3.1 Characterization By Properties Of Equilibrium Distribution*

Let $X$ be a random variable admitting absolutely continuous distribution function $F(x)$ with respect to Lebesgue measure in the support of the set of non-negative real numbers. Further, assume that $F(0) = 0$ and $m = \int_0^\infty x dF(x) < \infty$. Associated with $X$ a random variable $Y$ can be defined with probability density function,

$$g(y) = m^{-1} R(y), \quad y > 0,$$  \hspace{0.5cm} (3.1)

where $R(x) = P[X > x]$ is the survival function of $X$. The distribution specified by $X$ has special significance in renewal theory. Consider a set of components whose failure times are of interest to us and we start experimenting with a new component at time zero, replace it upon failure by a second component and so on. If the failure times $X_i$, $i = 1, 2, 3, \ldots$ of the components are independent and identically distributed random variables, then the sequence $(S_n)$ of points where $S_n = X_1 + X_2 + \ldots + X_n$ constitutes a renewal process. If $F(.)$ is the common distribution function

* Some results in this section have appeared in the J. Ind. Statist. Assoc. (Reference 66).
of the X's satisfying the conditions stipulated above and
Uy and Vy respectively denote the age and remaining life
time (residual life) of the component which is in use at
time y, then the limiting distributions of Uy or Vy is
shown in Cox (1967) to have density function (3.1). The
random variables Uy and Vy are called the backward and
forward recurrence times and their common asymptotic distribu-
tion (3.1), the equilibrium distribution. In this physical
situation, Y represents the residual life of the component
whose length of life is X. An alternative derivation of
the same model based on length biased sampling is also given
in Cox (1967).

Deshpande et. al. (1986), Singh (1989) and Bluementhal
(1967) have found several applications of the equilibrium
distribution in reliability studies. The probabilistic
comparison between the distribution functions of X and Y
were utilised to explain the phenomenon of ageing.

In the present section we develop some identities
connecting the failure rates and MRLF's of the random variables
X and Y. Eventhough we use them in the sequel to characterize
only the exponential, Pareto, Power and finite range distribu-
tions, the relationships are quite general in character and can be
used to characterize the distribution of any continuous non-
negative random variable.
3.1.1 Some Identities

Let the failure rates of X and Y be

\[ h(x) = \frac{f(x)}{R(x)}, \quad (3.2) \]

and

\[ k(x) = \frac{g(x)}{S(x)}, \quad (3.3) \]

where \( f(.) \) is the density of X and \( S(.) \) is the survival function of Y. If the MRL functions of X and Y are \( r(x) \) and \( s(x) \) respectively, we have from equation (2.5) specialised for \( r=1 \),

\[ r(x) = E(X-x|X>x), \]

\[ = \frac{1}{R(x)} \int_{x}^{\infty} R(t)dt, \quad (3.4) \]

and

\[ s(x) = \frac{1}{S(x)} \int_{x}^{\infty} S(t)dt. \quad (3.5) \]

Using the result (Gupta, 1979),

\[ k(x) = \frac{1}{r(x)} \equiv \frac{[r(x)]^{-1}}, \quad (3.6) \]

and the relationships,

\[ h(x) = \frac{1+r'(x)}{r(x)}, \quad (3.7) \]

\[ k(x) = \frac{1+s'(x)}{s(x)}, \quad (3.8) \]
the following identities connecting the various functions are easily proved.

\[ h(x) = k(x) - \frac{k'(x)}{k(x)} \quad (3.8) \]

\[ r(x) = \frac{s(x)}{1 + s'(x)} \quad (3.9) \]

The primes in the last two equations and elsewhere in the present study denotes differentiation. In view of the one-to-one relationship between failure rates, MRL function and the corresponding survival function, the distribution of \( X \) be inferred from that of \( Y \) as either

\[ R(x) = \exp \left[ - \int_0^x h(t) \, dt \right] \]

\[ = \left( \frac{k(x)}{k(0)} \right) \exp \left[ - \int_0^x k(t) \, dt \right], \quad (3.10) \]

in terms of the failure rates or as

\[ R(x) = \left( \frac{r(0)}{r(x)} \right) \exp \left[ - \int_0^x (r(t))^{-1} \, dt \right], \]

\[ = \frac{1 + s'(x)}{1 + s'(0)} \left( \frac{s(0)}{s(x)} \right)^2 \exp \left[ - \int_0^x (s(t))^{-1} \, dt \right], \quad (3.11) \]

in terms of the MRL function, provided the quantities on the right side exist. The equations from (3.6) through (3.11)
are quite useful in distribution theory as we shall see in the next few sections. Further, they can be applied to provide alternative simple proofs to many of the theorems in literature concerning the interrelationships between the various criteria for ageing and different modes of stochastic orderings. Since our main thrust is on characterizations, these aspects are not considered in the present study.

3.1.2 Characterizations

Since our main objective is to explore characterizations of the Pareto, exponential and finite range distributions, attention is confined only to MRL functions that are linear. However, the techniques employed can take care of functions of a quite general nature and therefore, be used in other cases as well.

Theorem 3.1.

The MRL function of X is linear if and only if the MRL function of Y is linear.

Proof.

Suppose that Y has linear MRL function.

Then

\[ s(x) = \ell x + m. \]
For $s(.)$ to be an MRL function it is necessary and sufficient that $s(o) = E(Y)$, $s'(x) > -1$ and

$$\lim_{x \to o} \frac{s(x)}{x \log x} = 0,$$

(Muth, 1980). Hence the constants $l$ and $m$ must satisfy the conditions $m > 0$ and $l > -1$.

Using the above form of $s(x)$ in equation (3.9), we find

$$r(x) = Lx + M,$$

where

$$L = \frac{l}{1+l} \text{ and } M = \frac{m}{1+l} \quad (3.12)$$

For $l > -\frac{1}{2}$, $L > -1$ and $M > 0$.

Conversely when $r(x)$ has the above form, from (3.9),

$$s'(x) + [s(x)-1] (Lx+M)^{-1} = 0.$$

This is a linear differential equation with integrating factor $(Lx+M)^{-1/L}$. Accordingly the general solution is

$$s(x) = \begin{cases} 
(1-L)^{-1} (Lx+M) + K(Lx+M)^{1/L}, & L \neq 1 \\
-(x+M) \log(x+M) + K_1(x+M), & L = 1
\end{cases}$$

where $K$ and $K_1$ are constants of integration.
K and \(K_1\) being the constants of integration. In view of the conditions on \(s(x)\), the second solution is inadmissible and in the first solution \(K\) must be zero. Thus,

\[
s(x) = lx + m,
\]

where

\[
l = L/(1-L) \text{ and } m = M/(1-L) \tag{3.13}
\]

in conformity with equation (3.12).

Theorem 3.1 will now be utilised to establish a closure property, for the distribution of \(X\) with respect to the formation of the distribution of \(Y\), in the sense that \(X\) and \(Y\) have the same form of distribution. For this purpose, we denote by \(E(b)\), the exponential distribution with density function,

\[
f(x) = b \exp[-bx]. \tag{3.14}
\]

Theorem 3.2.

The distribution of \(X\) is \(E(b)\), \((P\ II(a,\alpha), \ FR(c,R))\) if and only if \(Y\) is \(E(b)\), \((P\ II(a-1,\alpha), \ FR(c+1,R))\).

Proof.

When \(X\) is distributed as one of the above forms, we have from chapter II the following table of values of MRL, failure rate and means.

<table>
<thead>
<tr>
<th>Model</th>
<th>Failure rate</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>E(b)</td>
<td>(1/b)</td>
<td>(1)</td>
</tr>
<tr>
<td>P II(a,\alpha)</td>
<td>(x+a)/(a-1))</td>
<td>(a/(a-1))</td>
</tr>
<tr>
<td>Uniform</td>
<td>(1/(c+1))</td>
<td>(1/(c+1))</td>
</tr>
</tbody>
</table>
In the notations of Theorem 3.1, the values of $L$ for $E(b)$, $PII(a, \alpha)$ and $FR(c, R)$ are respectively 0, $(a-1)^{-1}$ and $-(c+1)^{-1}$ while corresponding values of $M$ are $b^{-1}$, $\alpha(a-1)^{-1}$ and $R(c+1)^{-1}$. Direct calculations from equation (3.13) give $I = 0$, $M = b^{-1}$ when $X$ is exponential, $I = (a-2)^{-1}$, $M = \alpha(a-2)^{-1}$ in the Pareto case, and $I = -(c+2)^{-1}$, $M = R(c+2)^{-1}$ for the finite range law.

By theorem 3.1 $Y$ has linear form and since $I = 0$ ($> 0$, $< 0$) $Y$ is $E(b)$, $(PII(a-1, \alpha)$, $FR(c+1, R)$). The conditions on the parameters of three distributions in the order in which they appear in the theorem for the result to be true are $b > 0$, $\alpha > 0$, $a > 1$, $R > 0$ and $c > -1$.

Table 1

Failure rates, MRLs and means.

<table>
<thead>
<tr>
<th>Model</th>
<th>MRL</th>
<th>failure rate</th>
<th>mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E(b)$</td>
<td>$1/b$</td>
<td>$b$</td>
<td>$1/b$</td>
</tr>
<tr>
<td>$PII(a, \alpha)$</td>
<td>$(x+\alpha)(a-1)^{-1}$</td>
<td>$\alpha(x+\alpha)^{-1}$</td>
<td>$\alpha(a-1)^{-1}$</td>
</tr>
<tr>
<td>$FR(c, R)$</td>
<td>$(R-x)(c+1)^{-1}$</td>
<td>$c(R-x)^{-1}$</td>
<td>$R(c+1)^{-1}$</td>
</tr>
<tr>
<td>Uniform$(o, R)$</td>
<td>$(R-x)2^{-1}$</td>
<td>$(R-x)^{-1}$</td>
<td>$R/2$</td>
</tr>
</tbody>
</table>
This theorem enables one to assert the distribution of \( Y \) in terms of the distribution of \( X \) and vice versa. It may be noticed that the closure property of the exponential distribution proved above is well known and has been properly exploited in literature. That the property holds for a class of distributions, points out to the possibility of a unified treatment of the tail behaviour of the different models involved therein. The present study being oriented more towards applications in reliability theory, some equivalent conditions in terms of the failure rates or MRL functions are proved in the following two theorems. In life length studies, models are often postulated by specifying the behaviour of the MRL's or failure rates and in such situations these theorems may help in identifying the appropriate models.

**Theorem 3.3.**

The relationship \( r(x) = ps(x) \) is satisfied for all \( x > 0 \) and positive constant \( p \) if and only if \( X \) is \( E(b) \) for \( p=1 \), \( P II(a,\alpha) \) for \( 0 < p < 1 \) and \( FR(c,R) \) for \( p > 1 \).

**Proof:**

When \( X \) follows (3.14), \( Y \) has an identical distribution with the same parameter. Hence \( r(x) = s(x) \). For the Pareto case also \( Y \) has the same distributional form \( P II(a-1,\alpha) \) and therefore

\[
\frac{s(x)}{1-s(x)} = ps(x),
\]

or

\[
s'(x) = p^{-1} s(x).
\]
from table 1, for \( a > 2 \), (condition for \( E(Y) \) to exist)

\[
\frac{r(x)}{s(x)} = \frac{(a-2)}{(a-1)} < 1.
\]

Likewise, for the finite range distribution,

\[
\frac{r(x)}{s(x)} = \frac{(c+2)}{(c+1)} > 1.
\]

This proves the necessity of the condition.

To prove the only if part, we note that from equations (3.6) and (3.7),

\[
\frac{r(x)}{s(x)} = \frac{s(x)}{(1+s'(x))}. \tag{3.15}
\]

When \( p=1 \), it follows that \( s'(x) = 0 \) or \( s(x) \) is a positive constant. Hence \( X \) follows (3.14) for some \( b > 0 \). When \( p < 1 \), equation (3.15) gives,

\[
s(x)/(1+s'(x)) = ps(x),
\]

or

\[
s'(x) = p^{-1} - 1.
\]

Thus,

\[
s(x) = (p^{-1} - 1)x + d,
\]

This completes the proof.
where, \( d = \text{E}(Y) \). Accordingly the survival function of \( X \) is

\[
R(x) = \exp \left[ -\int_0^x \frac{(1+r'(t))}{r(t)} \, dt \right],
\]

where, \( h(x) = pk(x) \) for all \( x > 0 \) and a positive constant \( p \) if and only if \( X \) is \( E(b) \) for \( p = 1 \), \( P \text{II}(a, \alpha) \) for \( p > 1 \) and \( FR(c, R) \) for \( 0 < p < 1 \).
Proof:

The proof is on similar lines as that of theorem 3.3 once we write

\[ k(x) = \frac{1}{r(x)} \text{ and } h(x) = \frac{1+r'(x)}{r(x)} \]

Corollary 3.1.

The conditions of Theorems 3.3 and 3.4 characterize the distribution of Y as either exponential or Pareto type II or finite range for values of p mentioned herein. The proof follows from Theorem 3.2.

Remark:

When the support of X is taken as \((a, b)\), \(b > a > 0\), the condition \(r(x) = ps(x)\) or \(k(x) = ph(x)\) for \(p > 1\) characterizes the distribution with survival function,

\[
R(x) = \left(\frac{b-x}{b-a}\right)^c, \quad c > 0,
\]

and the same relationships for \(p < 1\) is a unique property of the model specified by,

\[
E(X) = k E(Y), \quad k > 0,
\]

\[
R(x) = \left[\frac{(a+d)}{(x+d)}\right]^c, \quad x > a,
\]

which includes the Pareto type I as a particular case, when \(d = 0\).
The characterization theorems presented so far demands the required properties to hold for all points in the support of the random variable $X$. One problem that will be of interest is to inquire whether it is possible to identify a weaker set of criterion that can uniquely determine the above mentioned distributions. The following theorem answers the question in the affirmative, with the limitation that the models have to come from a subclass of the exponential family. This is in contrast with the earlier results where one had the freedom of choice from the class of all absolutely continuous distributions of non-negative random variables.

**Theorem 3.5.**

In the one parameter exponential family specified by,

$$f(x) = u(\Theta) v(x) \exp[\Theta \omega(x)], \quad (3.16)$$

where $\Theta$ lies in an open interval on the positive part of the real line, the condition

$$E(X) = k E(Y), \quad k > 0, \quad (3.17)$$

is satisfied, if and only if $X$ is distributed as
(i) exponential for \( k = 1 \) and \( \omega(x) = x \),

(ii) Pareto type I for \( k > 1 \) and \( \omega(x) = -\log x \) (\( k \), known)

and

(iii) Power with density,

\[
f(x) = (\Theta + 1) R^{-(\Theta + 1)} x^\Theta, \quad 0 < x < R,
\]

for \( 0 < k < 1 \) and \( \omega(x) = \log x \).

Proof:

The proof that \( k = 1 \) characterizes \( E(b) \) is given in Gupta (1979). To prove the other two cases we note that from equation (3.17) and from the definition of the random variable \( Y \) in (3.1),

\[
E(Y) = m^{-1} \int_0^\infty y R(y) dy,
\]

\[
= (2m)^{-1} E(X^2), \quad (3.18)
\]

by the usual integration by parts. Further in the support \( D \) of the family,

\[
\int_D u(\Theta) v(x) e^{\Theta \omega(x)} dx = 1.
\]

Changing \( \Theta \) to \( \Theta + 1 \) in the last expression,

\[
\int_D u(\Theta + 1) v(x) e^{(\Theta + 1) \omega(x)} dx = 1,
\]
and hence,
\[ u(\alpha + 1)/u(\alpha) \int u(\theta) v(x) e^{(\theta + 1)w(x)} \, dx = 1. \] (3.19)

When \( w(x) = \log x \), (3.19) reduces to
\[ E(X) = u(\alpha)/u(\alpha + 1), \] (3.20)

and similar calculations after replacing \( \alpha \) by \( \alpha + 2 \) provides,
\[ E(X^2) = u(\alpha)/u(\alpha + 2). \] (3.21)

Now, equation (3.17) along with (3.18) read,
\[ 2m E(X) = k E(X^2), \] (3.22)

Introducing equations (3.20) and (3.21) into (3.22),
\[ u(\alpha)/u(\alpha - 1) = p u(\alpha - 1)/u(\alpha - 2), \]

which is identical to
\[ b(\alpha) = p b(\alpha - 1), \quad \alpha > 1, \] (3.23)

where,
\[ b(\alpha) = u(\alpha)/u(\alpha - 1) \quad \text{and} \quad p = k/2. \] (3.24)
The next stage of the proof is the solution of the functional equation (3.23). Setting $\theta=1$ there $p=b(1)/b(o)$. Further transformation $c(\theta)=b(\theta)/b(o)$ yields

$$c(\theta) = c(1) c(\theta-1). \quad (3.25)$$

By successive application of $(3.25)$, for non-negative integer $n$,

$$c(\theta+n) = c(\theta) g(n),$$

in which

$$g(n) = [c(1)]^n.$$

Since $c(o)=1$, $c(n)=g(n)$ and hence,

$$u(o) = 1.$$

Taking $g(\theta) = \frac{p^n}{u(o)}$, once again we have the Cauchy functional equation with solution

$$c(\theta) = p^\theta, \quad p = c(1).$$

Thus,

$$b(\theta) = b(o) \ p^\theta$$

$$u(o) = q \ p^\theta, \quad b(o) = q > 0.$$
Substituting into (3.24),

\[ u(\Theta) = q \Theta u(\Theta - 1). \]

Recurrsively,

\[ u(\Theta + n) = q^n \ p^{n\Theta + n(n+1)/2} u(\Theta), \]

and in particular,

\[ u(n) = q^n \ p^{n(n+1)/2} u(0). \]

This gives,

\[ u(\Theta + n) = p^{n\Theta} u(n) u(\Theta) u(0), \tag{3.26} \]

and

\[ u(0) = 1. \]

Taking \( g(\Theta) = p^{\Theta^2/2} / u(\Theta), \) once again we have the Cauchy functional equation mentioned earlier with solution,

\[ g(\Theta) = \left[ g(1) \right]^\Theta, \]

so that \( X \) is \( P I_1(\Theta, q). \) The proof for \( X \) is on similar lines except for the steps in equations (3.27) and (3.28), our assertion follows from the fact that \( X \) is a member of the exponential family considered in Theorem 3.5. Theorem 3.5 has the following implications.
The density function of $X$ now becomes,

$$f(x) = p^{(1/2)}\theta(\theta+1) q^\theta v(x) x^{-\theta}.$$ 

Assuming the support of $X$ to be $(q,\infty)$, one must have,

$$p^{(1/2)}\theta(\theta+1) q^\theta \int_q^\infty v(x) x^{-\theta} \, dx = 1,$$

or equivalently,

$$p^{(1/2)}\theta(\theta+1) q^\theta \int_q^\infty v(x) x^{-\theta} \, dx = q^{-\theta}.$$

Differentiating both sides with respect to $q$,

$$v(q) = \frac{\theta}{q p^{(1/2)}\theta(\theta+1)}.$$

Using the expression for $v(.)$ in equation (3.16),

$$f(x) = q^\theta x^{-(\theta+1)}, \quad x > q > 0,$$

so that $X$ is $P I(\Theta,q)$. The proof for $k < 1$ is on similar lines except for the choice of $\omega(x) = \log x$ and our assertion follows.

Among the three sub-classes of the exponential family considered in Theorem 3.5 has the following implications.
(i) Since the coefficient of variation of $X$ is
\[ C = \left(\frac{2}{k} - 1\right)^{1/2}, \]
the values $C=1$, $C>1$ and $0 < C < 1$ characterize respectively $E(b)$, $P I(a,k)$ and the power distribution.

(ii) Muth (1980) has introduced the concept of memory of a distribution at a point $x$ in its support as $-r'(x)$, which we can see to be zero for $E(b)$, negative constant for $P I(a,k)$ and positive constant for power distribution at each point of its support. Our previous discussions show that these are the only continuous distributions possessing this property.

(iii) For arbitrary distributions of non-negative random variables Muth (1980) considers a weighted average of the memories at various points to arrive at a measure for global memory of a distribution as $M = 1 - C^2$ and classify the distributions as possessing no memory, positive memory or negative memory according as $M = 0$, $M > 0$ or $M < 0$. Since the weights he has chosen in the calculations are positive, it is evident that if a distribution has a particular type of memory at every point in the support it also has the same type of memory globally. But the converse need not be true as seen from the expression for $M$ and also from the fact that mixed types of memory at various points can produce a negative or a positive quantity as the average. Theorem 3.5 shows that the exponential, Pareto and Power models are the only distributions in the families considered above are characterized by a global lack of memory, negative memory and positive memory.
Another possible application of Theorem 3.5 lies in the measurement of income inequality. A random variable $Z$ has the first moment distribution corresponding to $X$, if it has distribution function,

$$F(z) = m^{-1} \int_0^z xf(x)dx . \quad (3.28)$$

When $X$ is the income of a unit, $F(z)$ is interpreted as the proportional share of total income of units having income upto $z$, and is extensively used to define and interpret the Lorenz curve and Gini index. (See Kakwani, 1980). We now prove

Theorem 3.6.

$F_Z(x) = F_Y(x)$ for all $x > \Theta$ if and only if $X$ is Pareto type I.

Proof:

The necessary part is easily verified. To prove the sufficiency part we observe as follows. Let $X$ ranges from $\Theta$ to $\infty$. In order to make the span of the distribution function 1, we change $E(X)$ to $E(X) - \Theta$, the mean of the distribution being $(m-\Theta)$. Thus if $f(.)$ is the density function of $Y$,

$$f_Y(x) = (m-\Theta)^{-1} R_X(x), \quad x > \Theta > 0, \quad (3.29)$$
then,
\[ F_Z(x) = m^{-1} \int_0^x t f(t) dt, \]
index \( E(Z)/E(Y) \) (see Arnold 1983) Theorem 3.6 offers a
characterization in that direction as well.
\[ = m^{-1} \int_0^x t(-dR(t)). \]

3.2 Characterization By Properties Of Bivariate Models.

Since \( X \) ranges from \( \Theta \) to \( \infty \), we have,

There have been several investigations concerning
bivariate distributions are determined uniquely by
conditional densities of specified forms. The papers by
Abrahams and Thomas (1944) Arnold (1987), Arnold and
Strauss (1988) and Nair (1989) discuss this problem
extensively. It is also possible to think in terms of
characterizing univariate models by assuming particular
forms of conditional distributions for certain
bivariate models. Seshadri and Patil (1964) has shown
that if the conditional density of \( X_1 \) given \( X_2 = x_2 \)
is of the form
\[ f(x) = -m^{-1} x(m-\Theta) f'(x), \]
so that
\[ f(x_1 | x_2) = [1+\Theta x_1][1+\Theta x_2]-\Theta \exp[-x_1-\Theta x_1 x_2], \]
then the distribution of \( x_1 \) is exponential if and only
identifying the bivariate density in the discrete domain
that characterizes the bivariate geometric law is avail-
able in Nair and Nair (1990). In the present section our
objective is to find out suitable conditional densities

When \( Z \) and \( Y \) have the same density function,
(3.30) leads to,
\[ f(x) = -m^{-1} x(m-\Theta) f'(x), \]
so that
\[ \log f(x) = \log C x^{-((m-\Theta)/\Theta)}, \]
where \( C \) is a constant, thus giving,
\[ f(x) = Cx^{-a}, \]
with \( a = m(m-\Theta)^{-1} > 0 \) and \( C = (a-1) \Theta^{(a-1)}. \)
When income inequality is measured in terms of the index $E(Z)/E(X)$ (See Arnold 1983) Theorem 3.6 offers a characterization in that direction as well.

### 3.2 Characterization By Properties Of Bivariate Models.

There have been several investigations concerning bivariate distributions that are determined uniquely by conditional densities of specified forms. The papers by Abrahams and Thomas (1984), Arnold (1987), Arnold and Strauss (1988) and Nair (1989) discuss this problem extensively. It is also possible to think in terms of characterizing univariate models by assuming particular forms of conditional distributions derived from certain bivariate models. Seshadri and Patil (1964) has shown that if the conditional density of $X_1$ given $X_2 = x_2$ is of the form,

$$f(x_1|x_2) = [(1+\theta x_1)(1+\theta x_2)-\theta] \exp[-x_1-\theta x_1 x_2],$$

then the distribution of $X_1$ is exponential if and only if that of $X_2$ is also exponential. An analogous result identifying the bivariate density in the discrete domain that characterizes the univariate geometric law is available in Nair and Nair (1990). In the present section our objective is to find out suitable conditional densities.
that guarantee unique Pareto and finite range densities for the component variates. The following results along with characterizations of a bivariate Pareto distribution has been reported in Hitha and Nair (1990).

Theorem 3.7.

Let \((X_1, X_2)\) be a vector of non-negative random variables with joint density function \(f(x_1, x_2)\) and marginal densities \(f_1(x_1)\) and \(f_2(x_2)\). If the conditional distribution of \(X_1\) given \(X_2=x_2\) is P II with density,

\[
f(x_1|x_2) = (c+1)(d(x_2))^{c+1}(x_1+d(x_2))^{-(c+2)}, \quad (3.31)
\]

where

\[
d(x_2) = a_1^{-1}(b+a_2x_2), \quad x_1, x_2 > 0, \quad a_1, a_2, b > 0, \quad c > -1,
\]

then the necessary and sufficient condition for \(X_1\) to follow P II\((c, b/a_1)\) is that \(X_2\) has the same type of distribution, P II\((c, b/a_2)\).

Proof:

When \(X_1\) given \(X_2=x_2\) is Pareto type II as in (3.31) and \(X_2\) is P II\((c, a_2/b)\), the relationship

\[
f_1(x_1) = \int_0^\infty f(x_1|x_2) f_2(x_2) dx_2, \quad (3.32)
\]
gives,

\[ f_1(x_1) = \int_0^\infty (c+1)(b+a_2x_2)^{c+1} a_1^{-1} a_1^{-(c+1)} [x_1 + a_1^{-(b+a_2x_2)}]^{-(c+2)} \]

\[ a_2 c^{(a_2x_2+b)-(c+1)} dx_2, \]

\[ = \int_0^\infty a_1 a_2 (c+1)(c+2) b^{c+1} (a_1 x_1 + a_2 x_2 + b)^{-(c+3)} dx_2, \]

\[ = a_1 c^{(a_1 x_1 + b)^-(c+1)}, \]

which establishes the necessary part.

Conversely, assuming that \( X_1 \) is \( P \Pi(c,b/a_1) \) so that,

\[ f_1(x_1) = a_1 c^{(a_1 x_1 + b)^-(c+1)}, \]

from equation (3.32)

\[ a_1 c^{(a_1 x_1 + b)^-(c+1)} = \int_0^\infty a_1^{(c+1)(a_2 x_2 + b)^{c+1}} \frac{f_2(x_2) dx_2}{(a_1 x_1 + a_2 x_2 + b)^{c+2}}. \]

Taking the Mellin transform of both sides with respect to \( x_1 \),

\[ \int_0^\infty \frac{cb^c x_1^{s-1} dx_1}{(a_1 x_1 + b)^{c+1}} = \int_0^\infty \int_0^\infty \frac{(c+1)(a_2 x_2 + b)^{c+1} x_1^{s-1}}{(a_1 x_1 + a_2 x_2 + b)^{c+2}} f_2(x_2) dx_1 dx_2. \]
Transforming $x_1$ to

$$y = a_1x_1 \left( a_1x_1 + a_2x_2 + b \right)^{-1}$$
on the right side and

$$z = a_1x_1 \left( a_1x_1 + b \right)^{-1}$$
on the left side leaves the equation,

$$\int_0^\infty (a_2x_2 + b)^{s-1} f_2(x_2) dx_2 = c b^{s-1} (c-s+1)^{-1}, c > s-1,$$

which is equivalent to

$$\int_0^\infty F(y) y^{s-1} dy = c b^{s-1} (c-s+1)^{-1},$$

where

$$F(y) = H(y-b) f_2 \left( \frac{y-b}{a_2} \right),$$

and $H(.)$ is the Heaviside unit function. Proceeding to the inverse Mellin transform,

$$f_2(x_2) = a_2 c b^c (a_2x_2 + b)^{-(c+1)}, x_2 > 0.$$
same component $X_1$ (or $X_2$). To show this, we assume that the distribution of $X_1$ given $X_2=x_2$ is as in equation (3.31) and further $X_1$ is $P II(c, b/a_1)$. By Theorem 3.7 $X_2$ is $P II(c, b/a_2)$ and therefore the joint distribution is computed as,

$$f(x_1, x_2) = a_1a_2(c+1)(c+2)b^{c+1} (a_1x_1+a_2x_2+b)^{-(c+3)},$$

$x_1, x_2 > 0, a_1, a_2, b > 0, c > -1,$

which is the bivariate distribution of Lindley and Singpurwalla (1986).

**Theorem 3.8.**

If $(X_1, X_2)$ is distributed such that $X_1$ given $X_2=x_2$ has finite range distribution

$$f(x_1|x_2) = (r-1)a_1(1-a_2x_2)^{-1}[1-a_1x_1/(1-a_2x_2)]^{r-2},$$

$0 < x_1 < a_1(1-a_2x_2)^{-1}, 0 < x_2 < a_2^{-1},$

$a_1, a_2 > 0, r > 2,$

then $X_1$ has finite range distribution with density,

$$f_1(x_1) = ra_1(1-a_1x_1)^{r-1}, 0 < x_1 < a_1^{-1},$$

if and only if $X_2$ is likewise distributed with density,

$$f_2(x_2) = ra_2(1-a_2x_2)^{r-1}, 0 < x_2 < a_2^{-1}.$$
Proof:
The if part follows from direct calculations as shown below. If $X_2$ has finite range distribution $FR(a_2, r)$ as in (3.35) then
\begin{equation}
f(x_1, x_2) = r(r-1) a_1 a_2 (1-a_1 x_1-a_2 x_2)^{r-2}, \tag{3.36}
a_1, a_2 > 0, a_1 x_1 + a_2 x_2 < 1, r > 2.
\end{equation}

Integrating out $x_2$, we recover (3.34).

On the other hand, assuming that $X_1$ has density (3.34), equation (3.32) gives
\begin{equation}
ra_1 (1-a_1 x_1)^{r-1} = \int f(x_1 | X_2 = x_2) f_2(x_2) \, dx_2.
\end{equation}
That is,
\begin{equation}
r(1-a_1 x_1)^{r-1} = \int_0^1 (r-1)(1-a_2 x_2)^{-1} \frac{d}{dx_2} \left[ \frac{1-a_1 x_1}{(1-a_2 x_2)} \right]^{r-2} \, dx_2,
\end{equation}
or
\begin{equation}
ra_2 y^{r-1} = \int_0^y (r-1)(y-x_2)^{r-2} g(x_2) \, dx_2, \tag{3.37}
\end{equation}
where
\begin{equation}
y = a_2^{-1}(1-a_1 x_1),
\end{equation}
and
\begin{equation}
g(x_2) = f_2(x_2) (1-a_2 x_2)^{1-r}.
\end{equation}
Equation (3.37) can be written as

\[(r/r-1) a_2^y r^{-1}/r^{-1} = I_g^{r-1},\]

with \(I_g^{r-1}\) standing for the Riemann-Liouville fractional integral of order \(r-1\) of the function \(g(.)\) (Erdélyi et. al. (1954) p.181) defined as

\[I_g^r = \frac{1}{r} \int_0^y (y-x_2)^{r-1} g(x_2) dx_2.\]

The operator \(I_g^r\) is connected with differential and integral operators as follows.

\[\frac{d}{dx} I_g^r (x) = I_g^{r-1}(x) \text{ and } \frac{d}{dx} I_g^2 (x) = \int_0^y g(t) dt.\]

Thus we get

\[\frac{d}{dx_2} I_g^{r-1}(x_2) = r(r-1)a_2^y r^{-2}/(r-1) \int (r-1),\]

\[= r(r-1) a_2^y r^{-2}/(r-1)\]

and

\[\frac{d}{dx_2} I_g^2 (x_2) = \frac{r(r-1)}{(r-1)!} a_2^y \int_0^y g(t) dt.\]

The unique inverse relation is, therefore,

\[g(x_2) = r a_2^y,\]
and thus,
\[ f_2(x_2) = ra_2(1-a_2x_2)^{r-1}. \]

This completes the proof of the theorem.

A characterization of the bivariate finite range distribution specified by the density (3.36), with the forms of \( X_1 \) and \( X_1 \) given \( X_2 = x_2 \) following the finite range distribution is also evident.

3.3 Characterization By Properties Of Partial Moments.

We recall from equation (2.8) that the \( r \)th partial moment of a random variable \( X \) about a point \( t \) is defined as
\[ p_r(t) = E[(X-t)^+]^r, \quad r = 0, 1, 2, \ldots, \]
where, \( (X-t)^+ = \max(0, X-t) \).

The properties of partial moments can be used to characterize probability distributions in the same way as truncated moments are employed by many authors in characterizing distributions like the exponential. Chong (1977) has characterized the exponential distribution by the property
\[ E[X-t-s]^+] E(X) = E(X-t)^+ E(X-s)^+ \]
of the partial means. In a recent paper Gupta and Gupta (1983) have made an extensive study of partial moments and established that one partial moment is sufficient to determine the parent distribution uniquely.

The random variable \((X-t)^+\) used in defining the partial moments are meaningful in the study of personal incomes. If \(X\) represents the income of an individual and \(t\) the tax-exemption level, \((X-t)^+\) represents the taxable income. Those incomes which fall short of \(t\) is of no consequence in the computation of taxes and therefore is as good as treated to be zero. Thus the study of partial moments is useful in analysing the incomes that exceeds the exempt level without truncating the distribution at \(t\).

In the following theorem Hitha (1990) shows that the Pareto distribution is characterized by the property that any partial moment at a given point is proportional to the product of partial moments at two other points which are factors of the original point.

**Theorem 3.8.**

Let \(X\) be a non-negative random variable in the support of \([k, \infty)\), \(k > 0\), having absolutely continuous distribution with respect to Lebesgue measure and with
E(X^r) < \infty. Then the partial moments satisfy the relation,

\[ p_r(t) p_r(s) = A(r) p_r(ts), \]  

(3.38)

for all \( t, s \geq 1, \ r = 0, 1, 2, \ldots, \) and only if \( X \) follows the \( P \ I(a, k), \) where \( A(r) \) satisfies the equation,

\[ a = r[1 + A(r-1)/A(r)], \ r \geq 1, a > r. \]  

(3.39)

Proof:

If \( X \) is distributed as \( P \ I(a, k), \)

\[ p_r(t) = r! k^{a r}/t^{a(a-1)(r)}, \ r = 0, 1, 2, \ldots, \]

so that

\[ p_r(t) p_r(s) = \frac{r! k^{a r}}{(a-1)(r)t^{a}} \cdot \frac{r! k^{a s}}{(a-1)(r)s^{a}}. \]

Now setting \( t = s = 1 \) in (3.38),

\[ A(r) = p_r(1) = \frac{r! k^{a}}{(a-1)(r)}, \]

(3.40)

and hence,

\[ p_r(t) p_r(s) = p_r(ts) \frac{r! k^{a}}{(a-1)(r)}, \]

\[ = A(r) p_r(ts). \]
Conversely, with \( p_\tau(1) > 0 \), (3.38) reduces to the functional equation,

\[
G(r,t) G(r,s) = G(r,ts), \quad t,s > 1,
\]

where

\[
G(r,t) = \frac{p_\tau(t)}{p_\tau(1)}.
\]

Considering the transformations \( u = \log t \) and \( v = \log s \) and writing \( G(r,e^u) = g(u) \) and \( G(r,e^v) = g(v) \), we get the Cauchy functional equation,

\[
g(u+v) = g(u) g(v), \quad u,v > 0,
\]

whose solution is

\[
g(v) = \lambda(r) e^{\alpha(r)v}.
\]

It follows that

\[
G(r,t) = \lambda(r) t^{\alpha(r)}.
\]

From \( G(r,1) = 1 \), we have \( \lambda(r) = 1 \) and hence,

\[
p_\tau(t) = p_\tau(1) t^{\alpha(r)}. \quad (3.40)
\]

By differentiating the integral form of \( p_\tau(t) \), viz.,
\[ p_r(t) = \int_t^\infty (x-t)^r \, dF, \quad (3.41) \]

\[ p_r'(t) = -r p_{r-1}(t), \quad r > 1. \quad (3.42) \]

Equations (3.40) and (3.42) mean that

\[ -r p_{r-1}(t) = \alpha(r) p_r(1) t^{\alpha(r)-1}, \quad (3.43) \]

and

\[ -r p_{r-1}(1) = \alpha(r) p_r(1). \quad (3.44) \]

Using (3.39) and \( A(r) = p_r(1) \) the conclusions

\[ (a-r) p_r(1) = r p_{r-1}(1), \quad a > 0, \quad k > 0, \quad (3.45) \]

and from (3.44),

\[ \alpha(r) = -(a-r) \]

follows. Theorem 3.8 provides a series of results relating

univariate families like exponential, power function, Burrr, logistic, etc. in terms of the monotone transformations connecting them. Using the logarithmic transformation to

the exponential distribution in

the result corresponding to (3.36) for

and from the recurrence relation (3.45),

\[ p_r(1) = a! \frac{p_0(1)}{\alpha(r)}. \]

\[ r > 1, \quad t, s > 1. \]
By definition,
\[
p_o(1) = \int_1^\infty dF,
\]
which is in fact a positive constant lying between 0 and 1 so that it can be written as \(ka\) for some \(k > 0\). Thus,

\[
p_r(t) = \int_t^\infty (x-t)^r dF = k^a t^{r-a}/(a-1)(r).
\]

Differentiating (3.46) successively with respect to \(t\), and using the relation (3.41) we arrive at,

\[
1-F(t) = (k/t)^a, \quad t > k, \quad a > 0, \quad k > 0,
\]

which completes the proof.

Theorem 3.8 provides a series of results relating to other univariate families like exponential, power function, Burr, logistic, etc. in terms of the monotone transformations connecting them. Using the logarithmic transformation to the Pareto variable, the result corresponding to (3.38) for the exponential distribution is

\[
E[(X-t-s)^+]^r = E[(X-t)^+]^r E[(X-s)^+]^r,
\]

\(r > 1\), \(t, s > 1\).
Setting \( r=1 \) in (3.47) we get the result due to Chong (1977) mentioned at the beginning of this section. There is a similar result that concerns the power distribution cited in Theorem 3.5 which is stated as follows.

Theorem 3.9.

Let \( X \) be a non-negative random variable in the support of \((0,R), R>0\), having absolutely continuous distribution with respect to Lebesgue measure such that \( EX^{-r} < \infty \). Then the partial moments

\[
U_r(t) = \mathbb{E}[(t-X)^+]^r,
\]

satisfy the relation

\[
U_r(t) U_r(s) = U_r(ts) B(r),
\]

(3.48)

for all \( 0 < t, s < R, r = 0,1,2,\ldots, B(r) > 0 \) if and only if \( X \) follows the power distribution with density,

\[
f(x) = (\Theta+1) x^{\Theta} / R^{\Theta+1}, \quad 0 < x < R,
\]

(3.49)

where \( B(r) \) satisfies the relation

\[
(r+1) - rB(r-1)/B(r) = \Theta, \quad r > 1.
\]

(3.50)
Proof:

For the distribution (3.49),

$$U_r(t) = r! \frac{t^{\theta+r+1}}{R^{\theta+1}(\theta+2) \ldots (\theta+r+1)},$$

so that from this expression and $B(r) = U_r(1)$, the necessity of the condition follows. The proof of sufficiency part is along the lines of the proof of the previous Theorem and therefore only the important steps are presented. As before $G(r, t) = U_r(t)/U_r(1)$ and subsequently $G(r, e^x) = g(x)$ produces the Cauchy functional equation whose solution turns out to be

$$G(r, t) = t^{\alpha(r)}$$

and

$$\alpha(r) = \theta + r + 1.$$

Further,

$$U_r(1) = r! \frac{U_0(1)}{(\theta+2) \ldots (\theta+r+1)}.$$

One can take $U_0(1)$ a positive constant to be $R^{\theta+1}$ for some $R > 0$ and this leads to

$$U_r(t) = r! \frac{R^{\theta+1}}{(\theta+2) \ldots (\theta+r+1)},$$

and whence the distribution (3.49).
3.4 Characterization By Truncated Reciprocal Moments

The $r^{th}$ truncated reciprocal moment of a random variable $X$ is given by equation (2.11) provided the expectation on the right side exists. The reciprocal moments exist only for those distributions which have a zero of order $r$ at least at the origin. A characteristic property of the Pareto distribution associated with the reciprocal moments will now be proved.

Theorem 3.10.

The truncated reciprocal moments $C_r(t)$ of the random variable $X$ given by equation (2.11) satisfy the relation

$$C_r(t) C_r(s) = M(r) C_r(ts),$$

for all $t, s > 1$, $r = 0, 1, 2, \ldots$, $C_r(1) > 0$, $M(r) > 0$, if and only if $X$ has $P I(a, k)$ distribution with

$$a = r[M(r-1)/M(r)-1], \ r > 1. \quad (3.52)$$

Proof:

If $X$ is $P I(a, k)$, direct calculation yields equation (2.12) and the relationship (3.51) is easily verified. Conversely, let (3.51) be true with the fact $C_r(1) = M(r)$. Proceeding exactly as in Theorem 3.8,
we arrive at the equation
\[ C_r(t) = M(r) t^a(r), \]  
which is analogous to (3.40).

Now,
\[ C_r(t) = (1/R(t))(-1)^F \int_{t}^{\infty} \left(\frac{1}{x}-(1/t)\right)^{r-1} \frac{1}{x^2} \frac{1}{1-F} dx. \]  

Thus,
\[ C_r(t) R(t) = (-r)(-1)^F \int_{t}^{\infty} \left(\frac{1}{x}-(1/t)\right)^{r-1} \left(\frac{1}{x^2}\right) dx. \]  

Differentiating the above equation with respect to \( t \),
\[ C_r'(t) R(t) + C_r(t) R'(t) = ((-r)/t^2)(r-1) \]
\[ \int_{t}^{\infty} \left(\frac{1}{x}-(1/t)\right)^{r-2} \left(\frac{1}{x^2}\right) R(x) dx \]
\[ = ((-r)/t^2) C_{r-1}(t) R(t). \]  

Thus we get
\[ [(r/t^2)C_{r-1}(t) + C_r'(t)]/C_r(t) = (-R'(t)/R(t)), \]
and so the recurrence relation

\[
\frac{(r/t^2)C_{r-1}(t) + C_r'(t))/C_r(t)}{C_r(t)} = \frac{(r-1)/t^2)C_{r-2}(t) + C_{r-1}'(t))/C_{r-1}(t)}{C_{r-1}(t)}.
\]

Using the relation (3.53) we will arrive at

\[
\frac{(r/t^2) M(r-1) t^{\alpha(r-1)} + \alpha(r) M(r) t^{\alpha(r)-1}}{M(r) t^{\alpha(r)}} = \frac{((r-1)/t^2) M(r-2) t^{\alpha(r-2)} + \alpha(r-1) M(r-1) t^{\alpha(r-1)-1}}{M(r-1) t^{\alpha(r-1)}}.
\]

Putting \( t=1 \),

\[
\alpha(r) + rM(r-1)/M(r) = \alpha(r-1) + (r-1) M(r-2)/M(r-1). (3.56)
\]

Further we have (3.52) to obtain,

\[
rM(r-1)/M(r) = a+r,
\]

so that (3.56) will become,

\[
\alpha(r) + a + r = \alpha(r-1) + a+r-1.
\]

That is,

\[
\alpha(r) = \alpha(r-1)-1.
\]
or

\[ \alpha(r) = -r, \text{ since } \alpha(0) = 1. \]

Thus,

\[ C_r(t) = C_r(1)t^{-r}. \]

Now,

\[ C_r(1) = (-1)^ra_rC_0(1)/a[r], \]

where

\[ C_0(t) = \left(\frac{1}{R(t)}\right) \int_t^\infty dF = 1, \quad k > 0, \]

for all \( t \).

Thus,

\[ C_r(t) = (-1)^ra_r/(a[r]t^r). \]

Further,

\[ r!R(t)a(-1)^r/a[r]t^r = \left(-1\right)^r \int_t^\infty \left((1/x)-(1/t)\right)^r dF, \]

\[ = \left(-1\right)^r(-r) \int_t^\infty \left((1/x)-(1/t)\right)^{r-1}(1/x^2) dF. \]

Differentiating with respect to \( t \), we get

\[ (-1)^r a_r/(a[r]\left( \frac{R'(t)}{t^r} - \frac{RR(t)}{t^{r+1}} \right) \]

\[ = (-1)^{r+1} \frac{r(r-1)}{t^2} \int_t^\infty ((1/x)-(1/t))^{r-2} \frac{R(x)}{x^2} dx, \]

also in Galambos and Kotz (1977) where the connection between...
\[ = \frac{(-r/t^2) R(t)(-1)^{r-1}a.(r-1)!}{a[r-1] t^{r-1}}. \]

That is,
\[
\frac{[R'(t) - (r/t) R(t)]}{(a+r) R(t)/t} = -\frac{R(t)}{t},
\]
or
\[
\frac{R'(t)}{R(t)} = -\frac{a}{t}.
\]

Hence,
\[
F(t) = 1-(k/t)^a, \quad t > k, \quad a > 0, \quad k > 0,
\]
and our result is completely proved.

3.5 Characterization By Additive Damage Model

The concept of damage models introduced by Rao and Rubin (1964) involves a random variable X reduced to another random variable U by some random mechanism represented by the conditional distribution of U given X = x. The quantity \( Y = X-U \) is the reduction in X and is called the damaged component in X and the objective in the formulation of such models, is to characterize the distribution of X in terms of the distribution of U. Instead of the additive model one can also have a multiplicative model of the form \( U = X Y \). A comprehensive survey of the various results in damage models is available in Patil and Retnaparki (1975) and also in Galambos and Kotz (1977) where the connection between
such models and those arrived at by geometric compounding
and rarefactions, is also explained.

In the present section, a result due to Revankar
et. al. (1974) that characterizes the P II(a,α) model will
be extended to cover the finite range and exponential variables.

Theorem 3.11.

If,

$$E(U|X=x) = a+bx,$$  \hspace{1cm} (3.57)

a necessary and sufficient condition that X has either an
exponential distribution with β = b>0 and α > a > 0 or a
Pareto type II distribution with β>b>0 and α>α or a finite
range distribution with β<b<0 and α>α is that

$$E(U|X>y) = α+βy.$$  \hspace{1cm} (3.58)

Proof:

The case when β>b>0 for the Pareto distribution is
proved in Revankar et.al. (1974).

Assuming X to be FR(c,R),

$$E(U|X>y) = \left(\frac{1}{R(y)}\right) \int \left(a-bx\right)\left(\frac{c}{R}\right)\left(1-x/R\right)^{c-1} dx,$$

$$= \left(a-bR/(c+1)\right) - bc/(c+1),$$

$$= α + βy,$$
where \( \beta = bc/(c+1) \) and \( a > \alpha \) and \( b < \beta < 0 \).

Conversely, with

\[
E(U|X>y) = \alpha + \beta y, \ b < \beta < 0,
\]

\[
R(y) (\alpha + \beta y) = - \int_{Y}^{\infty} (a-bx) \, dR(x),
\]

Differentiating both sides with respect to \( y \),

\[
R'(y)/R(y) = -\beta/[(\alpha-a) + (\beta+b)y].
\]

The solution is

\[
R(y) = K[(\alpha-a) + (\beta+b)y]^{-\beta/(\beta+b)}.
\]

Evaluating \( K \), using the condition \( R(0) = 1 \),

\[
K = (\alpha-a)^{\beta}/(\beta+b).
\]

Thus,

\[
R(y) = (1 - y/R)^{c},
\]

where

\[
R = (b+\beta)/(a-\alpha) \quad \text{and} \quad c = -\beta/(\beta+b) > 0,
\]

and \( Y \) is \( FR(c,R) \). The proof of the exponential case is trivial and our theorem is proved.
3.6 Residual Life Distributions

In section 2.5 we have examined the various characterizations of the Pareto and related distributions by properties of residual life time. These properties will appear in a natural way once we look at the entire distribution of the residual life time that is being currently discussed. The distribution function of the residual life $Y_x$ of a non-negative random variable $X$ with $F(0) = 0$, where $F(x)$ is the distribution function of $X$, is defined as (Arnold, 1983),

$$ G(y;x) = \frac{F(x+y) - F(x)}{1 - F(x)}, \quad y > 0 $$

(3.59)

The corresponding survival function is

$$ S(y;x) = \frac{R(x+y)}{R(x)} \cdot $$

(3.60)

It is easy to see that the mean of (3.60) is the MRL function $r(x)$ defined in equation (2.20), for,

$$ E(Y_x) = \int_0^\infty y \left( -\frac{\partial}{\partial y} S(y;x) \right) dy, $$

$$ = \left( \frac{1}{R(x)} \right) \int_0^\infty y \left( -\frac{\partial}{\partial y} R(x+y) \right) dy, $$

$$ = \left( \frac{1}{R(x)} \right) \int_0^\infty (z-x) dF(z) $$

$$ = E[X-x|X>x], $$

provided that $E[X] < \infty$. 

The first problem we investigate is the form of the random variable $Y_x$ when $X$ belongs to the class of models under consideration in this chapter. The answer is provided in the following

**Theorem 3.12.**

The random variable $X$ follows

(a) $E(b)$ if and only if $Y_x$ is $E(b)$

(b) $P I(a,k)$ if and only if $Y_x$ is $P I(a,k)$ with the origin shifted from $k$ to $k+x$.

(c) $P II(c,a)$ if and only if $Y_x$ is $P II(c,x+a)$

(d) $FR(d,R)$ if and only if $Y_x$ is $FR(d, R-x)$.

**Proof:**

When $Y_x$ is Pareto II $(c, x+a)$ it follows from (2.26) that

$$R(x+y) = \left( \frac{x+a}{x+y+a} \right)^c.$$

As $x$ tends to zero, $R(y) = \left( \frac{y}{y+a} \right)^c$ and $X$ is $P II(c,a)$. The if part is verified through direct calculations of $S(y;x)$ using (3.60). Proof for the other distributions follow suit.
As an alternative to the mean residual life function the median residual life has also been considered in literature, which stands for the time expected for half of the number of items that operate at time $x$ fail. The new measure enjoys relative superiority over the MRL in situations where the latter (a) becomes unstable in the presence of outliers (b) does not exist (but the median is always finite) (c) is less desirable for fat tailed distributions such as those considered here, and (d) the data is in the form of censored observations with at least half of those remaining have recorded failure times. Moreover, it has simple closed form expressions for many useful failure time models while the Mean Residual life has too complicated a functional form to be of use. We cite for example, the Weibull case where the median residual life is

$$\left( b^{-1} \log 2 + x^c \right)^{1/c} - x,$$

corresponding to the survival function $R(x) = e^{-bx^c}$, in contrast with the MRL which at best can be written only in terms of incomplete gamma function and is analytically intractable.

With respect to the residual life distribution the median residual life function is the solution for $y$ of the

\[ y = \frac{-1}{b} \int_{x}^{\infty} e^{-bt} \, dt. \]
equation

\[ P(Y > y) = \frac{1}{2}, \]

or

\[ S(y; x) = \frac{R(x+y)}{R(x)} = \frac{1}{2}, \]

which is in general a function of \( x \) to be denoted by \( M(x) \). Thus \( M(x) \) satisfies the functional equation

\[ R(x+M(x)) = \frac{1}{2} R(x). \]

When \( x \) tends to zero, the last expression gives,

\[ R(M(0)) = \frac{1}{2}, \]

so that \( M(0) = M \) becomes the median of the random variable \( X \). Then the median residual life is given by

\[ R(x+M(x)) = R(x) R(M). \]

Theorem 3.13.

Let \( R(x) \) be an absolutely continuous survival function with \( R(0) = 1 \) then the residual life distribution \( S(y; x) \) satisfies the equation,

\[ S(g(x)y; x) = R(y), \]

where \( g(x) = M(x)/M(0) \), if and only if \( X \) is distributed as either exponential or Pareto II or finite range.
Proof:

Lemma:

As a first step we show that the median residual life function is of the form $lt+m$, $m>0$ if and only if $X$ is $E(b)$ for $l=0$, $P II(c,\alpha)$ for $l>0$ and $FR(d,R)$ for $l<0$.

Proof:

By solving the equation (3.62) we see that

$$M(x) = \begin{cases} \frac{1}{b} \log 2, & \text{for } E(b), \\ (2^{1/c-1})(x+\alpha), & \text{for } P II(c,\alpha), \\ \left[1-\left(\frac{1}{2}\right)^{1/d}\right](R-x), & \text{for } FR(d,R), \end{cases}$$

satisfy the conditions of the Theorem. Conversely, when $l=0$, $M(x) = a$ constant so that (3.62) becomes the Cauchy functional equation,

$$R(x+M) = R(x) R(M),$$

whose only continuous solution that satisfy the probability requirements for $R(x)$ is

$$R(x) = e^{-bx},$$

where $R(M) = e^{-bM} = 1/2$. Then the result is true for the exponential distribution. The Pareto II case is proved in...
Schmittlein and Morrison (1981) and the finite range situation follows from the same proof.

In establishing the main result we first assume that (3.62) is true. Then with the help of (3.61) and (3.63) the equation

\[ R(g(x)y+x) = R(x) R(y) \]  

(3.64)
can be reached. Differentiating (3.64) with respect to \( x \),

\[ R'[g(x)y+x] g(x) = R(x) R'(y). \]

Hence,

\[ (g'(x)y+1)/g(x) = R'(x)R(y)/R(x) R'(y), \]

and,

\[ 1/g(x) = R'(x)/R(x) R'(y). \]

Combining the last two equations,

\[ g'(x)y+1 = R'(0) R(y)/R'(y). \]

The right side being independent of \( x \), so should be the left side also which implies \( g'(x) \) is a constant or \( g(x) = l x + m \).
From the definition of $g(x)$ this reduces to the linearity of $M(x)$ and therefore by lemma, $X$ has one of the distributions stated in the Theorem. In order to establish the converse, we observe that in the exponential case $g(x)=1$ so that relation (3.64) holds. For $P \Pi(c, \alpha)$ we replace $y$ by $yg(x) = y.(x/\alpha)/\alpha$ in

$$S(y; x) = [(x+c)/(x+y+c)]^c,$$

to get

$$S(yg(x); x) = (\alpha/(\alpha+y))^c = S(y).$$

The result for $FR(d, R)$ is established similarly and our Theorem stands proved.

The concept of median of residual life extends itself to the notion of percentile residual life if one wishes to have a finer set of summary measures of location. Such measures are used for inference in reliability studies by Joe and Proschan (1983). From the point of view of characterizations also they are quite handy and produces conclusions similar to that of the median.

The $q^{th}$ percentile residual life time is according to Haines and Singpurwalla (1974) is

$$M_q(x) = S^{-1}[qS(x)]^{-x}, 0 < q < 1.$$
For the family of distributions under investigation,

\[ M_q(x) = \begin{cases} \frac{1}{b} \log q, & \text{for } E(b), \\ (x+\alpha) \left( q^{1/c} - 1 \right), & \text{for } P II(c, \alpha), \\ \left[ 1 - \left( \frac{1}{q} \right)^{1/d} \right] \left[ R - x \right], & \text{for } FR(d, R). \end{cases} \]

Thus the form of the percentile residual life for the three distributions remains identical with that of the median residual life. Therefore the conclusions of the last theorem can be extended to involve the percentile residual life.