CHAPTER 2

ON STEADY STATE BEHAVIOUR OF A
NETWORK QUEUING MODEL WITH BISERIAL
AND PARALLEL CHANNELS

2.1 Introduction
This chapter deals with the study of the steady state analysis of a complex network of queueing model in which a common channel is linked in series with each of two systems, one containing two bi-serial channels and other three parallel channels in series. The arrival and service pattern both follow Poisson law. The generating function technique, laws of calculus and statistical tools have been used to find the various queue characteristics. The model finds its application in decision making in the process industries, in banking system, in networking, in many administrative setups and business service.

Jackson R.R.P [21] studied the behaviour of a queuing system containing phase type service. Maggu [30] introduced the concept of bitendom in theory of queues which corresponds to a practical situation arise in production concern. Later on this idea was developed by various authors with different modifications and argumentations. Khodadi Abutaleb
modified the queue system studied by Maggu by assuming the service parameter directly proportional to the queue number. Singh T.P.et.al [42] studied the transient behaviour of a queuing network with parallel biseries queue linked with a common channel. Singh T.P. et.al.[43] studied steady state behaviour of a queue model comprised of two subsystem with biserial channel linked with a common channel. Later Gupta Deepak, Singh T.P. et.al. [13] Studied a network queue model comprised of biserial and parallel channel linked with a common server.

The present queue model differ the study made by Gupta Deepak, Singh T.P. et al.[43] in the sense that in this model the first system consist of biserial channel while the second system consist of three parallel sub channels each of which are linked with the third system in series. The various queue characteristics have been obtained explicitly under the transient behaviour of the system.

### 2.2 Problem Formulation

The entire queue model is comprised of three service channels $S_1$, $S_2$ and $S_3$. The subsystem $S_1$ consist of two biserial service channels $S_{11}$ and $S_{12}$, the subsystem $S_2$ contain three parallel channels $S_{21}$, $S_{22}$ and $S_{23}$. The service channel $S_3$ is commonly linked in series with each of two servers $S_1$ and $S_2$ for competition of final phase service demanded either at a subsystem $S_1$ or $S_2$. The service time at $S_{ij}$ ($i=1,2$ and $j=1,2,3$) are
distributed exponentially. We assume the service mean rate \( \mu_i, \mu_2, \mu_i', \mu_2', \mu_3 \) at \( S_i, j (i=1,2 \text{ and } j =1,2,3) \) and \( \mu_3 \) at \( S_3 \) respectively. Queues \( Q_1, Q_2, Q_3, Q_4, Q_5 \) and \( Q_6 \) are said to be formed in front of the service channels \( S_{11}, S_{12}, S_{21}, S_{22}, S_{23} \) and \( S_3 \) respectively, if they are busy. Customers coming at the rate \( \lambda_i \) after completion of phase service at \( S_{11} \) will join \( S_{12} \) or \( S_3 \) (that is they may either go to the network of servers \( s_{11} \rightarrow s_{12} \rightarrow s_3 \) or \( s_{11} \rightarrow s_1 \) ) with the probabilities \( p_{12} \) or \( p_{13} \) such that \( p_{12} + p_{13} = 1 \) and those coming at the rate \( \lambda_2 \) after completion of phase service at \( S_{12} \) will join \( S_{11} \) or \( S_3 \) (that is they may either go to the network of servers \( s_{21} \rightarrow s_{11} \rightarrow s_3 \) or \( s_{21} \rightarrow s_3 \) ) with the probabilities \( p_{21} \) or \( p_{23} \) such that \( p_{21} + p_{23} = 1 \). The customers coming at the rate \( \lambda_i' \) go to the network of servers \( s_{21} \rightarrow s_3 \) and those coming at the rate \( \lambda_2' \) go to the network of servers \( s_{22} \rightarrow s_3 \) and those coming at the rate \( \lambda_3' \) go to the network of servers \( s_{23} \rightarrow s_3 \).

This section is based on the research paper “On Steady State Behaviour of a Network Queuing Model with Biserial and Parallel Channels Linked With a Common Server”, published in Computer Engineering and Intelligent System (U.S.A), Vol. 2 No. 3 pp 11-22.

### 2.3 Mathematical Analysis
Let \( P_{n_1,n_2,n_3,n_4,n_5,n_6} \) be the joint probability that there are \( n_1 \) units waiting in queue \( Q_1 \) in front of \( S_{11} \), \( n_2 \) units waiting in queue \( Q_2 \) in front of \( S_{12} \), \( n_3 \) units waiting in queue \( Q_3 \) in front of \( S_{21} \), \( n_4 \) units waiting in queue \( Q_4 \) in front of \( S_{22} \), \( n_5 \) units waiting in queue \( Q_5 \) in front of \( S_{23} \) and \( n_6 \) units waiting in queue \( Q_6 \) in front of \( S_3 \) (Figure 1). In each case the waiting includes a unit in service, if any.

Also, \( n_1, n_2, n_3, n_4, n_5, n_6 > 0 \).

![Network Queuing Model](image)

**Figure 2.1 (Network Queuing Model)**

The standard arguments lead to the following differential difference equations in transient form as

\[
P_{n_1,n_2,n_3,n_4,n_5,n_6}(t) = -\left(\lambda_1 + \lambda_2 + \lambda_3' + \lambda_4' + \lambda_5' + \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6\right) P_{n_1,n_2,n_3,n_4,n_5,n_6}(t)
+ \lambda_4 P_{n_1-1,n_2,n_3,n_4,n_5,n_6}(t) + \lambda_2 P_{n_1,n_2-1,n_3,n_4,n_5,n_6}(t) + \mu_1 (n_1+1) P_{n_1+1,n_2,n_3,n_4,n_5,n_6-1}(t) +
\mu_1 (n_1+1) p_{12} P_{n_1+1,n_2-1,n_3,n_4,n_5,n_6}(t) + \mu_2 (n_2+1) p_{23} P_{n_1,n_2+1,n_3,n_4,n_5,n_6-1}(t) +
\text{terms for other transitions}
\]
\[
\mu_2 \left( n_2 + 1 \right) p_{21} p_{n_1-1,n_2+1,n_3,n_4,n_5,n_6}(t) + \lambda_4' p_{n_1,n_2,n_3-1,n_4,n_5,n_6}(t) + \lambda_2' p_{n_1,n_2,n_3-1,n_4,n_5,n_6}(t) + \lambda_3' p_{n_1,n_2,n_3,n_4-1,n_5,n_6-1}(t) + \mu_3 \left( n_6 + 1 \right) p_{n_1,n_2,n_3,n_4,n_5,n_6+1}(t) + \mu_1 \left( n_3 + 1 \right) p_{n_1,n_2,n_3+1,n_4,n_5,n_6-1}(t) + \mu_2 \left( n_4 + 1 \right) p_{n_1,n_2,n_3,n_4+1,n_5,n_6-1}(t) + \mu_3 \left( n_5 + 1 \right) p_{n_1,n_2,n_3,n_4,n_5+1,n_6-1}(t).
\]

The steady state equation \( (t \to \infty) \) governing the model are depicted as

\[
\left( \lambda_4 + \lambda_2 + \lambda_3 + \mu_1 + \mu_2 + \mu_1 + \mu_2 + \mu_3 + \mu_3 \right) P_{n_1,n_2,n_3,n_4,n_5,n_6} = \\
\lambda_4 P_{n_1-1,n_2,n_3,n_4,n_5,n_6} + \lambda_2 P_{n_1,n_2-1,n_3,n_4,n_5,n_6} + \mu_1 p_{13} P_{n_1+1,n_2,n_3,n_4,n_5,n_6-1} + \\
\mu_1 p_{12} P_{n_1+1,n_2-1,n_3,n_4,n_5,n_6} + \mu_2 p_{23} P_{n_1,n_2+1,n_3,n_4,n_5,n_6-1} + \\
\mu_3 P_{n_1,n_2,n_3-1,n_4,n_5,n_6} + \lambda_4 P_{n_1,n_2,n_3-1,n_4,n_5,n_6} + \lambda_2 P_{n_1,n_2,n_3-1,n_4,n_5,n_6} + \\
\lambda_3 P_{n_1,n_2,n_3,n_4-1,n_5,n_6} + \mu_3 P_{n_1,n_2,n_3,n_4-1,n_5,n_6} + \mu_1 P_{n_1,n_2,n_3+1,n_4,n_5,n_6-1} + \\
\mu_2 P_{n_1,n_2,n_3,n_4+1,n_5,n_6-1} + \mu_3 P_{n_1,n_2,n_3,n_4,n_5+1,n_6-1}.
\] --- (1)

On taking \( n_1=0 \); we have

\[
\left( \lambda_4 + \lambda_2 + \lambda_3 + \mu_1 + \mu_2 + \mu_1 + \mu_2 + \mu_3 + \mu_3 \right) P_{0,n_2,n_3,n_4,n_5,n_6} = \\
\mu_1 p_{13} P_{n_1+1,0,n_3,n_4,n_5,n_6} + \mu_2 p_{23} P_{n_1,0,n_3,n_4,n_5,n_6-1} + \lambda_4 P_{0,n_2,n_3-1,n_4,n_5,n_6} + \lambda_2 P_{0,n_2,n_3-1,n_4,n_5,n_6} + \\
\lambda_3 P_{0,n_2,n_3,n_4-1,n_5,n_6} + \mu_3 P_{0,n_2,n_3,n_4-1,n_5,n_6} + \mu_1 P_{0,n_2,n_3+1,n_4,n_5,n_6-1} + \\
\mu_2 P_{0,n_2,n_3,n_4+1,n_5,n_6-1} + \mu_3 P_{0,n_2,n_3,n_4,n_5+1,n_6-1}.
\] --- (2)

On taking \( n_2=0 \); we have

\[
\left( \lambda_4 + \lambda_2 + \lambda_3 + \mu_1 + \mu_2 + \mu_1 + \mu_2 + \mu_3 + \mu_3 \right) P_{n_1,0,n_2,n_3,n_4,n_5,n_6} = \lambda_4 P_{n_1-1,0,n_3,n_4,n_5,n_6} + \\
\mu_1 p_{13} P_{n_1+1,0,n_3,n_4,n_5,n_6-1} + \mu_2 p_{23} P_{n_1,0,n_3,n_4,n_5,n_6-1} + \\
\mu_2 p_{21} P_{n_1-1,1,n_3,n_4,n_5,n_6} + \lambda_4 P_{n_1,0,n_3-1,n_4,n_5,n_6} + \lambda_2 P_{n_1,0,n_3-1,n_4,n_5,n_6} + \\
\lambda_3 P_{n_1,0,n_3,n_4-1,n_5,n_6} + \mu_3 P_{n_1,0,n_3,n_4-1,n_5,n_6} + \mu_1 P_{n_1,0,n_3+1,n_4,n_5,n_6-1} + \\
\mu_2 P_{n_1,0,n_3,n_4+1,n_5,n_6-1} + \mu_3 P_{n_1,0,n_3,n_4,n_5+1,n_6-1}.
\] --- (3)

On taking \( n_3=0 \); we have
\begin{align*}
(\lambda_1 + \lambda_2 + \lambda_3 + \mu_4 + \mu_5 + \mu_6 + \mu_7 + \mu_8 + \mu_9) P_{n_1,n_2,n_3,n_4} &= \lambda_1 P_{n_1-1,n_2,n_3,n_4,n_5} + \\
\lambda_2 P_{n_1,n_2-1,n_3,n_4,n_5} + \mu_4 P_{1,n_2+1,n_3,n_4,n_5} + \mu_5 P_{1,n_2,n_3+1,n_4,n_5} + \mu_6 P_{1,n_2,n_3,n_4+1,n_5} + \\
\lambda_3 P_{n_1,n_2,n_3,0,n_5,n_6} + \mu_3 P_{n_1,n_2,0,n_3,n_4,n_5} + \mu_4 P_{n_1,n_2,1,n_3,n_4,n_5} + \\
& + \lambda_4 P_{n_1,n_2,0,1,n_3,n_4,n_5} + \lambda_5 P_{n_1,n_2,0,n_3,1,n_4,n_5} + \lambda_6 P_{n_1,n_2,0,n_3,n_4,1,n_5}.
\end{align*}

--- (4)

On taking \( n_4 = 0 \); we have

\begin{align*}
(\lambda_1 + \lambda_2 + \lambda_3 + \mu_4 + \mu_5 + \mu_6 + \mu_7 + \mu_8 + \mu_9) P_{n_1,n_2,n_3,n_4} &= \lambda_1 P_{n_1-1,n_2,n_3,n_4,n_5} + \\
\lambda_2 P_{n_1,n_2-1,n_3,n_4,n_5} + \mu_4 P_{1,n_2+1,n_3,n_4,n_5} + \mu_5 P_{1,n_2,n_3+1,n_4,n_5} + \\
\lambda_3 P_{n_1,n_2,n_3,0,n_5,n_6} + \mu_3 P_{n_1,n_2,0,n_3,n_4,n_5} + \mu_4 P_{n_1,n_2,1,n_3,n_4,n_5} + \\
& + \lambda_4 P_{n_1,n_2,0,1,n_3,n_4,n_5} + \lambda_5 P_{n_1,n_2,0,n_3,1,n_4,n_5} + \lambda_6 P_{n_1,n_2,0,n_3,n_4,1,n_5}.
\end{align*}

--- (5)

On taking \( n_5 = 0 \); we have

\begin{align*}
(\lambda_1 + \lambda_2 + \lambda_3 + \mu_4 + \mu_5 + \mu_6 + \mu_7 + \mu_8 + \mu_9) P_{n_1,n_2,n_3,n_4} &= \lambda_1 P_{n_1-1,n_2,n_3,n_4,n_5} + \\
\lambda_2 P_{n_1,n_2-1,n_3,n_4,n_5} + \mu_4 P_{1,n_2+1,n_3,n_4,n_5} + \mu_5 P_{1,n_2,n_3+1,n_4,n_5} + \\
\lambda_3 P_{n_1,n_2,n_3,0,n_5,n_6} + \mu_3 P_{n_1,n_2,0,n_3,n_4,n_5} + \mu_4 P_{n_1,n_2,1,n_3,n_4,n_5} + \\
& + \lambda_4 P_{n_1,n_2,0,1,n_3,n_4,n_5} + \lambda_5 P_{n_1,n_2,0,n_3,1,n_4,n_5} + \lambda_6 P_{n_1,n_2,0,n_3,n_4,1,n_5}.
\end{align*}

--- (6)

On taking \( n_6 = 0 \); we have

\begin{align*}
(\lambda_1 + \lambda_2 + \lambda_3 + \mu_4 + \mu_5 + \mu_6 + \mu_7 + \mu_8 + \mu_9) P_{n_1,n_2,n_3,n_4} &= \lambda_1 P_{n_1-1,n_2,n_3,n_4,n_5} + \\
\lambda_2 P_{n_1,n_2-1,n_3,n_4,n_5} + \mu_4 P_{1,n_2+1,n_3,n_4,n_5} + \mu_5 P_{1,n_2,n_3+1,n_4,n_5} + \\
\lambda_3 P_{n_1,n_2,n_3,0,n_5,n_6} + \mu_3 P_{n_1,n_2,0,n_3,n_4,n_5} + \mu_4 P_{n_1,n_2,1,n_3,n_4,n_5} + \\
& + \lambda_4 P_{n_1,n_2,0,1,n_3,n_4,n_5} + \lambda_5 P_{n_1,n_2,0,n_3,1,n_4,n_5} + \lambda_6 P_{n_1,n_2,0,n_3,n_4,1,n_5}.
\end{align*}

--- (7)

On taking \( n_7 = 0 \); we have

\begin{align*}
(\lambda_1 + \lambda_2 + \lambda_3 + \mu_4 + \mu_5 + \mu_6 + \mu_7 + \mu_8 + \mu_9) P_{n_1,n_2,n_3,n_4} &= \lambda_1 P_{n_1-1,n_2,n_3,n_4,n_5} + \\
\lambda_2 P_{n_1,n_2-1,n_3,n_4,n_5} + \mu_4 P_{1,n_2+1,n_3,n_4,n_5} + \mu_5 P_{1,n_2,n_3+1,n_4,n_5} + \\
\lambda_3 P_{n_1,n_2,n_3,0,n_5,n_6} + \mu_3 P_{n_1,n_2,0,n_3,n_4,n_5} + \mu_4 P_{n_1,n_2,1,n_3,n_4,n_5} + \\
& + \lambda_4 P_{n_1,n_2,0,1,n_3,n_4,n_5} + \lambda_5 P_{n_1,n_2,0,n_3,1,n_4,n_5} + \lambda_6 P_{n_1,n_2,0,n_3,n_4,1,n_5}.
\end{align*}

--- (8)

On taking \( n_8 = 0 \); we have

\begin{align*}
(\lambda_1 + \lambda_2 + \lambda_3 + \mu_4 + \mu_5 + \mu_6 + \mu_7 + \mu_8 + \mu_9) P_{n_1,n_2,n_3,n_4} &= \lambda_1 P_{n_1-1,n_2,n_3,n_4,n_5} + \\
\lambda_2 P_{n_1,n_2-1,n_3,n_4,n_5} + \mu_4 P_{1,n_2+1,n_3,n_4,n_5} + \mu_5 P_{1,n_2,n_3+1,n_4,n_5} + \\
\lambda_3 P_{n_1,n_2,n_3,0,n_5,n_6} + \mu_3 P_{n_1,n_2,0,n_3,n_4,n_5} + \mu_4 P_{n_1,n_2,1,n_3,n_4,n_5} + \\
& + \lambda_4 P_{n_1,n_2,0,1,n_3,n_4,n_5} + \lambda_5 P_{n_1,n_2,0,n_3,1,n_4,n_5} + \lambda_6 P_{n_1,n_2,0,n_3,n_4,1,n_5}.
\end{align*}

--- (9)

On taking \( n_9 = 0 \); we have

\begin{align*}
(\lambda_1 + \lambda_2 + \lambda_3 + \mu_4 + \mu_5 + \mu_6 + \mu_7 + \mu_8 + \mu_9) P_{n_1,n_2,n_3,n_4} &= \lambda_1 P_{n_1-1,n_2,n_3,n_4,n_5} + \\
\lambda_2 P_{n_1,n_2-1,n_3,n_4,n_5} + \mu_4 P_{1,n_2+1,n_3,n_4,n_5} + \mu_5 P_{1,n_2,n_3+1,n_4,n_5} + \\
\lambda_3 P_{n_1,n_2,n_3,0,n_5,n_6} + \mu_3 P_{n_1,n_2,0,n_3,n_4,n_5} + \mu_4 P_{n_1,n_2,1,n_3,n_4,n_5} + \\
& + \lambda_4 P_{n_1,n_2,0,1,n_3,n_4,n_5} + \lambda_5 P_{n_1,n_2,0,n_3,1,n_4,n_5} + \lambda_6 P_{n_1,n_2,0,n_3,n_4,1,n_5}.
\end{align*}

--- (10)
\[ \mu_2 P_{21} P_{n_1-1,n_2+1,n_3,n_4,n_5,0} + \lambda_1 P_{n_1,n_2,n_3-1,n_4,n_5,0} + \lambda_2 P_{n_1,n_2,n_3-1,n_4,n_5,0} \\
+ \lambda_3 P_{n_1,n_2,n_3-1,n_4,n_5,0} + \mu_3 P_{n_1,n_2,n_3,n_4,n_5,1} \]  

--- (7)

On taking \( n_1=0, n_2=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \mu_1 + \mu_2 + \mu_1 + \mu_2 + \mu_3 + \mu_3 \right) P_{0,0,n_4,n_5,n_6} = \\
\mu_2 P_{1,0,n_3,n_4,n_5,n_6} + \mu_3 P_{0,1,n_3,n_4,n_5,n_6} + \lambda_1 P_{0,0,n_3-1,n_4,n_5,n_6} + \lambda_2 P_{0,1,n_3,n_4-1,n_5,n_6} \\
+ \lambda_3 P_{0,0,n_3,n_4-1,n_5,n_6} + \mu_3 P_{0,0,n_3+1,n_4,n_5,n_6} + \mu_2 P_{0,0,n_3+1,n_4,n_5,n_6} \\
+ \mu_2 P_{0,0,n_3+1,n_4,n_5,n_6} + \mu_3 P_{0,0,n_3+1,n_4,n_5,n_6} 
\]  

--- (8)

On taking \( n_1=0, n_3=0 \); we have

\[ \left( \lambda_1 + \lambda_2 + \lambda_3 + \mu_1 + \mu_2 + \mu_1 + \mu_2 + \mu_3 + \mu_3 \right) P_{0,n_2,0,n_4,n_5,n_6} = \\
\lambda_2 P_{0,n_2-1,0,n_4,n_5,n_6} + \mu_3 P_{1,n_2,0,n_4,n_5,n_6} - 1 + \mu_3 P_{0,n_2-1,0,n_4,n_5,n_6} + \mu_2 P_{2,n_2-1,0,n_4,n_5,n_6} + \\
\lambda_2 P_{0,n_2,0,n_4-1,n_5,n_6} + \mu_3 P_{0,n_2,0,n_4,n_5+1,n_6} + \mu_2 P_{0,n_2,0,n_4,n_5+1,n_6} \\
+ \mu_2 P_{0,n_2,0,n_4+1,n_5,n_6} + \mu_3 P_{0,n_2,0,n_4+1,n_5,n_6} 
\]  

--- (9)

On taking \( n_1=0, n_4=0 \); we have

\[ \left( \lambda_1 + \lambda_2 + \lambda_3 + \mu_1 + \mu_2 + \mu_1 + \mu_2 + \mu_3 + \mu_3 \right) P_{0,n_2,0,n_3,0,n_6} = \\
\lambda_2 P_{0,n_2-1,0,n_3,n_6} + \mu_1 P_{1,1,0,n_3,n_6} - 1 + \mu_1 P_{0,n_2-1,0,n_3,n_6} + \mu_2 P_{2,1,0,n_3,n_6} + \\
\mu_3 P_{0,n_2+1,0,n_3,n_6} + \lambda_1 P_{0,n_2-1,1,0,n_3,n_6} + \lambda_2 P_{0,n_2-1,1,0,n_3,n_6} \\
+ \mu_4 P_{0,n_2+1,0,n_3+1,n_6} + \mu_3 P_{0,n_2,0,n_3+1,n_6} - 1 
\]  

--- (10)

On taking \( n_1=0, n_5=0 \); we have

\[ \left( \lambda_1 + \lambda_2 + \lambda_3 + \mu_1 + \mu_2 + \mu_1 + \mu_2 + \mu_3 + \mu_3 \right) P_{0,n_2,0,n_3,0,n_6} = \\
\lambda_2 P_{0,n_2-1,0,n_3,n_6} + \mu_1 P_{1,1,0,n_3,n_6} - 1 + \mu_1 P_{0,n_2-1,0,n_3,n_6} + \mu_2 P_{2,1,0,n_3,n_6} + \\
\mu_3 P_{0,n_2+1,0,n_3,n_6} + \lambda_1 P_{0,n_2-1,1,0,n_3,n_6} + \lambda_2 P_{0,n_2-1,1,0,n_3,n_6} \\
+ \mu_4 P_{0,n_2+1,0,n_3+1,n_6} + \mu_3 P_{0,n_2,0,n_3+1,n_6} - 1 
\]
\[
\lambda_1 P_{n_1,n_2,n_3-1,n_4,0,n_6} + \lambda_2 P_{n_2,n_3,n_4-1,0,n_6} + \mu_3 P_{0,n_2,n_3,n_4,0,n_6+1} + \mu_4 P_{0,n_2,n_3+1,n_4,0,n_6-1} + \\
\mu_5 P_{0,n_2,n_3,n_4+1,0,n_6-1}.
\]  
--- (11)

On taking \(n_1=0\), \(n_6=0\); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 \right) P_{n_1,0,0,n_4,n_5,n_6} = \\
\lambda_1 P_{n_1-1,0,0,n_4,n_5,n_6} + \mu_4 P_{n_1+1,0,0,n_4,n_5,n_6-1} + \\
\mu_2 P_{n_1,1,0,0,n_4,n_5,n_6-1} + \\
\mu_5 P_{n_1,0,0,n_4+1,n_5,n_6-1}.
\]  
--- (12)

On taking \(n_2=0\), \(n_3=0\); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 \right) P_{n_1,0,0,n_4,n_5,n_6} = \\
\lambda_1 P_{n_1-1,0,n_2,n_4,n_5,n_6} + \\
\mu_4 P_{n_1+1,0,n_2,n_4,n_5,n_6-1} + \mu_2 P_{n_1,1,n_2,n_4,n_5,n_6-1} + \\
\mu_5 P_{n_1,0,n_2+1,n_4,n_5,n_6-1}.
\]  
--- (13)

On taking \(n_2=0\), \(n_4=0\); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 \right) P_{n_1,0,n_2,n_3,n_5,n_6} = \\
\lambda_1 P_{n_1-1,0,n_2,n_3,n_5,n_6} + \\
\mu_4 P_{n_1+1,0,n_2,n_3,n_5,n_6-1} + \mu_2 P_{n_1,1,n_2,n_3,n_5,n_6-1} + \\
\mu_5 P_{n_1,0,n_2,n_3+1,n_5,n_6-1}.
\]  
--- (14)

On taking \(n_2=0\), \(n_5=0\); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 \right) P_{n_1,0,n_2,n_3,n_4,0,n_6} = \\
\lambda_1 P_{n_1-1,0,n_2,n_3,n_4,0,n_6} + \\
\mu_2 P_{n_1,1,n_2,n_3,n_4,0,n_6-1} + \\
\mu_4 P_{n_1+1,0,n_2,n_3,n_4,0,n_6-1} + \mu_2 P_{n_1,1,n_2,n_3,n_4-1,0,n_6}.
\]
\[
+ \mu_3 P_{n_1,0,n_3,0,n_4,0,n_6,0} + \mu_1 P_{n_1,0,n_3,0,n_4,0,n_6,-1} \\
+ \mu_2 P_{n_1,0,n_3,0,n_4+1,0,n_6,-1} + \mu_3 P_{n_1,0,n_3,n_4+1,n_6,-1} 
\]
--- (15)

On taking \( n_2=0, n_6=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \mu_4 + \mu_2 + \mu_4 + \mu_2 + \mu_5 + \mu_3 \right) P_{n_1,0,0,0,0,0} = \\
\lambda_1 P_{n_1,-1,0,n_3,0,n_5,0} + \mu_2 P_{n_1,-1,1,n_3,0,n_5,0} + \mu_3 P_{n_1,0,n_3-1,n_5,0} + \lambda_3 P_{n_1,0,n_3,n_5-1,n_5,0} \\
+ \lambda_3 P_{n_1,0,n_3,n_5-1,0} + \mu_3 P_{n_1,0,n_3,n_5,1} 
\]
--- (16)

On taking \( n_3=0, n_4=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \mu_4 + \mu_2 + \mu_4 + \mu_2 + \mu_5 + \mu_3 \right) P_{n_1,n_2,0,0,n_6} = \\
\lambda_1 P_{n_1,-1,n_2,0,0,n_6} + \lambda_2 P_{n_1,n_2-1,0,0,n_6} + \mu_4 P_{n_1,n_2,0,0,n_6} + \mu_5 P_{n_1,n_2+1,0,0,n_6} + \mu_6 P_{n_1,n_2+1,0,0,n_6} \\
+ \mu_2 P_{n_1,n_2-1,0,0,n_5,n_6} + \lambda_3 P_{n_1,n_2,0,0,n_5-1,n_6} + \mu_3 P_{n_1,n_2+1,0,0,n_5,n_6} + \mu_4 P_{n_1,n_2,1,0,n_5,n_6} \\
+ \mu_5 P_{n_1,n_2,0,0,n_5-1,n_6-1} + \mu_5 P_{n_1,n_2,0,0,n_5,n_6-1} 
\]
--- (17)

On taking \( n_3=0, n_5=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \mu_4 + \mu_2 + \mu_4 + \mu_2 + \mu_5 + \mu_3 \right) P_{n_1,n_2,0,n_3,0,n_6} = \\
\lambda_1 P_{n_1,-1,n_2,0,n_3,0,n_6} + \lambda_2 P_{n_1,n_2-1,0,n_3,0,n_6} + \mu_4 P_{n_1,n_2,0,n_3,0,n_6} + \mu_5 P_{n_1,n_2+1,0,n_3,0,n_6} \\
+ \mu_2 P_{n_1,n_2-1,0,n_5,n_3,0,n_6} + \lambda_3 P_{n_1,n_2,0,n_5-1,n_3,0,n_6} + \mu_3 P_{n_1,n_2+1,0,n_5,n_3,0,n_6} + \mu_4 P_{n_1,n_2,1,n_5,n_3,0,n_6} \\
+ \mu_5 P_{n_1,n_2,0,0,n_5-1,n_3,0,n_6} + \mu_5 P_{n_1,n_2,0,0,n_5,n_3,0,n_6-1} 
\]
--- (18)

On taking \( n_3=0, n_6=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \mu_4 + \mu_2 + \mu_4 + \mu_2 + \mu_5 + \mu_3 \right) P_{n_1,n_2,0,n_3,0,n_6} = \\
\lambda_1 P_{n_1,-1,n_2,0,n_3,0,n_6} + \lambda_2 P_{n_1,n_2-1,0,n_3,0,n_6} + \mu_4 P_{n_1,n_2,0,n_3,0,n_6} + \mu_5 P_{n_1,n_2+1,0,n_3,0,n_6} \\
+ \mu_2 P_{n_1,n_2-1,0,n_5,n_3,0,n_6} + \lambda_3 P_{n_1,n_2,0,n_5-1,n_3,0,n_6} + \mu_3 P_{n_1,n_2+1,0,n_5,n_3,0,n_6} + \mu_4 P_{n_1,n_2,1,n_5,n_3,0,n_6} \\
+ \mu_5 P_{n_1,n_2,0,0,n_5-1,n_3,0,n_6} + \mu_5 P_{n_1,n_2,0,0,n_5,n_3,0,n_6-1} 
\]
--- (19)

On taking \( n_4=0, n_5=0 \); we have
\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6 \right) P_{n_1, n_2, n_3, 0, 0, n_6} = \lambda_1^P n_{1-n_2, n_3, 0, 0} + \\
\lambda_2 P_{n_1, n_2-1, n_3, 0, 0} + \mu_1 P_{n_1+1, n_2, 0, 0, n_6} + \mu_2 P_{n_1+1, n_2-1, n_3, 0, 0} + \\
\mu_2 P_{n_1-1, n_2, 0, 0, n_6} + \lambda_1^P n_{1-n_2, n_3, 0, 0} + \mu_3 P_{n_1, n_2, n_3, 0, 0, n_6-1} + \\
\mu_2 P_{n_1, n_2, n_3, 1, 0, n_6-1} + \lambda_1^P n_{1-n_2, n_3, 0, 0, n_6} + \mu_1 P_{n_1, n_2, n_3, 1, 0, n_6-1}.
\]

--- (20)

On taking \( n_4=0, n_6=0; \) we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \mu_1 + \mu_2 + \mu_3 \right) P_{n_1, n_2, n_3, 0, 0, n_6} = \lambda_1^P n_{1-n_2, n_3, 0, 0} + \\
\lambda_2 P_{n_1, n_2-1, n_3, 0, 0} + \mu_1 P_{n_1+1, n_2, 0, 0, n_6} + \\
\mu_2 P_{n_1+1, n_2-1, n_3, 0, 0} + \lambda_1^P n_{1-n_2, n_3, 0, 0} + \\
\mu_3 P_{n_1, n_2, n_3-1, 0, n_6-1} + \lambda_3 P_{n_1, n_2, n_3, 0, 0, n_6-1} + \mu_5 P_{n_1, n_2, n_3, 0, 0, n_6-1}.
\]

--- (21)

On taking \( n_5=0, n_6=0; \) we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \mu_1 + \mu_2 + \mu_3 \right) P_{n_1, n_2, n_3, 0, 0, n_6} = \lambda_1^P n_{1-n_2, n_3, 0, 0} + \\
\lambda_2 P_{n_1, n_2-1, n_3, 0, 0} + \mu_1 P_{n_1+1, n_2, 0, 0, n_6} + \\
\mu_2 P_{n_1+1, n_2-1, n_3, 0, 0} + \lambda_1^P n_{1-n_2, n_3, 0, 0} + \mu_3 P_{n_1, n_2, n_3, 0, 0, n_6-1} + \\
\mu_5 P_{n_1, n_2, n_3, 0, 0, n_6-1}.
\]

--- (22)

On taking \( n_1=0, n_2=0 \) and \( n_3=0; \) we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \mu_1 + \mu_2 + \mu_3 \right) P_{0,0,0, n_6} = \\
\mu_1 P_{0,1, 0, n_4, n_5, n_6-1} + \mu_2 P_{0,1, 0, n_4, n_5, n_6-1} + \\
\lambda_1^P P_{0,0,0, n_4, n_5, n_6-1} + \mu_3 P_{0,0,0, n_4, n_5, n_6-1} + \\
\mu_5 P_{0,0,0, n_4, n_5, n_6-1}.
\]

--- (23)

On taking \( n_1=0, n_2=0 \) and \( n_4=0; \) we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \mu_1 + \mu_2 + \mu_3 \right) P_{n_0,0, n_5, n_6} = \\
\mu_1 P_{0,0, n_4, n_5, n_6-1} + \mu_2 P_{0,0, n_4, n_5, n_6-1} + \\
\lambda_1^P P_{0,0, n_4, n_5, n_6-1} + \mu_3 P_{0,0, n_4, n_5, n_6-1}.
\]

--- (24)
On taking $n_1=0$, $n_2=0$ and $n_5=0$; we have

\[
\left( \hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_3 + \mu_4 + \mu_2 + \mu_1 + \mu_3 + \mu_3 \right) P_{0,0,n_1,0,n_6} = \\
\mu_1 P_{13} P_{0,0,n_1,n_4,0,n_6-1} + \mu_2 P_{23} P_{1,1,n_3,0,n_6-1} + \hat{\lambda}_2' P_{0,0,n_5-1,n_4,0,n_6} + \hat{\lambda}_2 P_{0,0,n_5,n_4-1,0,n_6} + \mu_3 P_{0,0,n_5,n_4,0,n_6+1} + \mu_4 P_{0,0,n_5+1,n_4,0,n_6-1} + \mu_5 P_{0,0,n_5,0,n_4+1,0,n_6-1}. \\
\]  

--- (25)

On taking $n_1=0$, $n_2=0$ and $n_6=0$; we have

\[
\left( \hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_3 + \mu_4 + \mu_2 + \mu_1 + \mu_3 + \mu_3 \right) P_{0,0,n_1,n_2,0} = \\
\hat{\lambda}_1' P_{0,0,n_3-1,n_4,0,n_5,0} + \hat{\lambda}_2' P_{0,0,n_3-1,n_4,0,n_6,0} + \hat{\lambda}_3' P_{0,0,n_5,n_6-1,0} + \mu_3 P_{0,0,n_5,n_6,0}. \\
\]  

--- (26)

On taking $n_2=0$, $n_3=0$ and $n_4=0$; we have

\[
\left( \hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_3 + \mu_4 + \mu_2 + \mu_1 + \mu_3 + \mu_3 \right) P_{n_1,0,0,n_2,n_6} = \hat{\lambda}_1 P_{n_1-1,0,0,n_5,n_6} + \\
\mu_1 P_{13} P_{n_1+1,0,0,n_5,n_6-1} + \mu_2 P_{23} P_{n_1,1,0,0,n_5,n_6-1} + \mu_2 P_{21} P_{n_1-1,1,0,0,n_5,n_6} + \\
\hat{\lambda}_3 P_{n_1,0,0,n_5,n_6-1} + \mu_4 P_{n_1,0,0,n_5,n_6+1} + \mu_3 P_{n_1,0,0,n_5,n_6-1}. \\
\]  

--- (27)

On taking $n_2=0$, $n_3=0$ and $n_5=0$; we have

\[
\left( \hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_3 + \mu_4 + \mu_2 + \mu_1 + \mu_3 + \mu_3 \right) P_{n_1,0,0,n_4,0,n_6} = \hat{\lambda}_1 P_{n_1-1,0,0,n_4,0,n_6} + \\
\mu_1 P_{13} P_{n_1+1,0,0,n_4,0,n_6-1} + \mu_2 P_{23} P_{n_1,0,n_4,0,n_6-1} + \mu_2 P_{21} P_{n_1-1,1,0,n_4,0,n_6} + \\
\hat{\lambda}_2 P_{n_1,0,0,n_4-1,0,n_6} + \mu_3 P_{n_1,0,0,n_4,0,n_6+1} + \mu_4 P_{n_1,0,1,n_4,0,n_6-1} + \mu_3 P_{n_1,0,0,n_4+1,0,n_6-1}. \\
\]  

--- (28)

On taking $n_2=0$, $n_3=0$ and $n_6=0$; we have
\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \mu_1 + \mu_2 + \mu_3 \right) P_{n_1,0,0,0,0,0} = \lambda_1 P_{n_1-1,0,0,0,0,0} + \\
\mu_2 P_{n_1-1,0,0,0,0,0} + \lambda_2 P_{n_1,0,0,0,0,0} + \lambda_3 P_{n_1,0,0,0,0,0} + \mu_3 P_{n_1,0,0,0,0,0}.
\]

--- (29)

On taking \( n_2=0, n_4=0 \) and \( n_5=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \mu_1 + \mu_2 + \mu_3 \right) P_{n_1,0,0,0,0,0} = \lambda_1 P_{n_1-1,0,0,0,0,0} + \\
\mu_2 P_{n_1-1,0,0,0,0,0} + \lambda_2 P_{n_1,0,0,0,0,0} + \lambda_3 P_{n_1,0,0,0,0,0} + \mu_3 P_{n_1,0,0,0,0,0}.
\]

--- (30)

On taking \( n_2=0, n_5=0 \) and \( n_6=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \mu_1 + \mu_2 + \mu_3 \right) P_{n_1,0,0,0,0,0} = \lambda_1 P_{n_1-1,0,0,0,0,0} + \\
\mu_2 P_{n_1-1,0,0,0,0,0} + \lambda_2 P_{n_1,0,0,0,0,0} + \lambda_3 P_{n_1,0,0,0,0,0} + \mu_3 P_{n_1,0,0,0,0,0}.
\]

--- (31)

On taking \( n_2=0, n_5=0 \) and \( n_6=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \mu_1 + \mu_2 + \mu_3 \right) P_{n_1,0,0,0,0,0} = \\
\lambda_1 P_{n_1-1,0,n_3,0,0,0} + \\
\mu_2 P_{n_1-1,0,n_3,0,0,0} + \lambda_2 P_{n_1,0,n_3,0,0,0} + \lambda_3 P_{n_1,0,n_3,0,0,0} + \\
\lambda_4 P_{n_1,n_3,0,0,0,0}.
\]

--- (32)

On taking \( n_3=0, n_4=0 \) and \( n_5=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \mu_1 + \mu_2 + \mu_3 \right) P_{n_1,0,0,0,0,0} = \lambda_1 P_{n_1-1,n_2,0,0,0,0} + \\
\lambda_2 P_{n_1,n_2-1,0,0,0,0} + \lambda_3 P_{n_1,n_2-1,0,0,0,0} + \\
\lambda_4 P_{n_1,n_2,0,0,0,0} + \\
\lambda_5 P_{n_1,n_2,0,0,0,0} + \\
\lambda_6 P_{n_1,n_2,0,0,0,0}.
\]

--- (33)

On taking \( n_3=0, n_4=0 \) and \( n_6=0 \); we have
\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6 \right) P_{n_1 n_2,0,0,0,0,0} = \lambda_1 P_{n_1-1,n_2,0,0,0,0} + \\
\lambda_2 P_{n_1,n_2-1,0,0,0,0} + \mu_4 P_{n_1+1,n_2-1,0,0,0,0} + \mu_2 P_{n_1-1,n_2+1,0,0,0,0} + \\
\lambda_3 P_{n_1,n_2,0,0,0,1} + \mu_3 P_{n_1,n_2,0,0,1,0,1}
\] --- (34)

On taking \( n_3=0, n_5=0 \) and \( n_6=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 \right) P_{n_1 n_2,n_3,0,0,0} = \lambda_1 P_{n_1-1,n_2,n_3,0,0,0} + \\
\lambda_2 P_{n_1,n_2-1,n_3,0,0,0} + \mu_4 P_{n_1+1,n_2-1,n_3,0,0,0} + \mu_2 P_{n_1-1,n_2+1,n_3,0,0,0} + \\
\lambda_3 P_{n_1,n_2,n_3-1,0,0,0} + \mu_3 P_{n_1,n_2,n_3,0,0,1} + \\
\lambda_4 P_{n_1,n_2,n_3,0,0,1}
\] --- (35)

On taking \( n_4=0, n_5=0 \) and \( n_6=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 \right) P_{n_1,n_2,n_3,n_4,0,0} = \lambda_1 P_{n_1-1,n_2,n_3,n_4,0,0} + \\
\lambda_2 P_{n_1,n_2-1,n_3,n_4,0,0} + \mu_4 P_{n_1+1,n_2-1,n_3,n_4,0,0} + \mu_2 P_{n_1-1,n_2+1,n_3,n_4,0,0} + \\
\lambda_3 P_{n_1,n_2,n_3-1,n_4,0,0} + \mu_3 P_{n_1,n_2,n_3,n_4-1,0,0} + \\
\lambda_4 P_{n_1,n_2,n_3,n_4,0,0,1} + \mu_4 P_{n_1,n_2,n_3,n_4,0,1,0,1}
\] --- (36)

On taking \( n_1=0, n_3=0 \) and \( n_4=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 \right) P_{n_1,0,0,0,n_5} = \\
\lambda_2 P_{n_1,0,0,0,n_5-1} + \mu_4 P_{n_1,0,0,0,n_5} + \mu_2 P_{n_1,0,0,1,0,0,0,0} + \\
\lambda_3 P_{n_1,0,0,1,0,0,0,1} + \mu_3 P_{n_1,0,0,0,0,0,1,0} + \\
\lambda_4 P_{n_1,0,0,0,0,0,1,0,0} + \mu_4 P_{n_1,0,0,0,0,0,1,1,0,1}
\] --- (37)

On taking \( n_1=0, n_3=0 \) and \( n_5=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 \right) P_{n_1,0,0,n_5,n_6} = \\
\lambda_2 P_{n_1,0,0,n_5,n_6-1} + \mu_4 P_{n_1,0,0,n_5,n_6} + \mu_2 P_{n_1,0,1,0,0,0,0,0} + \\
\lambda_3 P_{n_1,0,1,0,0,0,0,1} + \mu_3 P_{n_1,0,1,0,0,0,1,0} + \\
\lambda_4 P_{n_1,0,0,0,0,0,1,0,1} + \mu_4 P_{n_1,0,0,0,0,1,0,1,1,0,1}
\] --- (38)

On taking \( n_1=0, n_3=0 \) and \( n_6=0 \); we have
\[
\left( \lambda_4 + \lambda_2 + \lambda'_2 + \lambda'_3 + \mu_1 + \mu_2 + \mu'_1 + \mu'_2 + \mu_3 + \mu'_3 \right) P_{0,n_2,0,n_3,0} = \lambda_2 P_{0,n_2-1,0,n_4,n_5,0} + \\
\mu_4 P_{1,n_2-1,0,n_4,n_5,0} + \lambda'_2 P_{0,n_2,0,n_3-1,n_5,n_6} + \lambda'_3 P_{0,n_2,0,n_4,n_5-1,0} + \mu_3 P_{0,n_2,0,n_4,n_5,1}
\]

On taking \( n_1 = 0, n_4 = 0 \) and \( n_5 = 0 \); we have

\[
\left( \lambda_4 + \lambda_2 + \lambda'_2 + \lambda'_3 + \mu_1 + \mu_2 + \mu'_1 + \mu'_2 + \mu_3 + \mu'_3 \right) P_{0,n_2,n_3,0,0} = \\
\lambda_2 P_{0,n_2-1,n_3,0,n_6,0} + \\
\mu_4 P_{1,n_2-1,n_3,0,0,0,0} + \mu_2 P_{23} P_{0,n_2+1,n_3,0,0,0,0,0} + \\
\lambda'_1 P_{0,n_2,n_3-1,0,n_5,0} + \lambda'_3 P_{0,n_2,n_3-1,0,n_5,0} + \\
\mu_3 P_{0,n_2,n_3,0,1,0,0,0,0} \quad --- \ (41)
\]

On taking \( n_1 = 0, n_5 = 0 \) and \( n_6 = 0 \); we have

\[
\left( \lambda_4 + \lambda_2 + \lambda'_2 + \lambda'_3 + \mu_1 + \mu_2 + \mu'_1 + \mu'_2 + \mu_3 + \mu'_3 \right) P_{0,n_2,n_3,0,0} = \\
\lambda_2 P_{0,n_2-1,n_3,0,n_5,0} + \\
\mu_4 P_{1,n_2-1,n_3,0,0,0,0} + \mu_2 P_{23} P_{0,n_2+1,n_3,0,0,0,0,0} + \\
\lambda'_1 P_{0,n_2,n_3-1,0,n_5,0} + \lambda'_3 P_{0,n_2,n_3-1,0,n_5,0} + \\
\mu_3 P_{0,n_2,n_3,0,1,0,0,0,0} \quad --- \ (42)
\]

On taking \( n_1 = 0, n_2 = 0, n_3 = 0 \) and \( n_4 = 0 \); we have

\[
\left( \lambda_4 + \lambda_2 + \lambda'_2 + \lambda'_3 + \mu_1 + \mu_2 + \mu'_1 + \mu'_2 + \mu_3 + \mu'_3 \right) P_{0,0,0,0,n_5,n_6} = \\
\mu_4 P_{13} P_{0,0,0,0,n_5,n_6-1} + \\
\mu_2 P_{23} P_{0,1,0,0,n_5,n_6-1} + \\
\lambda'_3 P_{0,0,0,0,n_5-1,n_6} + \mu_1 P_{0,1,0,n_5,n_6-1} + \\
\mu'_2 P_{0,0,0,1,n_5,n_6-1} + \mu'_3 P_{0,0,0,0,n_5+1,n_6-1} \quad --- \ (43)
\]

On taking \( n_1 = 0, n_2 = 0, n_3 = 0 \) and \( n_5 = 0 \); we have
\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \mu_2 + \lambda_5 + \mu_3 \right) P_{0,0,0,0,0,0} = \mu_1 P_{1,0,0,0,0,0} + \\
\mu_2 P_{0,1,0,0,0,0} + \lambda_2 P_{0,0,0,0,1,0} + \mu_3 P_{0,0,0,0,0,1} + \mu_1 P_{0,1,0,0,0,0} + \\
\mu_2 P_{0,0,0,0,1,0} + \lambda_5 P_{0,0,0,1,0,0} - 1.
\]

--- (44)

On taking \( n_1=0, n_2=0, n_3=0 \) and \( n_6=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \mu_2 + \lambda_5 + \mu_3 \right) P_{0,0,0,0,0,0} = \\
\lambda_2 P_{0,0,0,0,1,0,0} + \lambda_3 P_{0,0,0,0,1,0,0} + \mu_3 P_{0,0,0,0,1,0,0} - 1
\]

--- (45)

On taking \( n_1=0, n_2=0, n_4=0 \) and \( n_5=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \mu_2 + \lambda_5 + \mu_3 \right) P_{0,0,0,0,0,0} = \\
\mu_1 P_{1,0,0,0,0,0} + \mu_2 P_{0,1,0,0,0,0} + \lambda_1 P_{0,0,0,1,0,0} + \\
\mu_3 P_{0,0,0,0,1,0,0} + \mu_1 P_{0,1,0,0,0,0} - 1
\]

--- (46)

On taking \( n_1=0, n_2=0, n_4=0 \) and \( n_6=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \mu_2 + \lambda_5 + \mu_3 \right) P_{0,0,0,0,0,0} = \\
\lambda_4 P_{0,0,0,1,0,0} + \lambda_5 P_{0,0,0,1,0,0} + \mu_3 P_{0,0,0,1,0,0} - 1
\]

--- (47)

On taking \( n_1=0, n_3=0, n_5=0 \) and \( n_6=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \mu_2 + \lambda_5 + \mu_3 \right) P_{0,0,0,0,0,0} = \\
\lambda_4 P_{0,0,0,1,0,0} + \lambda_5 P_{0,0,0,1,0,0} + \mu_3 P_{0,0,0,1,0,0} - 1
\]

--- (48)

On taking \( n_1=0, n_3=0, n_4=0 \) and \( n_5=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \mu_2 + \lambda_5 + \mu_3 \right) P_{0,0,0,0,0,0} = \\
\lambda_4 P_{0,0,0,1,0,0} + \lambda_5 P_{0,0,0,1,0,0} + \mu_3 P_{0,0,0,1,0,0} - 1
\]

--- (49)
On taking \( n_1=0, n_3=0, n_4=0 \) and \( n_6=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \mu_4 + \mu_2 + \mu_1 + \mu_2 + \mu_3 \right) P_{0,n_2,0,0,n_5,0} = \lambda_2 P_{0,n_2-1,0,0,n_5,0} + \\
\mu_1 p_{12} P_{1,n_2-1,0,0,n_5,0} + \lambda_3 P_{0,n_2,0,0,n_5-1,0} + \mu_2 P_{0,n_2,0,0,n_5,1} 
\] --- (50)

On taking \( n_1=0, n_3=0, n_5=0 \) and \( n_6=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \mu_4 + \mu_2 + \mu_1 + \mu_2 + \mu_3 \right) P_{0,n_2,0,0,n_6,0} = \lambda_2 P_{0,n_2-1,0,0,n_6,0} + \\
\mu_2 p_{12} P_{1,n_2-1,0,0,n_6,0} + \lambda_2 P_{0,n_2,0,0,n_6-1,0} + \mu_3 P_{0,n_2,0,0,n_6,1} 
\] --- (51)

On taking \( n_1=0, n_4=0, n_5=0 \) and \( n_6=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \mu_4 + \mu_2 + \mu_1 + \mu_2 + \mu_3 \right) P_{0,n_3,0,0,n_6,0} = \lambda_1 P_{0,n_3-1,0,0,n_6,0} + \\
\mu_1 p_{12} P_{1,n_3-1,0,0,n_6,0} + \lambda_1 P_{0,n_3,0,0,n_6-1,0} + \mu_2 P_{0,n_3,0,0,n_6,1} 
\] --- (52)

On taking \( n_2=0, n_3=0, n_4=0 \) and \( n_5=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \mu_4 + \mu_2 + \mu_1 + \mu_2 + \mu_3 \right) P_{n_1,0,0,0,0,n_6} = \lambda_1 P_{n_1-1,0,0,0,0,n_6} + \\
\mu_1 p_{12} P_{n_1-1,0,0,0,0,n_6} + \lambda_2 P_{n_1,0,0,1,0,n_6-1} + \mu_2 P_{n_1,0,0,1,0,n_6-1} + \mu_3 P_{n_1,0,0,1,0,n_6-1} 
\] --- (53)

On taking \( n_2=0, n_3=0, n_4=0 \) and \( n_6=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \mu_4 + \mu_2 + \mu_1 + \mu_2 + \mu_3 \right) P_{n_1,0,0,0,0,n_5} = \lambda_4 P_{n_1-1,0,0,0,0,n_5,0} + \\
\mu_2 p_{21} P_{n_1-1,0,0,0,n_5,0} + \lambda_3 P_{n_1,0,0,0,n_5-1,0} + \mu_3 P_{n_1,0,0,0,n_5,1} 
\] --- (54)

On taking \( n_2=0, n_3=0, n_5=0 \) and \( n_6=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \mu_4 + \mu_2 + \mu_1 + \mu_2 + \mu_3 \right) P_{n_2,0,0,0,0,n_4} = \lambda_4 P_{n_2-1,0,0,0,0,n_4,0} + \\
\mu_2 p_{21} P_{n_2-1,0,0,0,n_4,0} + \lambda_2 P_{n_2,0,0,0,n_4-1,0} + \mu_3 P_{n_2,0,0,0,n_4,1} 
\] --- (55)
\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 \right) P_{n_1,n_2,0,0,0,0} = \lambda_4 P_{n_1-1,n_2,0,0,0,0} + \\
\lambda_2 P_{n_1,n_2-1,0,0,0,0} + \mu_4 P_{12} P_{n_1+1,n_2-1,0,0,0,0} + \mu_2 P_{21} P_{n_1-1,n_2+1,0,0,0,0} + \mu_3 P_{n_1,n_2,0,0,0,1}
\]

--- (56)

On taking \( n_1=0, n_2=0, n_3=0 \) and \( n_6=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 \right) P_{0,0,0,0,n_6} = \lambda_1 P_{1,0,0,0,0,0} + \\
\mu_2 P_{23} P_{0,1,0,0,0,n_6-1} + \mu_1 P_{0,0,1,0,0,n_6-1} + \mu_2 P_{0,0,0,1,0,n_6-1} + \mu_3 P_{0,0,0,1,n_6-1}.
\]

--- (57)

On taking \( n_1=0, n_2=0, n_3=0, n_4=0 \) and \( n_5=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 \right) P_{0,0,0,0,n_5} = \\
+ \lambda_3 P_{0,0,0,0,n_5-1,0} + \mu_3 P_{0,0,0,0,n_5,1}
\]

--- (59)

On taking \( n_1=0, n_2=0, n_3=0 \) and \( n_6=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 \right) P_{0,0,0,0,n_4} = \lambda_2 P_{0,0,0,0,n_4-1,0,0} + \\
+ \mu_3 P_{0,0,0,0,n_4,0,1}
\]

--- (60)

On taking \( n_1=0, n_2=0, n_3=0, n_4=0 \) and \( n_5=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 \right) P_{0,0,0,0,n_3} = \lambda_4 P_{0,0,0,0,n_3-1,0,0,0} + \\
+ \mu_3 P_{0,0,0,0,n_3,0,0,1}
\]

--- (61)

On taking \( n_1=0, n_3=0, n_4=0, n_5=0 \) and \( n_6=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 \right) P_{0,n_2,0,0,0,0} = \lambda_2 P_{0,n_2-1,0,0,0,0} + \\
+ \mu_1 P_{12} P_{1,n_2-1,0,0,0,0} + + \mu_3 P_{0,n_2,0,0,0,1}
\]

--- (62)
On taking \( n_2=0, n_3=0, n_4=0, n_5=0 \) and \( n_6=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \mu_4 + \mu_2 + \mu_3 + \mu_5 + \mu_6 \right) P_{n_1,0,0,0,0,0} = \lambda_1 P_{n_1-1,0,0,0,0,0} + \\
\mu_2 P_{n_1-1,1,0,0,0,0} + \mu_3 P_{n_1,0,0,0,0,1} \tag{63}
\]

On taking \( n_1=0, n_2=0, n_3=0, n_4=0, n_5=0 \) and \( n_6=0 \); we have

\[
\left( \lambda_1 + \lambda_2 + \lambda_3 + \mu_4 + \mu_2 + \mu_3 + \mu_5 + \mu_6 \right) P_{0,0,0,0,0,0} = \mu_3 P_{0,0,0,0,0,1} \tag{64}
\]

Let us define the generating function as

\[
F(X,Y,Z,R,S,T) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \sum_{n_5=0}^{\infty} \sum_{n_6=0}^{\infty} P_{n_1,n_2,n_3,n_4,n_5,n_6} X^{n_1} Y^{n_2} Z^{n_3} R^{n_4} S^{n_5} T^{n_6}
\]

where \( |X|=|Y|=|Z|=|R|=|S|=|T|=1 \). \tag{65}

Also we define partial generating functions as

\[
F_{n_2,n_3,n_4,n_5,n_6}(X) = \sum_{n_1=0}^{\infty} P_{n_1,n_2,n_3,n_4,n_5,n_6} X^{n_1} \tag{66}
\]

\[
F_{n_3,n_4,n_5,n_6}(X,Y) = \sum_{n_2=0}^{\infty} P_{n_2,n_3,n_4,n_5,n_6}(X) Y^{n_2} \tag{67}
\]

\[
F_{n_4,n_5,n_6}(X,Y,Z) = \sum_{n_3=0}^{\infty} P_{n_3,n_4,n_5,n_6}(X,Y) Z^{n_3} \tag{68}
\]

\[
F_{n_5,n_6}(X,Y,Z,R) = \sum_{n_4=0}^{\infty} P_{n_4,n_5,n_6}(X,Y,Z) R^{n_4} \tag{69}
\]

\[
F_{n_6}(X,Y,Z,R,S) = \sum_{n_5=0}^{\infty} P_{n_5,n_6}(X,Y,Z,R) S^{n_5} \tag{70}
\]

\[
F(X,Y,Z,R,S,T) = \sum_{n_6=0}^{\infty} P_{n_6}(X,Y,Z,R,S) T^{n_6} \tag{71}
\]

Multiplying (1) by \( x^n \) and summing over from 0 to \( \infty \) and using (2) and (66), we get
\[
\left(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \mu_1 + \mu_2 + \mu_1' + \mu_2' + \mu_3\right)F_{n_0, n_1, n_2, n_3, n_4, n_5, n_6}(X) = \lambda_1 FN_{n_2, n_3, n_5, n_6}(X) + \lambda_2 FN_{n_1, n_3, n_5, n_6}(X) + \lambda_3 FN_{n_1, n_2, n_4, n_6}(X) + \lambda_4 FN_{n_2, n_3, n_4, n_6}(X) + \lambda_5 FN_{n_1, n_2, n_3, n_5}(X) + \lambda_6 FN_{n_1, n_2, n_3, n_4}(X) + \mu_1 P_{n_0, n_1, n_2, n_3, n_4, n_5, n_6} + \mu_1' P_{n_0, n_1, n_2, n_3, n_4, n_5, n_6} + \mu_2 P_{n_0, n_1, n_2, n_3, n_4, n_5, n_6} + \mu_2' P_{n_0, n_1, n_2, n_3, n_4, n_5, n_6} + \mu_3 P_{n_0, n_1, n_2, n_3, n_4, n_5, n_6} + \mu_3' P_{n_0, n_1, n_2, n_3, n_4, n_5, n_6}.
\]

Multiplying (3) by \(x^n\) and summing over from 0 to \(\infty\) and using (8) and (66), we get

\[
\left(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \mu_1 + \mu_2 + \mu_1' + \mu_2' + \mu_3\right)F_{n_0, n_1, n_2, n_3, n_4, n_5, n_6}(X) = \lambda_1 FN_{n_2, n_3, n_5, n_6}(X) + \mu_1 P_{n_0, n_1, n_2, n_3, n_4, n_5, n_6} + \mu_1' P_{n_0, n_1, n_2, n_3, n_4, n_5, n_6} + \mu_2 P_{n_0, n_1, n_2, n_3, n_4, n_5, n_6} + \mu_2' P_{n_0, n_1, n_2, n_3, n_4, n_5, n_6} + \mu_3 P_{n_0, n_1, n_2, n_3, n_4, n_5, n_6} + \mu_3' P_{n_0, n_1, n_2, n_3, n_4, n_5, n_6}.
\]

On combining the results (4), (9) and (66), we get

\[
\left(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \mu_1 + \mu_2 + \mu_1' + \mu_2' + \mu_3\right)F_{n_0, n_1, n_2, n_3, n_4, n_5, n_6}(X) = \lambda_1 FN_{n_2, n_3, n_5, n_6}(X) + \mu_1 P_{n_0, n_1, n_2, n_3, n_4, n_5, n_6} + \mu_1' P_{n_0, n_1, n_2, n_3, n_4, n_5, n_6} + \mu_2 P_{n_0, n_1, n_2, n_3, n_4, n_5, n_6} + \mu_2' P_{n_0, n_1, n_2, n_3, n_4, n_5, n_6} + \mu_3 P_{n_0, n_1, n_2, n_3, n_4, n_5, n_6} + \mu_3' P_{n_0, n_1, n_2, n_3, n_4, n_5, n_6}.
\]

On combining the results (5), (10) and (66), we get

\[
\left(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \mu_1 + \mu_2 + \mu_1' + \mu_2' + \mu_3\right)F_{n_0, n_1, n_2, n_3, n_4, n_5, n_6}(X) = \lambda_1 FN_{n_2, n_3, n_5, n_6}(X) + \mu_1 P_{n_0, n_1, n_2, n_3, n_4, n_5, n_6} + \mu_1' P_{n_0, n_1, n_2, n_3, n_4, n_5, n_6} + \mu_2 P_{n_0, n_1, n_2, n_3, n_4, n_5, n_6} + \mu_2' P_{n_0, n_1, n_2, n_3, n_4, n_5, n_6} + \mu_3 P_{n_0, n_1, n_2, n_3, n_4, n_5, n_6} + \mu_3' P_{n_0, n_1, n_2, n_3, n_4, n_5, n_6}.
\]
On combining the results (6), (11) and (66), we get

\[
\begin{align*}
\lambda_j + \lambda'_2 + \lambda''_2 + \lambda''_3 + \mu_j + \mu_2 + \mu'_1 + \mu'_2 + \mu'_3 + \mu_{ij} F_{n_0, n_0, 0, n_0} (X) - \mu_j P_{0, n_0, 0, n_0} = \lambda X \\
F_{n_0, n_0, 0, n_0} (X) + \frac{\partial P_{13}}{X} [F_{n_0, n_0, 0, n_0} (X) - P_{0, n_0, 0, n_0}] + \mu_2 P_{23} F_{n_0, n_0, 0, n_0} + \mu_2 P_{23} F_{n_0, n_0, 0, n_0} (X) + \mu_2 P_{23} X F_{n_0, n_0, 0, n_0} (X) \\
+ \mu_2 P_{23} [F_{n_0, n_0, 0, n_0} (X) - P_{0, n_0, 0, n_0}] + \mu_2 P_{23} F_{n_0, n_0, 0, n_0} - \mu_2 P_{23} F_{n_0, n_0, 0, n_0} = \lambda X
\end{align*}
\]

On combining the results (7), (12) and (66), we get

\[
\begin{align*}
\lambda_j + \lambda'_2 + \lambda''_2 + \lambda''_3 + \mu_j + \mu_2 + \mu'_1 + \mu'_2 + \mu'_3 + \mu_{ij} F_{n_0, n_0, 0, n_0} (X) - \mu_j P_{0, n_0, 0, n_0} = \\
\lambda_j F_{n_0, n_0, 0, n_0} (X) + \frac{\partial P_{13}}{X} [F_{n_0, n_0, 0, n_0} (X) - P_{0, n_0, 0, n_0}] + \mu_2 P_{23} F_{n_0, n_0, 0, n_0} + \mu_2 P_{23} F_{n_0, n_0, 0, n_0} (X) + \mu_2 P_{23} X F_{n_0, n_0, 0, n_0} (X) \\
+ \mu_2 P_{23} [F_{n_0, n_0, 0, n_0} (X) - P_{0, n_0, 0, n_0}] + \mu_2 P_{23} F_{n_0, n_0, 0, n_0} = \lambda X
\end{align*}
\]

On combining the results (13), (23) and (66), we get

\[
\begin{align*}
\lambda_j + \lambda'_2 + \lambda''_2 + \lambda''_3 + \mu_j + \mu_2 + \mu'_1 + \mu'_2 + \mu'_3 + \mu_{ij} F_{n_0, n_0, 0, n_0} (X) - \mu_j P_{0, n_0, 0, n_0} = \\
\lambda_j F_{n_0, n_0, 0, n_0} (X) + \frac{\partial P_{13}}{X} [F_{n_0, n_0, 0, n_0} (X) - P_{0, n_0, 0, n_0}] + \mu_2 P_{23} F_{n_0, n_0, 0, n_0} + \mu_2 P_{23} F_{n_0, n_0, 0, n_0} (X) + \mu_2 P_{23} X F_{n_0, n_0, 0, n_0} (X) \\
+ \mu_2 P_{23} [F_{n_0, n_0, 0, n_0} (X) - P_{0, n_0, 0, n_0}] + \mu_2 P_{23} F_{n_0, n_0, 0, n_0} = \lambda X
\end{align*}
\]

On combining the results (14), (24) and (66), we get

\[
\begin{align*}
\lambda_j + \lambda'_2 + \lambda''_2 + \lambda''_3 + \mu_j + \mu_2 + \mu'_1 + \mu'_2 + \mu'_3 + \mu_{ij} F_{n_0, n_0, 0, n_0} (X) - \mu_j P_{0, n_0, 0, n_0} = \\
\lambda_j F_{n_0, n_0, 0, n_0} (X) + \frac{\partial P_{13}}{X} [F_{n_0, n_0, 0, n_0} (X) - P_{0, n_0, 0, n_0}] + \mu_2 P_{23} F_{n_0, n_0, 0, n_0} + \mu_2 P_{23} F_{n_0, n_0, 0, n_0} (X) + \mu_2 P_{23} X F_{n_0, n_0, 0, n_0} (X) \\
+ \mu_2 P_{23} [F_{n_0, n_0, 0, n_0} (X) - P_{0, n_0, 0, n_0}] + \mu_2 P_{23} F_{n_0, n_0, 0, n_0} = \lambda X
\end{align*}
\]
\[ \mu_n F_{a, n, x, n+1} (x) + \mu_j F_{a, n, a, n, x+1} (X) + \mu_j F_{a, n, a, n, x+1} (x) \] --- (85)

On combining the results (15), (25) and (66), we get

\[ (\lambda_1 + \lambda_2 + \lambda'_1 + \lambda'_2 + \lambda'_3 + \mu_j + \mu'_1 + \mu'_2 + \mu_j ) F_{0, n+1, a, a, n+1} (X) - \mu_j P_{0, a, n, a, n+1} = \lambda_1 x F_{a, n, a, n} \]

\[ (X) + \frac{\mu_3 P_{13}}{x} [F_{a, n, a, n, -1} (X) - P_{0, a, n, a, n, a, -1}] + \mu_2 P_{23} F_{a, n, a, n, a, -1} (X) + \mu_2 P_{21} x F_{a, n, a, n, a, -1} (X) \]

\[ + \lambda'_1 F_{a, n+1, 0, 0, n, -1} (X) + \mu_1 F_{a, n, n, a, n, a, -1} (X) \] --- (86)

On combining the results (16), (26) and (66), we get

\[ (\lambda_1 + \lambda_2 + \lambda'_1 + \lambda'_2 + \lambda'_3 + \mu_j + \mu'_1 + \mu'_2 + \mu_j ) F_{0, n+1, a, a, n+1} (X) - \mu_j P_{0, a, n, a, n+1} = \lambda_1 x F_{a, n, a, n} \]

\[ (X) + \lambda_1 F_{a, n, n+1, n, a, -1} (X) + \lambda_2 F_{a, n, n+1, n, a, -1} (X) + \mu_1 F_{a, n+1, n, a, n, -1} (X) + \mu_1 F_{a, n, n+1, n, a, n, -1} (X) \]

\[ + \mu_1 F_{a, n, n, a, n, a, -1} (X) \] --- (87)

On combining the results (17), (37) and (66), we get

\[ (\lambda_1 + \lambda_2 + \lambda'_1 + \lambda'_2 + \lambda'_3 + \mu_j + \mu'_1 + \mu'_2 + \mu_j ) F_{n_2, 0, n, 0, n_6} (x) - \mu_j P_{0, a, n, a, n_6} = \]

\[ \lambda_1 x F_{n_2, n, a, n, a, -1} (X) + \lambda_2 F_{n_2, n, a, n, a, -1} (X) + \mu_1 F_{n_2+1, n, a, n, a, -1} (X) \]

\[ + \mu_1 F_{n_2+1, n, a, n, a, -1} (X) \] --- (88)

On combining the results (18), (38) and (66), we get

\[ (\lambda_1 + \lambda_2 + \lambda'_1 + \lambda'_2 + \lambda'_3 + \mu_j + \mu'_1 + \mu'_2 + \mu_j ) F_{n_2, 0, n, 0, n_6} (x) - \mu_j P_{0, a, n, a, n_6} = \]

\[ \lambda_1 x F_{n_2, n, a, n, a, -1} (X) + \lambda_2 F_{n_2+1, n, a, n, a, -1} (X) + \mu_1 F_{n_2+1, n, a, n, a, -1} (X) \]

\[ + \mu_1 F_{n_2+1, n, a, n, a, -1} (X) \] --- (89)
On combining the results (19), (39) and (66), we get

\[
(\lambda_j + \lambda_2 + \lambda'_2 + \lambda'_1 + \mu_3 + \mu_2 + \mu'_2 + \mu'_1) F_{n_0, n_0, 0, 0} (x) - \mu_j P_{0, 0, 0, 0, 0} = \lambda_j x
\]

\[
F_{n_0, n_0, n_0, 0} (X) + \lambda_j F_{n_0-1, n_0, n_0, 0} (X) + \frac{H_1 p_{12}}{x} \left[ F_{n_0-1, n_0, n_0, 0} (x) - P_{0, 0, 0, 0, 0} \right] + \mu_2 p_{21} x
\]

\[
F_{n_0+1, n_0, n_0, 0} (x) + \lambda'_2 F_{n_0, n_0+1, n_0, 0} (X) + \lambda'_3 F_{n_0, n_0, n_0-1, 0} (X) + \mu_3 F_{n_0, n_0, n_0, 0} (x) \quad (90)
\]

On combining the results (20), (40) and (66), we get

\[
(\lambda_j + \lambda_2 + \lambda'_2 + \lambda'_1 + \mu_3 + \mu_2 + \mu'_2 + \mu'_1) F_{n_0, n_0, 0, 0, 0} (x) - \mu_j P_{0, 0, 0, 0, 0} = \lambda_j x
\]

\[
F_{n_0, n_0, n_0, n_0, 0} (X) + \lambda_j F_{n_0-1, n_0, n_0, n_0, 0} (X) + \frac{H_1 p_{12}}{x} \left[ F_{n_0-1, n_0, n_0, n_0, 0} (x) - P_{0, 0, 0, 0, 0} \right] + \mu_2 p_{21} x F_{n_0+1, n_0, n_0, n_0, 0} (x)
\]

\[
+ \lambda'_2 F_{n_0, n_0+1, n_0, n_0, 0} (X) + \lambda'_3 F_{n_0, n_0, n_0-1, n_0, 0} (x) + \mu_3 F_{n_0, n_0, n_0, n_0, 0} (x) + \mu_3 F_{n_0, n_0, n_0, n_0, 0} (x) \quad (91)
\]

On combining the results (21), (41) and (66), we get

\[
(\lambda_j + \lambda_2 + \lambda'_2 + \lambda'_1 + \mu_3 + \mu_2 + \mu'_2 + \mu'_1) F_{n_0, n_0, n_0, n_0, 0} (x) - \mu_j P_{0, 0, 0, 0, 0} = x
\]

\[
F_{n_0, n_0, n_0, n_0, n_0, 0} (X) + \lambda_j F_{n_0-1, n_0, n_0, n_0, n_0, 0} (X) + \frac{H_1 p_{12}}{x} \left[ F_{n_0-1, n_0, n_0, n_0, n_0, 0} (x) - P_{0, 0, 0, 0, 0} \right]
\]

\[
+ \mu_2 p_{21} x F_{n_0+1, n_0, n_0, n_0, n_0, 0} (x) + \lambda'_2 F_{n_0, n_0+1, n_0, n_0, n_0, 0} (X) + \lambda'_3 F_{n_0, n_0, n_0-1, n_0, n_0, 0} (X) + \mu_3 F_{n_0, n_0, n_0, n_0, n_0, 0} (x) \quad (92)
\]

On combining the results (22), (42) and (66), we get

\[
(\lambda_j + \lambda_2 + \lambda'_2 + \lambda'_1 + \mu_3 + \mu_2 + \mu'_2 + \mu'_1) F_{n_0, n_0, n_0, n_0, n_0, 0} (x) - \mu_j P_{0, 0, 0, 0, 0} = \lambda_j x
\]

\[
F_{n_0, n_0, n_0, n_0, n_0, n_0, 0} (X) + \lambda_j F_{n_0-1, n_0, n_0, n_0, n_0, n_0, 0} (X) + \frac{H_1 p_{12}}{x} \left[ F_{n_0-1, n_0, n_0, n_0, n_0, n_0, 0} (x) - P_{0, 0, 0, 0, 0} \right]
\]

\[
+ \mu_2 p_{21} x F_{n_0+1, n_0, n_0, n_0, n_0, n_0, 0} (x) + \lambda'_2 F_{n_0, n_0+1, n_0, n_0, n_0, n_0, 0} (X) + \lambda'_3 F_{n_0, n_0, n_0-1, n_0, n_0, n_0, 0} (X) + \mu_3 F_{n_0, n_0, n_0, n_0, n_0, n_0, 0} (x) \quad (93)
\]

On combining the results (27), (43) and (66), we get

\[
(\lambda_j + \lambda_2 + \lambda'_2 + \lambda'_1 + \mu_3 + \mu_2 + \mu'_2 + \mu'_1) F_{n_0, n_0, n_0, n_0, n_0, n_0, 0} (x) - \mu_j P_{0, 0, 0, 0, 0, 0} = \lambda_j x
\]

\[
F_{n_0, n_0, n_0, n_0, n_0, n_0, n_0, 0} (X) + \lambda_j F_{n_0-1, n_0, n_0, n_0, n_0, n_0, n_0, 0} (X) + \frac{H_1 p_{12}}{x} \left[ F_{n_0-1, n_0, n_0, n_0, n_0, n_0, n_0, 0} (x) - P_{0, 0, 0, 0, 0, 0} \right]
\]

\[
+ \mu_2 p_{21} x F_{n_0+1, n_0, n_0, n_0, n_0, n_0, n_0, 0} (x) \quad (94)
\]
\[ \mu_2 P_{21} \times F_{1,0,0,n_0} (x) + \mu_3 P_{21} \times F_{1,0,1,n_0} (x) + \mu_4 P_{21} \times F_{1,1,0,n_0} (x) + \mu_5 P_{21} \times F_{1,1,1,n_0} (x) \]

--- (94)

On combining the results (28), (44) and (66), we get

\[
(\lambda_j + \lambda_2 + \lambda'_j + \lambda'_2 + \lambda'_j + \mu_j + \mu_j' + \mu_j') F_{0,0,0,n_0} (x) - \mu_j P_{0,0,0,n_0} = \lambda_j x F_{0,0,0,n_0} (x) + \mu_j P_{0,0,0,n_0} = \lambda_j x F_{0,0,0,n_0} (x) + \mu_j P_{0,0,0,n_0} \]

--- (95)

On combining the results (29), (45) and (66), we get

\[
(\lambda_j + \lambda_2 + \lambda'_j + \lambda'_2 + \lambda'_j + \mu_j + \mu_j' + \mu_j') F_{0,0,0,n_0} (x) - \mu_j P_{0,0,0,n_0} = \lambda_j x F_{0,0,0,n_0} (x) + \mu_j P_{0,0,0,n_0} = \lambda_j x F_{0,0,0,n_0} (x) + \mu_j P_{0,0,0,n_0} \]

--- (96)

On combining the results (30), (46) and (66), we get

\[
(\lambda_j + \lambda_2 + \lambda'_j + \lambda'_2 + \lambda'_j + \mu_j + \mu_j' + \mu_j') F_{0,0,0,n_0} (x) - \mu_j P_{0,0,0,n_0} = \lambda_j x F_{0,0,0,n_0} (x) + \mu_j P_{0,0,0,n_0} = \lambda_j x F_{0,0,0,n_0} (x) + \mu_j P_{0,0,0,n_0} \]

--- (97)

On combining the results (31), (47) and (66), we get

\[
(\lambda_j + \lambda_2 + \lambda'_j + \lambda'_2 + \lambda'_j + \mu_j + \mu_j' + \mu_j') F_{0,0,0,n_0} (x) - \mu_j P_{0,0,0,n_0} = \lambda_j x F_{0,0,0,n_0} (x) + \mu_j P_{0,0,0,n_0} = \lambda_j x F_{0,0,0,n_0} (x) + \mu_j P_{0,0,0,n_0} \]

--- (98)

On combining the results (32), (48) and (66), we get

\[
(\lambda_j + \lambda_2 + \lambda'_j + \lambda'_2 + \lambda'_j + \mu_j + \mu_j' + \mu_j') F_{0,0,0,n_0} (x) - \mu_j P_{0,0,0,n_0} = \lambda_j x F_{0,0,0,n_0} (x) + \mu_j P_{0,0,0,n_0} = \lambda_j x F_{0,0,0,n_0} (x) + \mu_j P_{0,0,0,n_0} \]

--- (99)

On combining the results (33), (49) and (66), we get
\( \lambda_j + \lambda_j' + \lambda_j'' + \lambda_j'' + \mu_j + \mu_j + \mu_j \) \( F_{n_0,0,0,0,0}(x) \) \( - \mu_j P_{n_0,0,0,0,0}(x) = \lambda_j x \( F_{n_0,0,0,0,0}(x) \) + \\
\lambda_j F_{n_0,0,0,0,0}(x) + \frac{\mu_j p_{n_0,0,0,0}}{x} \) 

On combining the results (34), (50) and (66), we get

\( \lambda_j + \lambda_j' + \lambda_j'' + \lambda_j'' + \mu_j + \mu_j + \mu_j' \) \( F_{n_0,0,0,0,0}(x) \) \( - \mu_j P_{n_0,0,0,0,0}(x) = \lambda_j x \( F_{n_0,0,0,0,0}(x) \) + \\
\lambda_j F_{n_0,0,0,0,0}(x) + \frac{\mu_j p_{n_0,0,0,0}}{x} \) \( - \mu_j P_{n_0,0,0,0,0}(x) \) 

On combining the results (35), (51) and (66), we get

\( \lambda_j + \lambda_j' + \lambda_j'' + \lambda_j'' + \mu_j + \mu_j + \mu_j' \) \( F_{n_0,0,0,0,0}(x) \) \( - \mu_j P_{n_0,0,0,0,0}(x) = \lambda_j x \( F_{n_0,0,0,0,0}(x) \) + \\
\lambda_j F_{n_0,0,0,0,0}(x) + \frac{\mu_j p_{n_0,0,0,0}}{x} \) \( - \mu_j P_{n_0,0,0,0,0}(x) \) 

On combining the results (36), (52) and (66), we get

\( \lambda_j + \lambda_j' + \lambda_j'' + \lambda_j'' + \mu_j + \mu_j + \mu_j' \) \( F_{n_0,0,0,0,0}(x) \) \( - \mu_j P_{n_0,0,0,0,0}(x) = \lambda_j x \( F_{n_0,0,0,0,0}(x) \) + \\
\lambda_j F_{n_0,0,0,0,0}(x) + \frac{\mu_j p_{n_0,0,0,0}}{x} \) \( - \mu_j P_{n_0,0,0,0,0}(x) \) 

On combining the results (53), (58) and (66), we get

\( \lambda_j + \lambda_j + \lambda_j + \lambda_j + \mu_j + \mu_j + \mu_j \) \( F_{n_0,0,0,0,0}(x) \) \( - \mu_j P_{n_0,0,0,0,0}(x) = \lambda_j x \( F_{n_0,0,0,0,0}(x) \) + \\
\lambda_j F_{n_0,0,0,0,0}(x) + \frac{\mu_j p_{n_0,0,0,0}}{x} \) \( - \mu_j P_{n_0,0,0,0,0}(x) \) 

On combining the results (54), (59) and (66), we get
\[
(\lambda_j + \lambda'_j + \lambda_j'^* + \lambda'_j + \mu_j + \mu_j'^*) F_{0,0,0,0,0,0}(x) - \mu_j P_{0,0,0,0,0,0}(x) = \lambda_j x F_{0,0,0,0,0,0}(x) + \mu_j P_{0,0,0,0,0,0}(x)
\]

On combining the results (55), (60) and (66), we get
\[
(\lambda_j + \lambda'_j + \lambda_j'^* + \lambda'_j + \mu_j + \mu_j'^*) F_{0,0,0,0,0,0}(x) - \mu_j P_{0,0,0,0,0,0}(x) = \lambda_j x F_{0,0,0,0,0,0}(x) + \mu_j P_{0,0,0,0,0,0}(x)
\]

On combining the results (56), (61) and (66), we get
\[
(\lambda_j + \lambda'_j + \lambda_j'^* + \lambda'_j + \mu_j + \mu_j'^*) F_{0,0,0,0,0,0}(x) - \mu_j P_{0,0,0,0,0,0}(x) = \lambda_j x F_{0,0,0,0,0,0}(x) + \mu_j P_{0,0,0,0,0,0}(x)
\]

On combining the results (57), (62) and (66), we get
\[
(\lambda_j + \lambda'_j + \lambda_j'^* + \lambda'_j + \mu_j + \mu_j'^*) F_{0,0,0,0,0,0}(x) - \mu_j P_{0,0,0,0,0,0}(x) = \lambda_j x F_{0,0,0,0,0,0}(x) + \mu_j P_{0,0,0,0,0,0}(x)
\]

On combining the results (63), (64) and (66), we get
\[
(\lambda_j + \lambda'_j + \lambda_j'^* + \lambda'_j + \mu_j + \mu_j'^*) F_{0,0,0,0,0,0}(x) - \mu_j P_{0,0,0,0,0,0}(x) = \lambda_j x F_{0,0,0,0,0,0}(x) + \mu_j P_{0,0,0,0,0,0}(x)
\]

Multiplying (78) \(Y^{n_2}\) and summing over \(n_2\) from 0 to \(\infty\) and using (79) & (67), we get
\[
(\lambda_j + \lambda'_j + \lambda_j'^* + \lambda'_j + \mu_j + \mu_j'^* + \mu_j' + \mu_j) F_{n_2, n_3, n_4, n_5, n_6, n_7}(x, y) - \mu_j F_{n_2, n_3, n_4, n_5, n_6, n_7}(x, y) = \lambda_j x F_{n_2, n_3, n_4, n_5, n_6, n_7}(x, y) + \lambda j' x F_{n_2, n_3, n_4, n_5, n_6, n_7}(x, y)
\]
Multiplying \((80)\) \(Y^{n_2}\) and summing over \(n_2\) from 0 to \(\infty\) and using \((84)\) & (67), we get

\[
(\lambda_1 + \lambda_2 + \lambda_3 + \mu_1 + \mu_2 + \mu_1' + \mu_2' + \mu_3 + \mu_1' + \mu_2' + \mu_3') F_{n_3, n_4, 0, n_6} (x, y) - \mu_1' F_{n_3, n_4, 0, n_6} (x, y) = \frac{\mu_1' p_{13}}{X} \left[F_{n_3, n_4, 0, n_6-1} (x, y) - F_{n_3, n_4, 0, n_6-1} (x)\right] + \frac{\mu_2' p_{23}}{Y} \left[F_{n_3, n_4, 0, n_6} (x, y) - F_{n_3, n_4, 0, n_6} (x)\right] + \frac{\mu_3' p_{31}}{X} \left[F_{n_3, n_4, 0, n_6} (x, y) - F_{n_3, n_4, 0, n_6} (x)\right].
\]

On combining the results \((81)\), \((85)\) and \((67)\), we get

\[
(\lambda_1 + \lambda_2 + \lambda_3 + \mu_1 + \mu_2 + \mu_1' + \mu_2' + \mu_3 + \mu_1' + \mu_2' + \mu_3') F_{n_3, n_4, n_5, n_6} (x, y) - \mu_1' F_{n_3, n_4, n_5, n_6} (x, y) = \frac{\mu_1' p_{13}}{X} \left[F_{n_3, n_4, n_5, n_6-1} (x, y) - F_{n_3, n_4, n_5, n_6-1} (x)\right] + \frac{\mu_2' p_{23}}{Y} \left[F_{n_3, n_4, n_5, n_6} (x, y) - F_{n_3, n_4, n_5, n_6} (x)\right] + \frac{\mu_3' p_{31}}{X} \left[F_{n_3, n_4, n_5, n_6} (x, y) - F_{n_3, n_4, n_5, n_6} (x)\right].
\]

On combining the results \((82)\), \((86)\) and \((67)\), we get

\[
(\lambda_1 + \lambda_2 + \lambda_3 + \mu_1 + \mu_2 + \mu_1' + \mu_2' + \mu_3 + \mu_1' + \mu_2' + \mu_3') F_{n_3, n_4, 0, n_6} (x, y) - \mu_1' F_{n_3, n_4, 0, n_6} (x, y) = \frac{\mu_1' p_{13}}{X} \left[F_{n_3, n_4, 0, n_6-1} (x, y) - F_{n_3, n_4, 0, n_6-1} (x)\right] + \frac{\mu_2' p_{23}}{Y} \left[F_{n_3, n_4, 0, n_6} (x, y) - F_{n_3, n_4, 0, n_6} (x)\right] + \frac{\mu_3' p_{31}}{X} \left[F_{n_3, n_4, 0, n_6} (x, y) - F_{n_3, n_4, 0, n_6} (x)\right].
\]

On combining the results \((82)\), \((86)\) and \((67)\), we get

\[
(\lambda_1 + \lambda_2 + \lambda_3 + \mu_1 + \mu_2 + \mu_1' + \mu_2' + \mu_3 + \mu_1' + \mu_2' + \mu_3') F_{n_3, n_4, n_5, n_6} (x, y) - \mu_1' F_{n_3, n_4, n_5, n_6} (x, y) = \frac{\mu_1' p_{13}}{X} \left[F_{n_3, n_4, n_5, n_6-1} (x, y) - F_{n_3, n_4, n_5, n_6-1} (x)\right] + \frac{\mu_2' p_{23}}{Y} \left[F_{n_3, n_4, n_5, n_6} (x, y) - F_{n_3, n_4, n_5, n_6} (x)\right] + \frac{\mu_3' p_{31}}{X} \left[F_{n_3, n_4, n_5, n_6} (x, y) - F_{n_3, n_4, n_5, n_6} (x)\right].
\]

--- (110)
On combining the results (83), (87) and (67), we get

\[
(\lambda_1 + \lambda_2 + \lambda_1' + \lambda_2' + \mu_1 + \mu_2 + \mu_1' + \mu_2') F_{n_0, n_0, 0}(x, y) - \mu_2 F_{0, n_0, 0, 0}(x) - \mu F_{0, n_0, n_0, 0}(y) = \lambda_1 x F_{n_0, n_0, 0}(x, y) + \lambda_2 y F_{n_0, n_0, 0}(x, y) + \mu_{12}^{p_{12}} X [F_{n_0, n_0, 0}(x, y) - F_{0, n_0, n_0, 0}]
\]

--- (114)

On combining the results (88), (94) and (67), we get

\[
(\lambda_1 + \lambda_2 + \lambda_1' + \lambda_2' + \mu_1 + \mu_2 + \mu_1' + \mu_2') F_{0, n_0, n_0, 0}(x, y) - \mu_1 F_{0, 0, n_0, 0}(x) - \mu F_{0, 0, n_0, n_0}(y) = \lambda_1 x F_{0, n_0, n_0, 0}(x, y) + \lambda_2 y F_{0, n_0, n_0, 0}(x, y) + \mu_{12}^{p_{12}} X [F_{0, n_0, n_0, 0}(x, y) - F_{0, 0, n_0, n_0}]
\]

--- (115)

On combining the results (89), (95) and (67), we get

\[
(\lambda_1 + \lambda_2 + \lambda_1' + \lambda_2' + \mu_1 + \mu_2 + \mu_1' + \mu_2') F_{0, n_0, n_0, n_0}(x, y) - \mu_1 F_{0, 0, n_0, n_0}(x) - \mu F_{0, 0, n_0, n_0, n_0}(y) = \lambda_1 x F_{0, n_0, n_0, n_0}(x, y) + \lambda_2 y F_{0, n_0, n_0, n_0}(x, y) + \mu_{12}^{p_{12}} X [F_{0, n_0, n_0, n_0}(x, y) - F_{0, 0, n_0, n_0, n_0}]
\]

--- (116)

On combining the results (90), (96) and (67), we get

\[
(\lambda_1 + \lambda_2 + \lambda_1' + \lambda_2' + \mu_1 + \mu_2 + \mu_1' + \mu_2') F_{0, n_0, n_0, n_0}(x, y) - \mu_1 F_{0, 0, n_0, n_0, 0}(x) - \mu F_{0, 0, n_0, n_0, 0}(n_0, n_0, 0, 0, 0) = \lambda_1 x F_{0, n_0, n_0, n_0}(x, y) + \lambda_2 y F_{0, n_0, n_0, n_0}(x, y) + \mu_{12}^{p_{12}} X [F_{0, n_0, n_0, n_0}(x, y) - F_{0, 0, n_0, n_0, n_0, n_0, 0}]
\]
\[
\frac{\mu_2 p_1 X}{Y} [F_{a_0, a_0, 0}(x, y) - F_{a_0, a_0, 0}(x)] + \lambda_1 F_{a_0, a_0, 0}(x, y) + \lambda_2 F_{a_0, a_0, 0}(x, y) + \mu_1 F_{a_0, a_0, 0}(x, y) \] --- (117)

On combining the results (91), (97) and (67), we get
\[
(\lambda_1 + \lambda_2 + \lambda_1' + \lambda_2' + \mu_1 + \mu_2 + \mu_1' + \mu_2') F_{n_0, a_0, 0}(x, y) - \mu_2 F_{a_0, a_0, 0}(x) - \mu_1 F_{a_0, a_0, 0}(x, y) \] --- (118)

On combining the results (92), (98) and (67), we get
\[
(\lambda_1 + \lambda_2 + \lambda_1' + \lambda_2' + \mu_1 + \mu_2 + \mu_1' + \mu_2') F_{n_0, a_0, 0}(x, y) - \mu_2 F_{a_0, a_0, 0}(x) - \mu_1 F_{a_0, a_0, 0}(x, y) \] --- (119)

On combining the results (93), (99) and (67), we get
\[
(\lambda_1 + \lambda_2 + \lambda_1' + \lambda_2' + \mu_1 + \mu_2 + \mu_1' + \mu_2') F_{n_0, a_0, 0}(x, y) - \mu_2 F_{a_0, a_0, 0}(x) - \mu_1 F_{a_0, a_0, 0}(x, y) \] --- (120)

On combining the results (100), (104) and (67), we get
\[
(\lambda_1 + \lambda_2 + \lambda_1' + \lambda_2' + \mu_1 + \mu_2 + \mu_1' + \mu_2') F_{a_0, a_0, 0}(x, y) - \mu_2 F_{a_0, a_0, 0}(x) - \mu_1 F_{a_0, a_0, 0}(x, y) \]
= \lambda_1 F_{x_0,x_0} (X,Y) + \lambda_2 F_{x_0,x_0} (X,Y) + \frac{\mu_1 P_{13}}{X} [F_{x_0,x_0} (X,Y) - F_{x_0,x_0} (Y)] + \\
\frac{\mu_2 P_{21}}{X} Y [F_{x_0,x_0} (X,Y) - F_{x_0,x_0} (Y)] + \frac{\mu_2 P_{23}}{Y} [F_{x_0,x_0} (X,Y) - F_{x_0,x_0} (X)] + \\
\frac{\mu_2 P_{21} X}{Y} [F_{x_0,x_0} (X,Y) - F_{x_0,x_0} (x_0)] + \lambda_1 \mu_{y_0} F_{x_0,x_0} (X,Y) + \mu_1 F_{x_0,x_0} (X,Y)

On combining the results (101), (105) and (67), we get

\begin{align*}
\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 &+ \mu_1 + \mu_2 + \mu_3 \text{ } F_{x_0,x_0} (X,Y) - \mu_2 F_{x_0,x_0} (X) - \mu_3 F_{x_0,x_0} (Y) \\
= \lambda_1 F_{x_0,x_0} (X,Y) + \lambda_2 F_{x_0,x_0} (X,Y) + \frac{\mu_1 P_{13}}{X} [F_{x_0,x_0} (X,Y) - F_{x_0,x_0} (Y)] + \\
\frac{\mu_2 P_{21} X}{Y} [F_{x_0,x_0} (X,Y) - F_{x_0,x_0} (x_0)] + \lambda_1 \mu_{y_0} F_{x_0,x_0} (X,Y) + \mu_1 F_{x_0,x_0} (X,Y) &--- (122)
\end{align*}

On combining the results (102), (106) and (67), we get

\begin{align*}
\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 &+ \mu_1 + \mu_2 + \mu_3 \text{ } F_{x_0,x_0} (X,Y) - \mu_2 F_{x_0,x_0} (X) - \mu_3 F_{x_0,x_0} (Y) \\
= \lambda_1 F_{x_0,x_0} (X,Y) + \lambda_2 F_{x_0,x_0} (X,Y) + \frac{\mu_1 P_{13}}{X} [F_{x_0,x_0} (X,Y) - F_{x_0,x_0} (Y)] + \\
\frac{\mu_2 P_{21} X}{Y} [F_{x_0,x_0} (X,Y) - F_{x_0,x_0} (x_0)] + \lambda_1 \mu_{y_0} F_{x_0,x_0} (X,Y) + \mu_1 F_{x_0,x_0} (X,Y) &--- (123)
\end{align*}

On combining the results (103), (108) and (67), we get

\begin{align*}
\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 &+ \lambda_5 + \mu_1 + \mu_2 + \mu_3 \text{ } F_{x_0,x_0} (X,Y) - \mu_2 F_{x_0,x_0} (X) - \mu_3 F_{x_0,x_0} (Y) = \lambda_1 F_{x_0,x_0} (X,Y) + \frac{\mu_1 P_{13}}{X} [F_{x_0,x_0} (X,Y) - F_{x_0,x_0} (Y)] + \\
\frac{\mu_2 P_{21} X}{Y} [F_{x_0,x_0} (X,Y) - F_{x_0,x_0} (Y)] + \lambda_1 \mu_{y_0} F_{x_0,x_0} (X,Y) + \mu_1 F_{x_0,x_0} (X,Y) &--- (124)
\end{align*}

On combining the results (107), (109) and (67), we get

\begin{align*}
\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 &+ \mu_1 + \mu_2 \text{ } F_{a_0,a_0} (X,Y) - \mu_2 F_{a_0,a_0} (X) - \mu_3 F_{a_0,a_0} (Y) = \lambda_1 F_{a_0,a_0} (X,Y) + \frac{\mu_1 P_{13}}{X} [F_{a_0,a_0} (X,Y) - F_{a_0,a_0} (Y)] + \\
\frac{\mu_2 P_{21} X}{Y} [F_{a_0,a_0} (X,Y) - F_{a_0,a_0} (Y)] + \lambda_1 \mu_{y_0} F_{a_0,a_0} (X,Y) + \mu_1 F_{a_0,a_0} (X,Y) &--- (125)
\end{align*}
On multiplying (110) by $Z^{n_3}$ and summing over $n_3$ from 0 to $\infty$ and using (111) & (68) we get

\begin{align}
(\lambda_1 + \lambda_2, + \lambda_3, + \mu_1, + \mu_2, + \mu_3, + \mu_4, + \mu_5) F_{n_0, n_1, n_2} (X, Y, Z) - \mu_1 F_{0, n_0, n_1} (X, Y) - \mu_2 F_{n_0, n_1, n_2} (X, Y) = \lambda_1 x F_{n_0, n_1, n_2} (X, Y, Z) + \lambda_2 y F_{n_0, n_1, n_2} (X, Y) + \frac{\mu_1 P_{l_3}}{X} [F_{n_0, n_1, n_2} (X, Y, Z) - F_{n_0, n_1, n_2} (X, Y)] + \frac{\mu_2 P_{l_3}}{Y} [F_{n_0, n_1, n_2} (X, Y, Z) - F_{n_0, n_1, n_2} (X, Y)] + \lambda'_1 F_{n_0, n_1, n_2} (X, Y, Z) + \lambda'_2 F_{n_0, n_1, n_2} (X, Y, Z) + \lambda'_3 F_{n_0, n_1, n_2} (X, Y, Z) + \mu_1 F_{n_0, n_1, n_2} (X, Y, Z) + \mu_2 F_{n_0, n_1, n_2} (X, Y, Z)
\end{align}

--- (126)

On combining the results (112), (115) and (68), we get

\begin{align}
(\lambda_1 + \lambda_2, + \lambda_3, + \mu_1, + \mu_2, + \mu_3, + \mu_4, + \mu_5) F_{n_0, n_1, n_2} (X, Y, Z) - \mu_1 F_{0, n_0, n_1} (X, Y) - \mu_2 F_{n_0, n_1, n_2} (X, Y) = \lambda_1 x F_{n_0, n_1, n_2} (X, Y, Z) + \lambda_2 y F_{n_0, n_1, n_2} (X, Y) + \frac{\mu_1 P_{l_3}}{X} [F_{n_0, n_1, n_2} (X, Y, Z) - F_{n_0, n_1, n_2} (X, Y)] + \frac{\mu_2 P_{l_3}}{Y} [F_{n_0, n_1, n_2} (X, Y, Z) - F_{n_0, n_1, n_2} (X, Y)] + \lambda'_1 F_{n_0, n_1, n_2} (X, Y, Z) + \lambda'_2 F_{n_0, n_1, n_2} (X, Y, Z) + \lambda'_3 F_{n_0, n_1, n_2} (X, Y, Z) + \mu_1 F_{n_0, n_1, n_2} (X, Y, Z) + \mu_2 F_{n_0, n_1, n_2} (X, Y, Z)
\end{align}

--- (127)

On combining the results (113), (116) and (68), we get

\begin{align}
(\lambda_1 + \lambda_2, + \lambda_3, + \mu_1, + \mu_2, + \mu_3, + \mu_4, + \mu_5) F_{n_0, n_1, n_2} (X, Y, Z) - \mu_1 F_{0, n_0, n_1} (X, Y) - \mu_2 F_{n_0, n_1, n_2} (X, Y) = \lambda_1 x F_{n_0, n_1, n_2} (X, Y, Z) + \lambda_2 y F_{n_0, n_1, n_2} (X, Y) + \frac{\mu_1 P_{l_3}}{X} [F_{n_0, n_1, n_2} (X, Y, Z) - F_{n_0, n_1, n_2} (X, Y)] + \frac{\mu_2 P_{l_3}}{Y} [F_{n_0, n_1, n_2} (X, Y, Z) - F_{n_0, n_1, n_2} (X, Y)] + \lambda'_1 F_{n_0, n_1, n_2} (X, Y, Z) + \lambda'_2 F_{n_0, n_1, n_2} (X, Y, Z) + \lambda'_3 F_{n_0, n_1, n_2} (X, Y, Z) + \mu_1 F_{n_0, n_1, n_2} (X, Y, Z) + \mu_2 F_{n_0, n_1, n_2} (X, Y, Z)
\end{align}

--- (128)
\[(X,Y,Z) - F_{\alpha,\alpha,\alpha-1}(X,Z) + \frac{\mu_2 p_{21} X}{Y} \left[ F_{\alpha,\alpha,\alpha}(X,Y,Z) - F_{\alpha,\alpha,\alpha}(X,Z) \right] + \lambda^' F_{\alpha,\alpha,\alpha}(X,Y,Z) + \mu_j F_{\alpha,\alpha,\alpha}(X,Y,Z) \] -- (128)

On combining the results (114), (117) and (68), we get
\[
(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 + \lambda_9) F_{\alpha,\alpha,\alpha}(X,Y,Z) - \mu_1 F_{\alpha,\alpha,\alpha}(X,Y,Z) - \mu_2 F_{\alpha,\alpha,\alpha}(X,Z) - \mu_3 F_{\alpha,\alpha,\alpha}(Y,Z) = \lambda_1 F_{\alpha,\alpha,\alpha}(X,Y,Z) + \mu_2 F_{\alpha,\alpha,\alpha}(X,Z) + \mu_3 F_{\alpha,\alpha,\alpha}(Y,Z) -- (129)

On combining the results (118), (121) and (68), we get
\[
(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 + \lambda_9) F_{\alpha,\alpha,\alpha}(X,Y,Z) - \mu_1 F_{\alpha,\alpha,\alpha}(X,Y,Z) - \mu_2 F_{\alpha,\alpha,\alpha}(X,Z) - \mu_3 F_{\alpha,\alpha,\alpha}(Y,Z) = \lambda_1 F_{\alpha,\alpha,\alpha}(X,Y,Z) + \mu_2 F_{\alpha,\alpha,\alpha}(X,Z) + \mu_3 F_{\alpha,\alpha,\alpha}(Y,Z) -- (130)

On combining the results (119), (122) and (68), we get
\[
(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 + \lambda_9) F_{\alpha,\alpha,\alpha}(X,Y,Z) - \mu_1 F_{\alpha,\alpha,\alpha}(X,Y,Z) - \mu_2 F_{\alpha,\alpha,\alpha}(X,Z) - \mu_3 F_{\alpha,\alpha,\alpha}(Y,Z) = \lambda_1 F_{\alpha,\alpha,\alpha}(X,Y,Z) + \mu_2 F_{\alpha,\alpha,\alpha}(X,Z) + \mu_3 F_{\alpha,\alpha,\alpha}(Y,Z) -- (131)
On combining the results (120), (123) and (68), we get
\[
(\lambda_1 + \lambda_2 + \lambda_3 + \mu_1 + \mu_2 + \mu_3 + \mu_1') \cdot F_{n_{a_0}}(X, Y, Z) - \mu_1' \cdot F_{0, n_{a_0}}(X, Y) - \mu_2' \cdot F_{0, n_{a_0}}(X, Z)
\]
\[
- \mu_3' \cdot F_{0, n_{a_0}}(Y, Z) = \lambda_1 \cdot F_{n_{a_0}}(X, Y, Z) + \lambda_2 \cdot F_{n_{a_0}}(X, Y, Z) + \frac{\mu_{12}}{X} Y [F_{n_{a_0}}(X, Y, Z) -
\]
\[
F_{0, n_{a_0}}(Y, Z)] + \frac{\mu_{21}}{Y} [F_{n_{a_0}}(X, Y, Z) - F_{0, n_{a_0}}(X, Z)] + \lambda_1' \cdot F_{0, n_{a_0}}(X, Y, Z) + \lambda_2' \cdot F_{n_{a_0}}(X, Y, Z) -
\]
\[
\lambda_3' \cdot F_{n_{a_0}}(X, Y, Z) - \mu_1' \cdot F_{0, n_{a_0}}(Y, Z) - \mu_2' \cdot F_{0, n_{a_0}}(X, Z) -
\]
\[
\mu_3' \cdot F_{0, n_{a_0}}(X, Z)
\]

--- (123)

On combining the results (124), (125) and (68), we get
\[
(\lambda_1 + \lambda_2 + \lambda_3 + \mu_1 + \mu_2 + \mu_3 + \mu_1') \cdot F_{0, a_0, a_0}(X, Y, Z) - \mu_1' \cdot F_{0, a_0, a_0}(X, Y) - \mu_2' \cdot F_{0, a_0, a_0}(X, Z)
\]
\[
- \mu_3' \cdot F_{0, a_0, a_0}(Y, Z) = \lambda_1 \cdot F_{0, a_0, a_0}(X, Y, Z) + \lambda_2 \cdot F_{0, a_0, a_0}(X, Y, Z) + \frac{\mu_{12}}{X} Y [F_{0, a_0, a_0}(X, Y, Z) -
\]
\[
F_{0, a_0, a_0}(Y, Z)] + \frac{\mu_{21}}{Y} [F_{0, a_0, a_0}(X, Y, Z) - F_{0, a_0, a_0}(X, Z)] + \lambda_1' \cdot F_{0, a_0, a_0}(X, Y, Z) + \lambda_2' \cdot F_{0, a_0, a_0}(X, Y, Z) +
\]
\[
\lambda_3' \cdot F_{0, a_0, a_0}(X, Y, Z)
\]

--- (132)

On Multiplying (126) by \(R^{n_4}\) and summing over \(n_4\) from \(0\) to \(\infty\) and using (127) & (69) we get
\[
(\lambda_1 + \lambda_2 + \lambda_3 + \mu_1 + \mu_2 + \mu_3 + \mu_1') \cdot F_{n_{a_0}}(X, Y, Z, R) - \mu_1' \cdot F_{0, n_{a_0}}(X, Y, Z, R) - \mu_2' \cdot F_{0, n_{a_0}}(X, Y, R) -
\]
\[
\mu_3' \cdot F_{0, n_{a_0}}(X, Z, R) = \lambda_1 \cdot F_{n_{a_0}}(X, Y, Z, R) + \lambda_2 \cdot F_{n_{a_0}}(X, Y, Z, R) + \lambda_3 \cdot F_{0, n_{a_0}}(X, Y, Z, R) + \lambda_1' \cdot F_{0, n_{a_0}}(X, Y, Z, R) +
\]
\[
\lambda_2' \cdot F_{0, n_{a_0}}(X, Y, Z, R) + \lambda_3' \cdot F_{0, n_{a_0}}(X, Y, Z, R)
\]

--- (133)

On combining the results (128), (130) and (69), we get
\[
(\lambda_1 + \lambda_2 + \lambda_3 + \mu_1 + \mu_2 + \mu_3 + \mu_1') \cdot F_{n_{a_0}}(X, Y, Z) - \mu_1' \cdot F_{0, n_{a_0}}(X, Y) - \mu_2' \cdot F_{0, n_{a_0}}(X, Z)
\]
\[
- \mu_3' \cdot F_{0, n_{a_0}}(Y, Z) = \lambda_1 \cdot F_{n_{a_0}}(X, Y, Z) + \lambda_2 \cdot F_{n_{a_0}}(X, Y, Z) + \lambda_3 \cdot F_{0, n_{a_0}}(X, Y, Z) + \lambda_1' \cdot F_{0, n_{a_0}}(X, Y, Z) +
\]
\[
\lambda_2' \cdot F_{0, n_{a_0}}(X, Y, Z) + \lambda_3' \cdot F_{0, n_{a_0}}(X, Y, Z)
\]

--- (134)
\[
(\lambda_j + \lambda_j' + \lambda_j'' + \lambda_j'' + \mu_1 + \mu_2 + \mu_j + \mu_j' + \mu_j'' + \mu_j''' + \mu_j') F_{n_0,0} (X, Y, Z, R) - \mu_j' F_{n_0,0} (X, Y, Z, R) - \mu_j F_{n_0,0} (X, Y, Z, R) = \lambda_j X F_{n_0,0} (X, Y, Z, R) + \lambda_j Y F_{n_0,0} (X, Y, Z, R) + \lambda_j' Z F_{n_0,0} (X, Y, Z, R) + \lambda_j'' R F_{n_0,0} (X, Y, Z, R) + \lambda_j''' F_{1,0} (X, Y, Z, R) + \mu_j F_{n_0+1,0} (X, Y, Z, R)
\]

--- (135)

On combining the results (129), (131) and (69), we get

\[
(\lambda_j + \lambda_j' + \lambda_j'' + \mu_1 + \mu_2 + \mu_j + \mu_j') F_{n_0,0} (X, Y, Z, R) - \mu_j' F_{n_0,0} (X, Y, Z, R) - \mu_j F_{n_0,0} (X, Y, Z, R) = \lambda_j X F_{n_0,0} (X, Y, Z, R) + \lambda_j Y F_{n_0,0} (X, Y, Z, R) + \lambda_j' Z F_{n_0,0} (X, Y, Z, R) + \lambda_j'' R F_{n_0,0} (X, Y, Z, R) + \lambda_j''' F_{1,0} (X, Y, Z, R) + \mu_j F_{n_0+1,0} (X, Y, Z, R)
\]

--- (136)

On combining the results (132), (133) and (69), we get

\[
(\lambda_j + \lambda_j' + \lambda_j'' + \mu_1 + \mu_2 + \mu_j + \mu_j') F_{n_0,0} (X, Y, Z, R) - \mu_j' F_{n_0,0} (X, Y, Z, R) - \mu_j F_{n_0,0} (X, Y, Z, R) = \lambda_j X F_{n_0,0} (X, Y, Z, R) + \lambda_j Y F_{n_0,0} (X, Y, Z, R) + \lambda_j' Z F_{n_0,0} (X, Y, Z, R) + \lambda_j'' R F_{n_0,0} (X, Y, Z, R) + \lambda_j''' F_{1,0} (X, Y, Z, R) + \mu_j F_{n_0+1,0} (X, Y, Z, R)
\]

--- (137)

On Multiplying (134) by \(S^{n_5}\) and summing over \(n_5\) from 0 to \(\infty\) and using (135) & (70) we get

\[
(\lambda_j + \lambda_j' + \lambda_j'' + \mu_1 + \mu_2 + \mu_j + \mu_j' + \mu_j'' + \mu_j''' + \mu_j') F_{n_0} (X, Y, Z, R, S) - \mu_j' F_{n_0} (X, Y, Z, R, S) - \mu_j F_{n_0} (X, Y, Z, R, S) = \lambda_j X F_{n_0} (X, Y, Z, R, S) + \lambda_j Y F_{n_0} (X, Y, Z, R, S) + \lambda_j' Z F_{n_0} (X, Y, Z, R, S) + \lambda_j'' R F_{n_0} (X, Y, Z, R, S) + \lambda_j''' F_{1,0} (X, Y, Z, R, S)
\]

--- (138)
\[-\mu_2 F_{a,n_6}(X, Y, Z, S) - \mu_1 F_{a,n_6}(X, Y, R, S) - \mu_2 F_{a,n_6}(X, Y, R, S) - \mu_j F_{a,n_6}(Y, Z, R, S) = \lambda_j X F_{a,n_6}(X, Y, Z, R, S) + \lambda_2 Y F_{a,n_6}(X, Y, Z, R, S) + \frac{\mu_1 P_{31}}{X} F_{a,n_1}(X, Y, Z, R, S) - F_{a,n_1}(Y, Z, R, S)
\]
\[= \lambda_j X F_{a,n_6}(X, Y, Z, R, S) + \lambda_2 Y F_{a,n_6}(X, Y, Z, R, S) + \frac{\mu_1 P_{31}}{X} [F_{a,n_1}(X, Y, Z, R, S) - F_{a,n_1}(Y, Z, R, S)] + \lambda_j Z F_{n_6}(X, Y, Z, R, S) + \lambda_2 R F_{n_6}(X, Y, Z, R, S) + \lambda_3 S F_{n_6}(X, Y, Z, R, S) + \mu_j F_{n_6+1}(X, Y, Z, R, S)\]

--- (138)

On combining the results (136), (137) and (70), we get

\[(\lambda_j + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \mu_j + \mu_j + \mu_j + \mu_j + \mu_j) F_0(X, Y, Z, R, S) - \mu_j F_{0,0}(X, Y, Z, R) - \mu_j F_{0,0}(Y, Z, R, S) - \mu_j F_{0,0}(Z, R, S) - \mu_j F_{0,0}(R, S) - \mu_j F_{0,0}(S, T) + \lambda_j X F_0(X, Y, Z, R, S) + \lambda_2 Y F_0(X, Y, Z, R, S) + \lambda_3 Z F_0(X, Y, Z, R, S) + \lambda_4 R F_0(X, Y, Z, R, S) + \lambda_5 S F_0(X, Y, Z, R, S) + \mu_j F_1(X, Y, Z, R, S)
\]

--- (139)

On Multiplying (138) by $T^{n_6}$ and summing over $n_6$ from 0 to $\infty$ and using (139) & (71) we get

\[(\lambda_j + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \mu_j + \mu_j + \mu_j + \mu_j + \mu_j) F(X, Y, Z, R, S, T) - \mu_j F_{0,0}(X, Y, Z, R, S, T) - \mu_j F_{0,0}(X, Y, Z, R, S, T) - \mu_j F_{0,0}(X, Y, Z, R, S, T) - \mu_j F_{0,0}(X, Y, Z, R, S, T) - \mu_j F_{0,0}(X, Y, Z, R, S, T) + \lambda_j X F(X, Y, Z, R, S, T) + \lambda_2 Y F(X, Y, Z, R, S, T) + \lambda_3 Z F(X, Y, Z, R, S, T) + \lambda_4 R F(X, Y, Z, R, S, T) + \lambda_5 S F(X, Y, Z, R, S, T) + \mu_j F_1(X, Y, Z, R, S, T)
\]

--- (138)

On combining the results (136), (137) and (70), we get

\[(\lambda_j + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \mu_j + \mu_j + \mu_j + \mu_j + \mu_j) F_0(X, Y, Z, R, S, T) - \mu_j F_{0,0}(X, Y, Z, R, S, T) - \mu_j F_{0,0}(X, Y, Z, R, S, T) - \mu_j F_{0,0}(X, Y, Z, R, S, T) - \mu_j F_{0,0}(X, Y, Z, R, S, T) + \lambda_j X F_0(X, Y, Z, R, S, T) + \lambda_2 Y F_0(X, Y, Z, R, S, T) + \lambda_3 Z F_0(X, Y, Z, R, S, T) + \lambda_4 R F_0(X, Y, Z, R, S, T) + \lambda_5 S F_0(X, Y, Z, R, S, T) + \mu_j F_1(X, Y, Z, R, S, T)
\]

--- (139)

\[\text{--- (140)}\]

For convenience, let us denote

\[F(Y, Z, R, S, T) = F_1\]
\[F(X, Z, R, S, T) = F_2\]
\[F(X, Y, R, S, T) = F_3\]
\[F(X, Y, Z, S, T) = F_4\]
\[F(X, Y, Z, R, T) = F_5\]
\[F(X, Y, Z, R, S) = F_6\]

Also \(F(1,1,1,1,1) = 1\), being the total probability.

On taking \(X = 1\) as \(Y, Z, R, S, T \to 1\), \(F(X, Y, Z, R, S, T)\) is of \(\frac{0}{0}\) indeterminate form.

Now, on differentiating numerator and denominator of (141) separately w.r.t \(X\), we have

\[\text{--- (141)}\]
\[1 = \frac{\mu_1 (p_{13} + p_{12}) F_1 + \mu_2 (-p_{21}) F_2}{-\lambda_1 + \mu_1 (p_{12} + p_{13}) + \mu_2 (-p_{21})}\]

\[\Rightarrow \mu_1 F_1 - \mu_2 p_{12} F_2 = -\lambda_1 + \mu_1 - \mu_2 p_{21} \quad (\because p_{12} + p_{13} = 1) \quad \text{--- (142)}\]

Similarly, on Diff. numerator and denominator of (141) separately w.r.t Y, on taking \(Y=1\) and \(X,Z,R,S,T \rightarrow 1\) we have

\[1 = \frac{\mu_1 (-p_{12}) F_1 + \mu_2 (p_{23} + p_{21}) F_2}{-\lambda_2 + \mu_1 (-p_{12}) + \mu_2 (p_{23} + p_{21})}\]

\[\Rightarrow \mu_1 p_{12} F_1 - \mu_2 F_2 = -\lambda_2 - p_{12} \mu_1 + \mu_2 \quad (\because p_{23} + p_{21} = 1) \quad \text{--- (143)}\]

Again, on Diff. numerator and denominator of (141) separately w.r.t Z, on taking \(Z=1\) and \(X,Y,R,S,T \rightarrow 1\) we have

\[1 = \frac{\mu_1^i F_3}{-\lambda_1^i + \mu_1^i} \Rightarrow \mu_1^i F_3 = -\lambda_1^i + \mu_1^i \quad \text{--- (144)}\]

Again, on Diff. numerator and denominator of (141) separately w.r.t R, on taking \(R=1\) and \(X,Y,Z,S,T \rightarrow 1\) we have

\[1 = \frac{\mu_2^i F_4}{-\lambda_2^i + \mu_2^i} \Rightarrow \mu_2^i F_4 = -\lambda_2^i + \mu_2^i \quad \text{--- (145)}\]

Again, on Diff. numerator and denominator of (141) separately w.r.t S, on taking \(S=1\) and \(X,Y,Z,R,T \rightarrow 1\) we have

\[1 = \frac{\mu_3^i F_5}{-\lambda_3^i + \mu_3^i} \Rightarrow \mu_3^i F_5 = -\lambda_3^i + \mu_3^i \quad \text{--- (146)}\]

Again, on Diff. numerator and denominator of (141) separately w.r.t T, on taking \(T=1\) and \(X,Y,Z,R,S \rightarrow 1\) we have
\[
1 = \frac{\mu_1 p_{12} F_1 - \mu_2 p_{23} F_2 + \mu_1 (-F_3) + \mu_2 (-F_4) + \mu_3 (-F_5) + \mu_3 F_6}{\mu_1 (-p_{13}) + \mu_2 (-p_{23}) + \mu_1 (-1) + \mu_2 (-1) + \mu_3 (-1) + \mu_3}
\]

\[\Rightarrow -\mu_1 p_{13} F_1 - \mu_2 p_{23} F_2 - \mu_1 F_3 - \mu_2 F_4 - \mu_3 F_5 + \mu_3 F_6 = -p_{13} \mu_1 - p_{23} \mu_2 - \mu_1 - \mu_2 - \mu_3 - \mu_3\] \hspace{1cm} -- (147)

On multiplying (143) with \(p_{21}\) and adding to (142), we get

\[F_1 (\mu_1 - \mu_1 p_{12} p_{21}) = -\lambda_1 + \mu_1 (1 - p_{12} p_{21}) - \lambda_2 p_{21}\]

\[\Rightarrow F_1 = 1 - \frac{\lambda_1 + \lambda_2 p_{21}}{(1 - p_{12} p_{21}) \mu_1}\] \hspace{1cm} --- (148)

\[F_3 = 1 - \frac{\lambda_1}{\mu_1} \quad \text{(Using (144))} \] \hspace{1cm} --- (149)

\[F_4 = 1 - \frac{\lambda_2}{\mu_2} \quad \text{(Using (145))} \] \hspace{1cm} --- (150)

\[F_5 = 1 - \frac{\lambda_3}{\mu_3} \quad \text{(Using (146))} \] \hspace{1cm} --- (151)

On multiplying (142) with \(p_{21}\) and adding to (143), we get

\[\mu_2 (1 - p_{12} p_{21}) F_2 = -\lambda_2 - \lambda_1 p_{12} + \mu_2 (p_{12} - p_{21}) + \mu_1 (1 - p_{21} p_{12})\]

\[F_2 = 1 - \frac{\lambda_2 + \lambda_1 p_{12}}{(1 - p_{12} p_{21}) \mu_2}\] \hspace{1cm} ---- (152)

Now on putting the values of \(F_1, F_2, F_3, F_4, F_5\) in (147), we get

\[F_6 = 1 - \left[ \frac{\lambda_1 + \lambda_2 + \lambda_3}{\mu_3} + \frac{(\lambda_1 + \lambda_2 p_{12}) p_{13} + (\lambda_2 + \lambda_4 p_{21}) p_{23}}{\mu_3 (1 - p_{12} p_{21})} \right]\] \hspace{1cm} --- (153)

On using the values of \(F_1, F_2, F_3, F_4, F_5\) and \(F_6\), the joint probability is given by
\[ P_{n_1,n_2,n_3,n_4,n_5,n_6} = \rho_1^{n_1} \rho_2^{n_2} \rho_3^{n_3} \rho_4^{n_4} \rho_5^{n_5} \rho_6^{n_6} (1-\rho_1)(1-\rho_2)(1-\rho_3)(1-\rho_4)(1-\rho_5)(1-\rho_6) \]

Where \( \rho_1 = 1-F_1, \rho_2 = 1-F_2, \rho_3 = 1-F_3, \rho_4 = 1-F_4, \rho_5 = 1-F_5, \rho_6 = 1-F_6. \)

Further the solution in a steady state condition exist if \( \rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6 < 1. \)

### 2.4 Mean Queue Length

Average number of the customer (L)

\[
L = L_1 + L_2 + L_3 + L_4 + L_5 + L_6
\]

Where

\[
L_1 = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \sum_{n_5=0}^{\infty} \sum_{n_6=0}^{\infty} n_1 P_{n_1,n_2,n_3,n_4,n_5,n_6}
\]

\[
= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \sum_{n_5=0}^{\infty} \sum_{n_6=0}^{\infty} n_1 \rho_1^{n_1} \rho_2^{n_2} \rho_3^{n_3} \rho_4^{n_4} \rho_5^{n_5} \rho_6^{n_6} (1-\rho_1)(1-\rho_2)(1-\rho_3)(1-\rho_4)(1-\rho_5)(1-\rho_6)
\]

\[
= \frac{\rho_1}{1-\rho_1}
\]

Similarly \( L_2 = \frac{\rho_2}{1-\rho_2}, L_3 = \frac{\rho_3}{1-\rho_3}, L_4 = \frac{\rho_4}{1-\rho_4}, L_5 = \frac{\rho_5}{1-\rho_5}, L_6 = \frac{\rho_6}{1-\rho_6}. \)
2.5 Variance of Queue

\[
V(n_1+n_2+n_3+n_4+n_5+n_6)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty}(n_1+n_2+n_3+n_4+n_5+n_6)^2P_{n_1,n_2,n_3,n_4,n_5,n_6}-\bar{L}^2
\]

\[
=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty}(n_1)^2P_{n_1,n_2,n_3,n_4,n_5,n_6}+\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty}(n_2)^2P_{n_1,n_2,n_3,n_4,n_5,n_6}+\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty}(n_3)^2P_{n_1,n_2,n_3,n_4,n_5,n_6}+\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty}(n_4)^2P_{n_1,n_2,n_3,n_4,n_5,n_6}+\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty}(n_5)^2P_{n_1,n_2,n_3,n_4,n_5,n_6}+\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty}(n_6)^2P_{n_1,n_2,n_3,n_4,n_5,n_6}+\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}\sum_{n_6=0}^{\infty}(n_1n_2n_3n_4n_5n_6)P_{n_1,n_2,n_3,n_4,n_5,n_6}-\bar{L}^2
\]

\[
V = \frac{\rho_1}{(1-\rho_1)^2} + \frac{\rho_2}{(1-\rho_2)^2} + \frac{\rho_3}{(1-\rho_3)^2} + \frac{\rho_4}{(1-\rho_4)^2} + \frac{\rho_5}{(1-\rho_5)^2} + \frac{\rho_6}{(1-\rho_6)^2}
\]

2.6 Algorithm

The following algorithm provides the procedure to determine the joint probability and various queues characteristics of above discussed queueing model:

Step 1: Obtain the number of customers \(n_1, n_2, n_3, n_4, n_5, n_6\).

Step 2: Obtain the values of mean service rate \(\mu_1, \mu_2, \mu_1', \mu_2', \mu_3, \mu_3\).

Step 3: Obtain the values of mean arrival rate \(\lambda_1, \lambda_2, \lambda_1', \lambda_2', \lambda_3\).

Step 4: Obtain the values of the probabilities \(P_{11}, P_{13}, P_{21}, P_{23}\).

Step 5: Calculate the values of
i. \[ F_1 = 1 - \frac{\lambda_1 + \lambda_2 p_{21}}{(1 - p_{12} p_{21}) \mu_1}. \]

ii. \[ F_2 = 1 - \frac{\lambda_2 + \lambda_4 p_{12}}{(1 - p_{12} p_{21}) \mu_2}. \]

iii. \[ F_3 = 1 - \frac{\lambda_1'}{\mu_1}. \]

iv. \[ F_4 = 1 - \frac{\lambda_2'}{\mu_2}. \]

v. \[ F_5 = 1 - \frac{\lambda_3'}{\mu_3}. \]

vi. \[ F_6 = 1 - \left[ \frac{\lambda_1' + \lambda_2' + \lambda_3'}{\mu_3} + \frac{(\lambda_1 + \lambda_2 p_{12}) p_{13} + (\lambda_2 + \lambda_4 p_{21}) p_{23}}{\mu_3 (1 - p_{12} p_{21})} \right]. \]

vii. \[ \lambda = \lambda_1 + \lambda_2 + \lambda_1' + \lambda_2' + \lambda_3'. \]

Step 6: Calculate:

i. \[ \rho_1 = 1 - F_1. \]

ii. \[ \rho_2 = 1 - F_2. \]

iii. \[ \rho_3 = 1 - F_3. \]

iv. \[ \rho_4 = 1 - F_4. \]

v. \[ \rho_5 = 1 - F_5. \]

vi. \[ \rho_6 = 1 - F_6. \]
Step 7: Check: $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6 < 1$, if so then go to step no 8, else steady state condition does not holds good.

Step 8: The joint probability

$$P_{n_1,n_2,n_3,n_4,n_5,n_6} = \rho_1^{n_1}\rho_2^{n_2}\rho_3^{n_3}\rho_4^{n_4}\rho_5^{n_5}\rho_6^{n_6} (1-\rho_1)(1-\rho_2)(1-\rho_3)(1-\rho_4)(1-\rho_5)(1-\rho_6)$$

Step 9: Calculate average no. of customers (Mean Queue Length)

$$L = \frac{\rho_1}{1-\rho_1} + \frac{\rho_2}{1-\rho_2} + \frac{\rho_3}{1-\rho_3} + \frac{\rho_4}{1-\rho_4} + \frac{\rho_5}{1-\rho_5} + \frac{\rho_6}{1-\rho_6}.$$  

Step 10: Calculate Variance of queue:

$$V = \frac{\rho_1}{(1-\rho_1)^2} + \frac{\rho_2}{(1-\rho_2)^2} + \frac{\rho_3}{(1-\rho_3)^2} + \frac{\rho_4}{(1-\rho_4)^2} + \frac{\rho_5}{(1-\rho_5)^2} + \frac{\rho_6}{(1-\rho_6)^2}.$$  

Step 11: Calculate average waiting time for customer $E(w) = \frac{L}{\lambda}$.

2.6 Programme

#include<iostream.h>
#include<stdio.h>
#include<conio.h>
#include<process.h>
#include<math.h>

int n[6],u[6],L[5];
float p[4],r[6];
float a1,b1,a2,b2,a3,b3,c1,c2,c3,P,Q,V,W,M;

void main()
{
    clrscr();
}
cout<<"Enter the number of customers and values of mean service rate"
for(int i=1;i<=6;i++)
{
    cout<<"Enter the number of customers "<<i<<":";
    cin>>n[i];
    cout<<"Enter the value of mean service rate"<<i<<":";
    cin>>u[i];
}
cout<<"Enter the values of mean Arrival rate"
for(int j=1;j<=5;j++)
{
    cout<<"Enter the value of mean arrival rate"<<j<<":";
    cin>>L[j];
}
cout<<"Enter the values of probabilities"
for(int k=1;k<=4;k++)
{
    cout<<"Enter the value of probability"<<k<<":";
    cin>>p[k];
}
a1=L[1]+L[2]*p[3];
b1=(1-p[1]*p[3])*u[1];
r[1]=a1/b1;
a2=L[2]+L[1]*p[1];
b2=(1-p[1]*p[3])*u[2];
\[ c_1 = \frac{a_3}{u[6]}; \]
\[ c_2 = u[6]*(1-p[1]*p[3]); c_3 = b_3/c_2; r[6] = c_1 + c_3; \]
\[ \text{if}(r[1], r[2], r[3], r[4], r[5], r[6] > 1) \]
\{ 
  \text{cout}<<"Steady state condition does not holds good...\nExiting"; 
  \text{getch}(); 
  \text{exit}(0); 
\}
\[ P = (\text{pow}(r[1],n[1]))*(\text{pow}(r[2],n[2]))*(\text{pow}(r[3],n[3]))*(\text{pow}(r[4],n[4]))*(\text{pow}(r[5],n[5]))*(\text{pow}(r[6],n[6]))*(1-r[1])*(1-r[2])*(1-r[3])*(1-r[4])*(1-r[5])*(1-r[6]); \]
\[ \text{cout}<<"The joint probability is:}"<<P<<"\n"; \]
\[ Q = (r[1]/(1-r[1]))+(r[2]/(1-r[2]))+(r[3]/(1-r[3]))+(r[4]/(1-r[4]))+(r[5]/(1-r[5]))+(r[6]/(1-r[6])); \]
\[ \text{cout}<<"The mean queue length is :}"<<Q<<"\n"; \]
\[ V = (r[1]/\text{pow}((1-r[1]),2))+(r[2]/\text{pow}((1-r[2]),2))+(r[3]/\text{pow}((1-r[3]),2))+(r[4]/\text{pow}((1-r[4]),2))+(r[5]/\text{pow}((1-r[5]),2))+(r[6]/\text{pow}((1-r[6]),2)); \]
\[ \text{cout}<<"The Variance of Queue is :}"<<V<<"\n"; \]
\[ W = Q/M; \]
\[ \text{cout}<<"Average waiting time for the customer is:"]<<W<<"\n"; \]
\text{getch}();
\}

2.7 Numerical Illustration
Given customers coming to three servers out of which one server consist two biserial channels and other consist of three parallel service channels and further these two service channels are linked with common server.

The number of customers, mean service rate, mean arrival rate and associated probabilities are given as follows:

<table>
<thead>
<tr>
<th>S.No.</th>
<th>No. of Customers</th>
<th>Mean Service Rate</th>
<th>Mean arrival rate</th>
<th>Probabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( n_1 = 5 )</td>
<td>( \mu_1 = 10 )</td>
<td>( \lambda_1 = 4 )</td>
<td>( p_{12} = 0.4 )</td>
</tr>
<tr>
<td>2</td>
<td>( n_2 = 8 )</td>
<td>( \mu_2 = 9 )</td>
<td>( \lambda_2 = 5 )</td>
<td>( p_{13} = 0.6 )</td>
</tr>
<tr>
<td>3</td>
<td>( n_3 = 6 )</td>
<td>( \mu_3 = 7 )</td>
<td>( \lambda_3 = 3 )</td>
<td>( p_{21} = 0.5 )</td>
</tr>
<tr>
<td>4</td>
<td>( n_4 = 3 )</td>
<td>( \mu_4 = 6 )</td>
<td>( \lambda_4 = 5 )</td>
<td>( p_{23} = 0.5 )</td>
</tr>
<tr>
<td>5</td>
<td>( n_5 = 4 )</td>
<td>( \mu_5 = 5 )</td>
<td>( \lambda_5 = 4 )</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>( n_6 = 20 )</td>
<td>( \mu_6 = 22 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| Tableau 2.1 |

Find the joint probability, mean queue length, variance of queue and average waiting time for customers.

Solution:- We have

\[
\rho_1 = \frac{\lambda_1 + \lambda_2 p_{21}}{(1 - p_{12} p_{21}) \mu_1} = \frac{4 + 5 \times 0.5}{(1 - 0.4 \times 0.5)0.1} = \frac{6.5}{8} = 0.8125
\]

\[
\rho_2 = \frac{\lambda_2 + \lambda_4 p_{12}}{(1 - p_{12} p_{21}) \mu_2} = \frac{5 + 4 \times 0.4}{(1 - 0.4 \times 0.5)9} = \frac{6.6}{7.2} = 0.9166
\]

\[
\rho_3 = \frac{\lambda_3'}{\mu_3} = \frac{3}{7} = 0.4285
\]
\[ \rho_4 = \frac{\lambda_2}{\mu_2} = \frac{5}{6} = 0.8333 \]
\[ \rho_5 = \frac{\lambda_3}{\mu_3} = \frac{4}{5} = 0.8 \]
\[ \rho_6 = \frac{\lambda_3 + \lambda_2 + \lambda_3}{\mu_5} + \frac{(\lambda_4 + \lambda_2 p_{12}) p_{13} + (\lambda_5 + \lambda_4 p_{21}) p_{23}}{\mu_5 (1-p_{12} p_{21})} = \frac{12}{22} + \frac{(4+5 \times 0.5)0.6 + (5+4 \times 0.4)0.5}{22(1-0.4 \times 0.5)} = 0.9545 \]

The joint probability is
\[ P_{n_1,n_2,n_3,n_4,n_5,n_6} = \rho_1^{n_1} \rho_2^{n_2} \rho_3^{n_3} \rho_4^{n_4} \rho_5^{n_5} \rho_6^{n_6} (1-\rho_1)(1-\rho_2)(1-\rho_3)(1-\rho_4)(1-\rho_5)(1-\rho_6) \]
\[ = \left( \frac{6.5}{8} \right)^5 \left( \frac{66}{72} \right)^8 \left( \frac{3}{6} \right)^6 \left( \frac{5}{4} \right)^3 \left( \frac{4}{5} \right)^4 (0.9545)^{20} \left( 1-\frac{6.5}{8} \right) \left( 1-\frac{3}{7} \right) \left( 1-\frac{5}{6} \right) \left( 1-\frac{4}{5} \right)(1-0.9545) \]
\[ = 1.3833 \times 10^{-9} \]

The Mean queue length (Average no. of customers)
\[ L = \frac{\rho_1}{1-\rho_1} + \frac{\rho_2}{1-\rho_2} + \frac{\rho_3}{1-\rho_3} + \frac{\rho_4}{1-\rho_4} + \frac{\rho_5}{1-\rho_5} + \frac{\rho_6}{1-\rho_6} \]
\[ = \frac{0.8125}{1-0.8125} + \frac{0.9166}{1-0.9166} + \frac{0.4285}{1-0.4285} + \frac{0.8333}{1-0.8333} + \frac{0.8}{1-0.8} + \frac{0.9545}{1-0.9545} \]
\[ = 46.0833 \]

Variance of queue
\[ V = \frac{\rho_1}{(1-\rho_1)^2} + \frac{\rho_2}{(1-\rho_2)^2} + \frac{\rho_3}{(1-\rho_3)^2} + \frac{\rho_4}{(1-\rho_4)^2} + \frac{\rho_5}{(1-\rho_5)^2} + \frac{\rho_6}{(1-\rho_6)^2} \]
\[ = \frac{0.8125}{(1-0.8125)^2} + \frac{0.9166}{(1-0.9166)^2} + \frac{0.4285}{(1-0.4285)^2} + \frac{0.8333}{(1-0.8333)^2} + \frac{0.8}{(1-0.8)^2} + \frac{0.9545}{(1-0.9545)^2} \]
\[ = 668.4245 \]

Average waiting time for customer
\[ E(w) = \frac{L}{\lambda} = \frac{46.0833}{21} = 2.1944. \]

### 2.9. Conclusions

In this chapter the steady state analysis of a complex network of queueing model in which a common channel is linked in series with each of two systems, one containing two bi-serial channels and other three parallel channels in series is studied. The model finds its application in decision making in the process industries, in banking system, in networking and many administrative setups and business service.