H' \,(u)P(u) \, du \right\} \, dt \right],$$

where $c = \frac{1}{\alpha} - 1$. 107
Letting $A = C = 1$ and $B = D = -1$, we have

**Corollary 4.3.5** If $f \in \mathcal{C}_{1,5}(\alpha)$, then

\[
|a_2| \leq \frac{1}{(1 + \alpha)},
\]

\[
|a_3| \leq \frac{7 + 2\alpha}{9(1 + 2\alpha)},
\]

\[
|a_4| \leq \frac{2(1 + 2\alpha)}{3(1 + \alpha)(1 + 3\alpha)},
\]

\[
|a_5| \leq \frac{43 + 158\alpha + 24\alpha^2}{75(1 + 2\alpha)(1 + 4\alpha)},
\]

\[
|a_6| \leq \begin{cases} 
\frac{69 + 488\alpha + 523\alpha^2}{135(1 + \alpha)(1 + 3\alpha)(1 + 5\alpha)}, & 0 \leq \alpha \leq 1, \\
\frac{69 + 448\alpha + 563\alpha^2}{135(1 + \alpha)(1 + 3\alpha)(1 + 5\alpha)}, & \alpha \geq 1.
\end{cases}
\]