CHAPTER-1

INTRODUCTION

1.1 Overview of Operation Research

The changing Business environment, globalization and the Information revolution has posed increasing challenges for the organization to be competitive and productive. A new information revolution is well under way. It will surely engulf all institutions of society. It will radically change the meaning of information for both enterprises and individuals. It is not only a revolution in technology, machinery, techniques, software or speed, but also it is a revolution in concepts, it is a revolution in system approach, it is a revolution in the organization as a whole. In fact globalization becomes the reality today.

The research in operation, research in activity at every level is must. Moreover availability of high speed computer have made it possible and easier both in terms of time and cost to apply quantitative models /O.R. models to various real life problems in different types of organization like Business, Industry, Production planning and inventory control, Transportation, Military, Government sector, Health & planning, Computers network and information systems and so on. The computer software packages are useful for rapid and effective calculation which is
necessary part of O.R. approach to solve the problem. Since decisions making is a key managerial responsibility of today’s manager under intense competitions than ever before. Hence the need of effective decision making in business has been more apparent. A sound & useful decision requires a vigorous & scientific method to provide criteria and repetitive criteria fore decision regarding man machine system involving repetitive operations provides managers with quantitative basis for decision-making. More clearly, an understanding (a system of actions ) having a clear defined objective may be organized in several alternative steps by making a decision based upon choosing from a set of possible alternatives. Each choice offers its own advantages and disadvantages. Their aim is to analyze the situation critically and prepare a decision for those bearing the responsibility for a final choice which is in the best interest of the organization as a whole.

1.2 Basics of Queuing Theory

Queues (waiting lines) are a part of everyday life. We all wait in queues to buy a movie ticket, make a bank deposit, pay for groceries, mail a package, obtain food in a cafeteria, start a ride in an amusement park and vehicles waiting at petrol pumps or to pass a road bottle neck etc. A queue (waiting line) is formed when persons (units) commonly referred
as customers waiting for service when the total number of customers requiring service exceeds the number of service facilities. An amazing number of waiting time situations exist as when

- Computer programs are waiting to be processed at a computer center.
- Customers are waiting to be served at a bank teller’s window.
- Parts are waiting to be processed at a manufacturing operation.
- Machines are waiting to be repaired at a maintenance shop.
- Ships at sea, forced to wait outside the harbor because there are no free Berths at which to unload their cargoes (The delay could be very expensive indeed, due to the cost of the fuel consumed).
- Liquid entering in a sudden volume into a tank having only a small outlet pipe can be said to be queuing tank.

Queues and queuing systems have been the subject of considerable research since the appearance of the first telephone system. In addition to models originating in biology and genetics (branching processes), they have been the principal examples of realistic discrete space random process. In the years immediately following the Second World War, the problems of operation research, that is inventory and production control, aroused a new interest in this subject area. The modeling of computer system and data transmission system opened the way in the sixties to
study the queues characterized by complex service disciplines and have created the need to analyze interconnected systems. Progress in this area has been rapid and its industrial applications have been widely accepted since the seventies. At present in the computer industry, queuing network models have resulted in software packages for the automatic solution of problems arising in the design of new computers and in the evaluation and in the improvement of existing systems. The methods of queuing networks have always been a basic component of the study of the communication systems. The widespread introduction of computers into these systems have introduced their uses, in a systematic manner and new results on queuing networks in studies of the performance of large communication network are introduced.

1.2.1 Economical Aspect of Queuing theory

Waiting for service is usually an unpleasant experience, which creates competition among customers who may switch to competitive service offerings. Having to wait is not just a petty personal annoyance. Waiting time represents the loss of valuable resources and this loss translates into psychological as well as economical costs of waiting. The longer customers wait, the more dissatisfied they are to be. In fact, the customer becomes dissatisfied because of the total length of time they have to be
spending before being served. If customer decision making can be predicted using the standard queuing theory results, a powerful set of tools would be available to customer service managers. The amount of time that a nation’s population waste by waiting in queues in a major factor in both the quality of life there and the efficiency of the nation’s economy. Here we find an observation before its dissolution; the U.S.S.R. was notorious for the tremendously long queues to purchase basic necessities as breads etc. Even in our country, it has been estimated that Indians spend about 5000-6000 crore hours per year waiting in queues. If this time could be spent productively instead, It would about nearly 20 million person years of useful work each year. Taking machines wait to be repaired may result in loss of production. Airlines waiting to take off or land may disrupt later travel schedule. Delaying service jobs extend the due date may disrupt subsequent production and hence future business. Solving problems in an industry are more serious than those experienced in daily life. Similarly, the cost of working of a workman at the tool crib for few minutes per day along may run into few thousands of rupees per month on additional wages, production losses and crest of unabsorbed overheads. One can shorten or prevent queues by employing addition service but that may not be always desirable. The cost of addition service facilities reduces margin of profit while excessive waiting time results in
loss of sales /loss of customers. Queueing theory gives mathematical
treatment to the waiting time problems. It helps to strike an optimum
balance between ‘the cost associated with waiting’ and ‘the cost
associated with prevention of waiting’.

The optimum balance may be arrived at either by:

a) Introducing more facilities at extra cost

b) Displacing less effective service facilities by more effective ones

c) Changing the pattern of the arrival of customers for service

d) Rescheduling arrival of customers

e) Changing pattern of servicing

f) Effective method to reduce service time.

Thus the main reason for the study of queues, is to enable us, through
our models, to predict what will happen if certain changes are made. The
changes that can be made could be in the pattern of arrivals, reducing the
mean length of service time, increasing the number of servers or
increasing the amount of room to accommodate waiting customers etc.
Typical measure of congestion is the average numbers of customer
waiting for the service.

1.2.2 The Arrival - Pattern
This is the element concerned with the pattern in which the customer arrives and join the system. An arrival pattern is characterized by:

(i) Its size.
(ii) The arrival time distribution of the customers
(iii) The attitude of the customer.

An arrival pattern is infinite or finite. It is considered infinite if the rate at which the source generates the customer in the service system (customer in queue and those in service is not affected by the number of customers in service). Alternatively, the source is considered finite if the rate is affected by the no. of consumers in the system. Arrival pattern in which the value of parameters of a probability distribution describing the input process are time independent, is called a stationary arrival pattern and the pattern which is not time independent is termed as a non-stationary arrival pattern.

1.2.3 The Service Pattern

The service pattern takes a variety of patterns and can be regular (e.g. machining identical part) or virtually instant with periods of no services at all (e.g. passengers boarding a bus stop) or using a negative exponential distribution. Arrivals normally involve discrete distribution but service time involves continuous distribution. Consequently we cannot talk of the
probability of the service time being an exact period, but only the probability of the service being completed during a given period. This probability is obtained by finding the area below the distribution curve that relates to the period given.

A further important term in queuing theory is the average service rate (symbolized as $\mu$). This is simply the average number of queue components that can be serviced per minute or hour or day etc.

1.2.4 The Traffic Intensity

Traffic intensity ($\rho$) is the ratio of the average arrival rate to the average service rate i.e. $\rho = \lambda / \mu$.

1.2.5 The Number of Service – Channels

Arrival and service pattern are often beyond our immediate control but one thing the executives usually have full control over is the number of service channels (check out counters, maintenance men, bus schedules etc.). Indeed, the object of most queue analysis is to determine the optimum number of service channels. The number of service channels refers to the number of parallel service station which can serve customers simultaneously.
1.2.6 The Queue Discipline

Although most queues are based on first come first server (FCFS) system. This is not always so, sometimes certain queue components have priorities (e.g. ambulances at traffic lights), sometimes last in first out (LIFO) system works (as when a tax inspector always takes the top document first form his tray), and sometimes components are served at random.

Therefore, there are different methods of determining the order of service. The method appertaining to any situation is known as **Queue – Discipline**.

1.2.7 Service facility

Service facility represents servers-clerks, maintenance crew and machines which render service. To analyse a service facility, both “number of servers” and “arrangements of servers” (in parallel or in series etc.) need to be considered. The Service facility can be arranged in any one of the following types:

1.2.7.1 Single Queue: Single Server

Under this arrangement, there is one queue and one server in the service facility
1.2.7.2 Single Queue: Multiple Servers (Series Arrangement)

Following queue system self explained as in following figure:

1.2.7.3 Single Queue: Multiple servers (Parallel Arrangement)

Under this arrangement, there is one queue but more servers in the service facility are placed in a parallel arrangement.
1.2.7.4 Multiple Queue : Multiple servers

The following queuing system is selfexplained as in the following figure.

1.2.8 Customers Behavior
The human behavior and the facility of servicing in any system are also necessary to know for the development of queuing problem. Customer’s behavior can be classified in following categories:

**1.2.8.1 Balking**

A customer may not like to join the queue as he may not like to wait. First of all, Height introduced the concept of Balking in the problem for a single queue in equilibrium with Poisson input and exponential holding time. Finch studied the similar problem for general input distribution.

**1.2.8.2 Reneging**

The customer may leave the queue due to impatience. The reneging process has been introduced by Barrer.

**1.2.8.3 Collusion**

Only few of new customers may join the queue and hence, some of them may demand service on their behalf as well as on behalf of some other customers. This action is called ‘collusion’.

**1.2.8.4. Jockeying**
A customer may help on switching from one queue to another. This happens when there are more than one service counter. The tendency so observed called jockeying. These situations all relate to examples of queues with impatient customers.

1.2.9 Steady Transient and Explosive States in a Queue System

In the case of arrivals in a queue system three states of nature are possible, namely, the steady state, transient state and explosive state. A short description of these states is given below:

1.2.9.1 Transient State

A system is said to be in transient state when its operating characteristics are dependent on time. Usually a system is in transient state during the early stages of its operation when its behavior of the system is dependent on the initial state of queue, i.e. when the probability distribution of arrivals waiting time and servicing time are dependent on time the system the system is said to be in transient state.

1.2.9.2 Steady State
If the average rate of arrival is less than the average rate of service, and both are constant, the system will eventually settled down into steady state and become independent of the initial state of queue. The probability of finding a particular length of queue at any time will be same. The size of queue will fluctuate in the steady state but the statistical behavior of the queue remains steady.

A necessary condition for the steady state to be reached is that the elapsed time since the start of the operation becomes sufficiently large i.e. \( t \to \infty \). If the rate of arrival is greater than the rate of service than a steady state can not be reached. Here we assume that the system acquires a steady state as the number of arrivals during it certain interval becomes independent of time.

Hence in the steady state system the probability distributions of arrivals, waiting time and servicing time does not depend on time.

1.2.9.3 Explosive State

In the explosive state waiting line increases indefinitely with time. For example, arrivals in the restaurant during rush hours.

1.2.10 Stochastic Processes
A stochastic process \( \{X(t), t \in T\} \) is called a collection of random variables that is, for each \( t \in T \), there is a random variable \( X(t) \) denotes the state of the system at time \( t \). If \( t \) assume discrete values then the process \( \{X(t), t = 1, 2, \ldots\} \) is called a discrete parametric stochastic process. If \( t \) assumes continues values then the process \( \{X(l), l \geq 0\} \) is called as a continuous parametric stochastic process. The set of possible values of \( X(t) \) is called state space and set of possible values of \( t \), denoted by \( T \) is called the parametric space or the index set.

Generally, in the stochastic processes encountered in the analysis of a specified complex system can be identified as one of the following:

1.2.10.1 Poisson Process

It is applied to a system when the changes are independent of time i.e. the factors which affects the changes remain independent of time. An infinite sequence of independent events occurring at an instant of time from a Poisson process, if the following conditions are satisfied:

1. The total number of events in any interval \( X \) does not depend on the events which occurred before the beginning of the period i.e. the number of arrivals in non-overlapping interval are statistically independent and the process has independent and identically distributed increments.
2. The probability of an event occurring in a small time interval $\Delta t$ is $\lambda \Delta t + O(\Delta t)^2$ where $\lambda$ is some constant and $O(\Delta t)^2$ means some function of $\Delta t$ of order $\geq 2$, i.e. $f(\Delta t)/\Delta t < K$ as $\Delta t \to 0$ for any constant $K \geq 0$.

3. Two or more units can not arrived or serviced at the same time i.e. the probability of the occurrence of more than one event in the interval $t$ and $t + \Delta t$ is of the order $O(\Delta t)$ which is negligible.

1.2.10.2 Markov Process

A stochastic process $\{X(t) : t \in T\}$ is said to be a Markov Process if for any set of $n$ points $t_1 \leq t_2 \leq \ldots \leq t_n$

$$P[X(t_n) \leq x_n \mid X(t_1) = x_1, \ldots, x(t_{n-1}) = x_{n-1}]$$

$$= P[X(t_n) \leq x_n \mid X(t_{n-1}) = (x)]$$

A stochastic process in a physical system $S$ is known as a Markov Process (or a process without after effect) if the occurrence of any future state of the system is independent of any past state and depends only the present state. It means for a given present, the future is independent of the past.

A Markov process whose state space is a discrete is called Markov chain.

A Markov chain is called a discrete parameter Markov chain or a continuous parameter Markov chain according as the parametric space discrete or continuous.
1.2.11 Literature Survey

1.2.11.1 Multiple- Input Queuing

The Interesting and more practical idea of multi- input in queuing theory were stated by Payday as follows:

In the classical queuing modals some Poisson arrivals with service time distribution obeying negative exponential law. Further, we assume that that probability of more than one arrival in the small interval of time is zero. However in practice arrivals to a single facility may be form a number of sources. These sources may be independent of each other working at different service rate and in such cases the possibility of one or more unit arriving simultaneously for the service to the same station can not be ruled out. In this thesis waiting line situation has been considered where multiple arrivals at different rates take place to a single service facility. It has been assumed that more than one arrival but from different sources, take place simultaneously.

1.2.11.2 System with Priority

In manipulating queuing situations one component which it may be possible to adjust in order to improve performance in the situation is queue discipline and adjustment (if any) is the introduction of a
classification of customer and an associated system of priorities. For example, two classes of customer given preferential treatment as compared to those in the second. One extreme kind of preferential would be to ensure that a class-I customer never had to queue except when the service mechanism was busy serving customers of the same class. In this case the arrival of class-I customer in the system at a time when class-II customer was in service would cause the interruption of class-II customer’s service in order to serve the class-I customer. This kind of priority is pre-emptive in that the arriving class-I customer preempts the position in service of class--II customer who only returns to complete his service when there is no class-I customer requiring service with system. The pre-emptive priority queuing problem with exponential arrival and service time has been discussed by several authors. In particular Heathcoat, Miller has given the steady state solution for the pre-emptive resume problem and an approximate solution for the repeat discipline. For a detailed account of priority disciplines, one may refer to Jaiswal. There is no. of priority discipline according to which the customers are divided into two or more priority classes and a higher priority unit is chosen for service before a lower priority unit. Further under a preemptive priority rule, a lower priority unit is preempted or taken out of service whenever a higher priority unit arrives during its service the
preempted lower priority unit being attended only after serving all the units of higher priority classes.
Under non preemptive priority rule a service once started is allowed continuing up to completion, irrespective of arrival of higher priority units, and they are attended only after completion of the service in progress. There are also dynamic priority disciplines where the priority increases due to waiting.

1.2.11.3. Multiple Queuing Systems
A multistage queuing system would be like a physical examination procedure, where each patient must precede through several stages such as medical examination of ears, nose and throat examination blood test, electrocardiogram, eye examination etc.; in certain multistage queuing processes recycling may start. Recycling is common in manufacturing-processes where quality/control inspection are performed after certain stages and parts that do not meet quality standard are send back for reprocessing.

1.2.11.4 Queues in Series
There are situations where an arriving unit must pass successively through several distinct phase service channels before its service is
completed. Obrein [32] was the first to tackle the problem of queue in series taking Poisson input and exponential holding time. The queues in series is illustrated in the following figure

![Queue in series diagram](image)

**Fig: 1.6: Queue in series**

It was be noted that in the queue with Erlangian phase type service a customer is admitted into the service only after its predecessor has completed all its phases of service of where as forming the queues in tandem/ series, the different parts of the service are served at different successive service stations with unlimited/limited queuing allowed before each service station.

### 1.2.11.5 Bitandem/Biseries Queues

A Bitandem / Biseries queue has been observed practically in a queuing system having two queues Q₁ and Q₂ with servers S₁ and S₂. There are situation where the output from Q₁ or the part of it constitute the input to Q₂ and vice versa i.e. the output from Q₂ or part of it forms the input to Q₁ such an arrangement of two queues in series is termed as queues in...
biseries/ bitendem. This concept in queuing theory was introduced by Maggu.

![Queue in Biseries Diagram]

**Fig. 1.7: Queue in Biseries**

In the present study the concept of biseries has been frequently used. Hence it is better to discuss in brief the complete original model discussed by Maggu in biseries as follows:-

Let there are two services channels $S_1$ and $S_2$ assembling type 1 and type 2 components. Let the queuing system consist of two queues $Q_1$ and $Q_2$ which is being attended by $S_1$ and $S_2$. Poisson arrivals from outside the system occur with mean rates $\lambda_1$ and $\lambda_2$ and join the respective queues $Q_1$ and $Q_2$ before the servers $S_1$ and $S_2$. It is assumed that the service time distributions for the queues $Q_1$ and $Q_2$ are mutually independent, negative exponentially distributed with respective parameters $\mu_1$ and $\mu_2$. Further it is assumed that $P_1, P_2 \geq 0$ (or $P_1 + P_2 = 1$) denote respectively the probabilities. If a unit completing service at $S_1$ to either leaves the system or enters the subsequent service channels $S_2$. Also let $q_1, q_2 \geq 0$ (or $q_1 + q_2$
=1) be the respective probabilities of a unit having received service at $S_1$ to either depart from the system or go through the subsequent phase channel $S_2$. Maggu obtained the following differential equation

$$P(m,n,t) = - (\lambda_1 + \lambda_2 + \mu_1 + \mu_2) p_{m,n}(t) + \lambda_1 p_{m-1,n}(t) + \lambda_2 p_{m,n-1}(t) + \mu_1 p_{m+1,n}(t) + \mu_2 q_{1} p_{m+1,n+1}(t) + \mu_2 q_{2} p_{m-1,n+1}(t)$$

for $m,n \geq 0$

By using generating function technique (g.f.t), he obtained the implicit form of probability generating function of the queues lengths in terms of Laplace transforms. In the steady state of following expression has been obtained for the expected queues length of the system.

$$E(Q) = \frac{\lambda_1 + \lambda_2 q_2}{\mu_1 (1 - p_1 q_1) - (\lambda_1 + \lambda_2 q_2)} + \frac{\lambda_2 + \lambda_1 p_2}{\mu_2 (1 - p_2 q_2) - (\lambda_2 + \lambda_1 p_2)}$$

The biseries concept was further developed by T.P. Singh discussing different types of models. In this thesis the concept of biseries is used frequently.

### 1.2.11.6 Cyclic –Queues

A Cyclic queuing system consists of several service channels arranged in series/random, the ends of which are joined to form a closed circuit so that the output of last channels becomes input of first. For example, a coal
cutting problem, where in a face worker worked on by cutting machine, a drilling machine a blasting crew etc., each of which on finishing and in the same order proceeds to work on another face and when finishing with all n-faces, returns to first face. The problem is considered as a set of queues in series serving to customers (faces) use ordered service, where a customer leaving the last phase of service waits for service in the queue of first phase, hence the name cyclic queues. Finch [ ] has also studied the same system of cyclic queue as described by Koingsberg [ ] except that he allowed a customer certain probability \( p_i \) to return to the \( i \)th phase. T.P. Singh studied a feedback queue model in which departure rate was taken into bulk.

If we consider a closed network of \( k \) nodes such that:

\[
P_{ij} = \begin{cases} 
1; (j = i + 1, 1 \leq i \leq k - 1) \\
1; (i = k, j = 1) \\
o; (\text{elsewhere}) 
\end{cases}
\]

Then we have a cyclic queue which is a sort of series queue in a “circle”, where the input of the last node feeds back to the first node. For single servers at each node, equations are

\[
P_{n_1, n_2, \ldots, n_k} = C p_{1}^{n_1} p_{2}^{n_2} \ldots \ldots \ldots \ldots p_{k}^{n_k} \quad \ldots \ldots (2)
\]

with \( \mu_i \rho_i = \sum_{j=1}^{k} \mu_j [n_j] p_j \) Using (1) in the traffic equation results in

\[
\mu_i \rho_i = \begin{cases} 
\mu \rho_i - 1 \rho_i - 1 (1 - z_i, \ldots, k) \\
\mu k \rho_i (1 - 1)
\end{cases}
\]
Thus we have

$$\rho_i = \begin{cases} \frac{1}{\mu_i - 1/\mu_i} \rho_1 - \frac{1}{(1 - \rho_1) \mu_i} (1 = 2, 3, \ldots, k) \\ \rho_1 \frac{\mu_i}{\mu_1} \end{cases}$$  \quad \text{--- (3)}$$

From (3) we see that

$$\rho_2 = \frac{\mu_2}{\mu_1} \rho_1, \rho_3 = \frac{\mu_3}{\mu_2} \rho_2, \ldots, \rho_{k-1} = \frac{\mu_{k-1}}{\mu_{k-2}} \rho_{k-2} = \rho_1, \rho_k = \frac{\mu_k}{\mu_{k-1}} \rho_{k-1}$$

Since one $\rho$ can be set equal to one due to redundancy, we select $\rho_1 = 1$, and substituting into (2) we obtain

$$\rho_{n_1 \ldots n_k} = \frac{\mu_{1N-n_1}}{G(N) \mu_2 n_2 \mu_3 n_3 \ldots \mu_k n_k}$$  \quad \text{.....(4)}$$

Again $G(N)$ can be found by summing over all cases $n_1 + n_2 + \ldots + n_k = N$ or by Husen’s algorithm.

### 1.2.12 Development of Queuing Network Theory

The queue is a current area of great research and application interest with many extremely difficult problems. In view of their increased applicability to modeling manufacturing facilities and computer/communication networks the reader interested in developing into this topic further is refer to Walrand.

The partial development of queuing networking theory may be seen since 1957 (tendon queues) and 1957 (queuing network). The literature on the subject has work very rapidly. We trace here the directions and
development of queuing network literature in recent years. Queuing networks theory can be generally divided into four main groups-

(i) Open unrestricted networks;
(ii) Close unrestricted networks;
(iii) Open restricted networks; and
(iv) Closed restricted networks.

We briefly review the development of the multistage queuing networks models for all the above four groups separately.

Some terms used in queuing networks theory are defined below-

1.2.12.1 Blocking

Blocking is the situation that takes place when a customer has completed service in the first stage but can’t proceed because the second station & queues are completely filled. There are many types of blocking, but the most often dealt with in the literature are- blocking after service and blocking before service. Blocking after service occurs when a station has completed the service of customer while there is no space for it in the next queue. In this case the first station will house this customer and can’t start serving the next one. So the customer will remain there (at the first station) until a customer departs to the next station. In blocking before service, a station can serve the customer only when there is a space for it
in the next queue. These two types of blocking are sometime called manufacturing blocking and communication blocking.

1.2.12.2 A Cyclic Network

It is an open network in which each customer can visit any one only once.

1.2.12.3 Sojourn Time

The Sojourn time of a customer in a queue, is the sum of its delay, if any, plus its service time.

1.2.12.4 Erogodicity

The queuing system is said to be erogodic if the joint distribution of waiting times of the \( n^{th} \) customer in the first and the second queues, converges, as \( n \to \infty \), to a probability distribution.

1.2.12.5 Zero- Switching Rule

The server stay until its queue becomes empty and then it switches to the other stage.

1.2.13 Usages
Queuing networks models appear in many important and diverse areas such as manufacturing systems, communication systems, multi programmed computer systems, scheduling networks of queues, congestion hospital facilities, machines, serial production lines, satellite network, air traffic control, maintenance and repair facilities, etc. The development of this area can be seen in the warland.

1.2.14 Two-Stage Queuing Networks

Tandem queuing systems, in which the input from one queuing process serves as the input to another, have been studied by many authors. The first work of sequences of queues in tandem starts with work of Jackson’s & Taylor and Jackson Taylor & Jackson [ ] applied queuing theory to the provisioning of spare engines. They studied an open queue in tandem system with a finite no. of customers. So they were considered the first in introduce the idea of cyclic queues. Jackson [ ] dervies the differential- difference equations characterizing the system of two queues in series and obtained the steady state solution:

\[ P_{(n_1,n_2)} = \left( \frac{\lambda}{\mu_1} \right)^{n_1} \left( \frac{\lambda}{\mu_2} \right)^{n_2} P_{0,0} \]

Where \( P_{0,0} = \left( 1 - \frac{\lambda}{\mu_1} \right) \left( 1 - \frac{\lambda}{\mu_2} \right) \)

The average no. of customers in the system is
The no. of customer waiting for service in the system (not including in the service is

\[ \frac{\lambda/\mu_1}{1-\lambda/\mu_1} + \frac{\lambda/\mu_2}{1-\lambda/\mu_2} \]

Probability that there are \( n \) customers in phase one is

\[ \left( \frac{\lambda}{\mu_1} \right)^n \left( 1 - \frac{\lambda}{\mu_1} \right) \]

Probability that there are \( n \) customers in phase two is

\[ \left( \frac{\lambda}{\mu_2} \right)^n \left( 1 - \frac{\lambda}{\mu_2} \right) \]

Jackson was considered the first two introduce the concept of “Product form in the theory of queues. Also he studied the problem of two queues in tandem with required number of customers. He derived the differential difference equation describing this model and obtained the steady state solution which and are similar to the unrestricted case with different value for \( P_{0,0} \).

Hunt [17] studied some exponential service cases for a limited no. of servers and obtained few results concerning the whole system. He studied queuing system consisting of two servers in tandem with infinite no. of customer before the two servers. The inter-arrival times are eventfully
distributed, mutually independent random variables with distribution function.

\[ F(x) = \{1-e\} \]

The service times at both stations are mutually independent. He derived the queue time and waiting time distribution at the second serves. He also considered a tandem queue model with blocking where he studied a Markov chain imbedded in the process. The transient solutions of the above model were obtained by Prabhv. Tander queues with blocking can be studied in the terms of an imbedded Markov process.

### 1.3 Basics of Scheduling Theory

This section of the introductory chapter deals with the second category of waiting time models pertaining to scheduling /sequencing a given no. of jobs for processing them on a finite number of machines. Owing to combinatorial nature of scheduling and the specific key characteristic i.e. uncertainly faced by schedulers, exact practical solution procedures to most scheduling problems have not yet been found, so far, even though computational abilities of the electronic data processing equipments are frequently advancing to date.

Scheduling n jobs on m machines is one of the classical combinational search problems in the field of operation research and combinatorial
Interests in mathematical formulation and solution of various type of scheduling problems have reported a considerable research papers in this field. Though a lot of work on scheduling has been done to this date, yet there are practical productions scheduling problems which can’t be optimized effectively and efficiently by the recent well known techniques in the field. In this thesis efforts are made to deal with real scheduling problems through heuristic techniques.

Sequencing simply refers to the determination of order in which the jobs are to be processed on various machines to find a definite performance measure while scheduling refers to the time table that includes the start time and completion time of jobs on machines. Scheduling contains time tabling as well as sequencing information. The two terms, viz, sequencing and scheduling though distinct to some extent are used as synonymous terms in this thesis.

The scheduling/sequencing problems are common occurrence in our daily life e.g. ordering of jobs for processing in a manufacturing plant, waiting aircraft for landing clearance, programs to be run in a sequence at a computer center etc. Such problems exist whenever there is an alternative choice in which a no. of jobs can be done. The selection of an appropriate order or sequence in which to receive waiting customer is called sequencing. As with others operational research problems the objective is
to optimize the use of available facilities to effectively process the item or the jobs.

It is also essential to select from the situation and its environment a criterion or objective function. If it is assumed that time required to prepare the processing of a particular item at a specific machine is given, the question of optimizing becomes the question to ordering the items at each facility so that the objective of minimization of total elapsed time or minimize of total waiting time or cost associated with operations, is achieved.

In industries man, machines, material and money are involved for the production of them. The manager of an industry is interested to use man, machines, material and money in an economic manner so that the cost associated with the production of an item is not increased or it is the most minimum in the competitive market.

In the last three and half decades deterministic sequencing and scheduling has received considerable attention in the literature. The scheduling field can be divided into several major areas, namely single machine scheduling, parallel machines, scheduling flow shop and open shop scheduling etc. In the last decade stochastic sequencing and scheduling has begun to receive attention in the literature too. In an industry or business activity scheduling techniques are more useful and give
scientific systems to approach to utilize machines idle times or operator’s
times to cut down or reduce the cost of production. The scheduling
methods can also suggest to a decision maker the interval of alternative
purpose of the industry or in business unit in an economical manner.

This section is divided into following sub-sections:

- Classification of scheduling models.
- Basic assumptions in flow-shop.
- Various performance measures.
- Brief literature survey.
- Summary of the thesis.

1.3.1 Classification of Scheduling Models

We now describe the different types of scheduling models:

1. Flow Shop Scheduling Model

In this model we have m machines and n jobs. The order of processing on
the various machines is same for all jobs. Also the sequence in which the
job go through the first machine has to be the same any of the subsequent
machine, i.e. a job may not ‘pass’ another job while waiting for
processing on machine. A flow-shop with this restriction is often called a
permutation flow shop.
In this flow shop there is an intermediate storage of infinite capacity between all machines. If machine \( k+1 \) is busy when a job has been completed on machine \( k \), this job will be stored between machines \( k \& k+1 \). The time for which this job is stored is called in process inventory time for the job and the model is called Flow-Shop with Infinite Intermediate Storage (FSIIS).

2. Job-Shop Scheduling Model

In this model we have \( n \) jobs and \( m \) machines and each job has its own machine order specified.

3. Open Shop Scheduling Model

In this model we have \( n \) jobs and \( m \) machines. Any given job requires an execution on each of the \( m \) machines. The order in which the job passes through a machine is immaterial.

4. Project Scheduling

It is one of the types of a project where all the resources are brought to the job.

1.3.2 Basic Assumption in Flow-Shop
1.3.2.1 Assumption Regarding Machines

- No machine processes more than one operation at a time.
- Each operation on a machine once started must be performed to its completion.
- Each operation takes finite time and it must be completed before any other operation begins. The given operation time includes set up time, until unless specified.
- There is only one machine of each type.
- Each machine operates independently of the other.

1.3.2.7 Assumption Regarding Jobs

- All jobs are available for processing at time zero.
- All jobs allow the same sequence of operations.
- Jobs are independent of each other.
- No job is processed more than once on any machine.
- Each job consists of specified number of operations and each operations performed by only one machine.
- The processing times of the jobs are independent of the order in which jobs are performed.
- Each job once started must be processed to completion.
1.3.2.7 Assumption Regarding Operating Process

- Each job is processed as early as possible.
- Each job is considered as indivisible entity even though it may consist of a number of individual units.
- Each machine is provided with sufficient waiting space for allowing job to wait before starting their processing.
- Each machine process jobs in the same sequence. i.e. no passing or overtaking of jobs is permitted.
- Transportation time of job between machines is negligible, until unless specified.
- Set up times are sequences-independent.
- The given operation times include set-up time, until unless specified.

1.3.5 Various Performance Measures

Let the job be identifies as 1,2,……..n and the m machines by M_1,M_2…..M_m.

I. **Release Time (r_i):** This is the time at which a job is released to the shop by some external job generating process it is the earliest time at
which processing of the first generation of the job. It is also known as ready time or arrival time.

II. Due-Date \( (d_i) \): It is the promise delivery date of job. It is the time when the processing of the last operation should be completed i.e., the time when the job should be completed.

III. Completion Time \( (C_i) \): It is the time at which the processing time of the last operation of the job is completed.

IV. Flow-time \( (F_i) \): It is total time that the job \( i \) spends in the shop.

\[
F_i = C_i - r_i
\]

The flow time \( (F_i) \) is also called the manufacturing interval and the shop time. When \( r_i = 0 \) then \( F_i = C_i \) i.e. completion time of job \( i \) is equal to the flow-time of job \( i \).

V. Lateness \( (L_i) \): It is the difference between the completion time of job \( i \) and its due-date.

\[
L_i = C_i - d_i
\]

VI. Tardiness \( (T_i) \): The tardiness of job = Max \( \{0, L_i\} \).

VII. Earliness \( (E_i) \): The Earliness of job = Max \( \{0, -L_i\} \).

Note: Lateness, Tardiness and Earliness are three different ways of comparing the actual completion time with the desired completion time. Lateness considers the algebraic of each job regardless of the sign of difference. Tardiness considers only positive difference, i.e. jobs which are completed after there due-dates and earliness considers only negative
difference i.e. jobs completed ahead of their due-dates. Note that when a job is early, i.e. when it completes before its due-dates, \( L \) is negative and when a job is completed after its due-dates, \( L \) is positive and non-zero and the job is tardy.

VIII. **Total Elapsed time (\( C_{\text{max}} \))**: It is defined as total completion time with the set of all jobs finish processing on all the machines. It is also known as Makespan. \( C_{\text{max}} = \text{Max}\{c_i\} : i = 1,2,\ldots,n. \)

IX. **Idle time (\( I_k \))**: It is idle time on machine \( M_k \) is denoted by \( I_k \) and is defined as:

\[
I_k = C_{\text{max}} - \sum_{i} p_{i,k}
\]

Where \( p_{i,k} \) is the processing time of job \( i \) on machine \( M_k \).

X. **Mean completion Time (\( C \))**: It is the average completion time of any job. \( C = \frac{\sum c_i}{n} \)

XI. **Mean Flow –time (\( F \))**: It is the average time of a job spends in the shop. \( F = \frac{\sum r_i}{n} \)

XII. **Penalty Cost**: It is defined as the total penalty paid by virtue of jobs being late in completion by their due-dates.

XIII. **Total Production Cost**: It is defined as the total production cost for the production of a set of products on machines.

Two performances measures are equivalent if a schedule which is optimal with respect to one performance is also optimal with respect to other
performance measure and vise-versa the performances C and F are equivalent while the measure $C_{max}$ and $F_{max}$ are not equivalent.

**XIV. Minimization of Total Rental Cost**

When machines are taken on rent, the following renting policies generally exit.

**P₁:** All the machines are taken on rent at the same time and are return also at same time.

**P₂:** All the machines are taken on rent at the same time and are returned as and when they are no longer required for processing the jobs.

**P₃:** All the machines are taken on rent as and when they are required and are returned as and when they are no longer required for processing the jobs.

We have adopted policy P₃ in this thesis and the work is carried out in chapters 3 to 6 of the thesis for the general and the special cases of the Flow-shop problems.

If the present thesis , we have considered mainly total elapsed time, mean flow-shop (for mean completion time), idle time of machine, combination of mean –flow shop and total elapsed time, total Rental cost etc. as the performance of measures in the class of sequencing problems.

**1.3.4 Components of the Production Cost**
We discuss the four components of the production cost as follows:

- **Operation Cost:** That component of cost which represents the cost incurred in actual production and which may be treated as the processing cost of the jobs on all machines is defined as the operation cost.

- **Job Waiting Cost:** That component of cost which (also called as the in-process inventory cost) reflects the opportunity cost due to the waiting of the semi-finished jobs in the shop for processing on some machines. The job which waits in the shop in the form of capital could have been utilized to produce additional return on capital.

- **Machine Ideal Cost:** when machines are ideal, some opportunity is lost because by utilizing this idle capacity of the machines, some return on machine could be obtained. Determination of machine idle cost may be divided into two categories.
  
  (a) The idle time of the machines can be utilized to perform some other work which may be as profitable as the exiting work.
  
  (b) The idle time of the machines cannot be utilized to perform any other useful work.

For case (1), the machine idle cost is the difference between the expected rates of return from the machine which is obtained by utilizing the idle capacity of the machine on a subordinate job. For case (2), the idle cost of the machine is the respected rate of return.
I. Penalty Cost of Jobs

If the jobs are not completed by their dates, certain costs are incurred. These costs includes –

- Direct dealing with the customer’s paperwork, telephone calls, taken up execution time taken up.
- Penalty clause in the contract, if any.
- Loss of good –will result in a increased probability for loosing the customer for some or all of his future business; or perhaps a damaged reputation which will turn customer’s away.
- Expediting because the jobs may have to be moved quickly through the shop at an extra set-up cost and indifferent use of work men and machinery

II. Total Opportunity Cost

The total opportunity cost of a schedule is the sum of the component of the opportunity cost i.e. operation cost, job waiting cost, machine idle cost penalty cost of jobs.

1.3.5 Idle Waiting Operating $O_{i,w}$

Let $\mathbb{R}_+$ be the set of non negative real numbers. Let $D = \mathbb{R}_+ \times \mathbb{R}_+$. Then $O_{i,w}$ defined as a mapping from $D \times D \rightarrow D$ given by

\[ O_{i,w} [(x_1,y_1), (x_2,y_2)] = (x_1,y_1) O_{i,w} (x_2,y_2) \]
\[
= \{ x_1 + \max(x_2 - y_1,0), y_2 + \max(y_1 - x_2,0) \}
\]

Where \( x_1, x_2, y_1, y_2 \in \mathbb{R}^+ \).

### 1.3.6 Concept of Group Technology in Scheduling

There are various situations where some sets of specified jobs are required to be processed together as block sequence either by a virtue of technological constraints or some extremely imposed restrictions. This type of situation is known as group technology or equivalent job block introduced by Maggu & Das it has very wide applications to a variety of production system for the purpose of improving productivity.

#### 1.3.6.1 Equivalent-Job Concept in Sequencing Theory

In the two machines, \( n \) job make-span problem the concept of equivalent job for job block in job sequencing was introduced by Maggu & Das and is as follows:

Consider the job sequence \( S = \{a_1, a_2, a_3, \ldots, a_n\} \) of \( n \)-jobs with the condition that job \( i \) and \( i+1 \) must occur in the sequence as a block i.e. if \( a_i \) is the \( i \)th job then \( a_{i+1} \) must be the \((i+1)\)th job. Now it is possible to define job \( \beta \) (say) with the processing time \( t_{\beta A} \) and \( t_{\beta B} \) on two machine A&B respectively which can replace the job \( i \) and \( i+1 \) for the purpose of finding the minimum scheduled time, \( \beta \) replace \( i \) and \( i+1 \) jobs to produce new
sequence $S'$, the completion time on both the machine is changed by a value which is independent of particular sequence $S$. Hence the substitution does not change the relative merits of different sequence.

### 1.3.7 Job – Block Theorem

The job block theorem given by Maggu and Das is as follows:

In processing a schedule $S=(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k, \alpha_m, \ldots, \alpha_n)$ of $n$-jobs on two machines $A&B$ in the order AB with no passing allowed. The job-block $(\alpha_k, \alpha_m)$ having processing times $\{A_{\alpha_k}, B_{\alpha_k}, A_{\alpha_m}, B_{\alpha_m}\}$ is equivalent to the single job (called equivalent – job $\beta$).

Now the processing times of jobs $\beta$ on the machines $A&B$ denoted respectively by $A_{\beta}$ and $B_{\beta}$ are given by:

- $A_{\beta} = A_{\alpha_k} + A_{\alpha_m} \cdot \min\{B_{\alpha_k}, A_{\alpha_m}\}$
- $B_{\beta} = B_{\alpha_k} + B_{\alpha_m} \cdot \min\{B_{\alpha_k}, A_{\alpha_m}\}$

### 1.3.8 Transportaation Time

In many practical situations of scheduling theory the machines which are considered in the production schedule are placed on the same place. Therefore the transportation time becomes significant which includes:

- Loading time of jobs.
- Moving time of jobs.
• Unloading time of job.

The sum of all the above times is designated as transportation time of jobs denoted by $I_{ij}$ for job $i$ to machine $j$. Singh T.P. has applied the transportation time in various scheduling models.

1.3.9 Methodology

1.3.9.1 Branch and Bound Technique

Branch and bound technique is most widely used in scheduling. It is an enumeration technique and is applied to optimization. It is a useful method for solving many combinatorial problems and is a general-purpose strategy for installed enumeration. As its name implies the approach consist of two fundamental procedures.

**Branching:** It is the process of partitioning a large number into two or more sub-problems.

**Bounding:** It is the process of calculating a lower bound on the optimal solution of a given sub-problem. The branching procedure replaces an original problem by a set of new problems that are mutually exclusive and exhaustive sub-programs of the original problems. Partially solved versions of the original problem and smaller problems then the original problems.
Further more, the sub-problems can themselves be portioned in a similar fashion. Branching–and-Bounding from a branching tree with vertices or nodes representing partial sequence of jobs which has been assigned positions sequence that is south to optimize the given criterion. The partial sequences represent the subject of sub-problems formed by partitioning. The first level node (p) corresponds to not having assigned any job to any position in the required sequence. From this node (p), n different branches emanate with a job fixed in the First position. Lower bounds, on the performance measure for the given problem, are calculated separately for the partial sequence starting with 1, 2, … n respectively where lower bound for any partial schedule is the lowest value of the objective function irrespective of the order in which the remaining jobs are to be possessed. The nodes or vertex with the lower bound is branched further in (n-1) nodes with the second position being occupied by the remaining (n-1) jobs. Again lower bounds are calculated with the partial sequences with the first two positions being filled. The vertex with the least lower bound among the unbranched vertices is branched further. This process continues till vertex with complete sequence is reached. If the lower bound attached with this node is less than or equal to the lower bounds attached with unbranched vertices, the complete sequence represented by this node is an optimal sequences.
otherwise the node with lesser lower bound is branched further and the optimal sequences is obtained.

The minimum number of nodes in Branch and bond technique, which can be created is \( \frac{n(n+1)}{2} \). This happens when each stage of further branching only one is chosen for the purpose and the value of the complete sequence so obtained is less than all lower bonds. The maximum no. of nodes, in Branch-and-bound method, for being created is \( 1 + n + n(n-1) + n(n-1)(n-2) + \ldots + n! \), which is when all the vertices are branched further where \( n \) is the no. of jobs.

### 1.3.9.2 Heuristic Approaches

The Branch and Bound approach and the Elimination approach have two inevitable disadvantages, which are typical of implicit enumeration methods. First, the computational requirement will be severe for long problems. Second, even for relatively small problems, there is no guarantee that the solution can be obtain quickly, since the extent of the partial enumeration depends on the data in the problem. Heuristic algorithms avoid these two drawbacks. They can obtain solution to large problems with limited computation effort, and their computational requirements are predictable for problems of given size. The drawback of heuristic approaches is that they do not guarantee optimally, in some
instances it may even be difficult to judge their effectiveness. Heuristic methods often are based on relatively simple common-sense ideas for how to search for a good solution.

These ideas need to be carefully tailored to fit the specific problem of interest. Thus Heuristic methods tend to be ad-hoc in nature. That is, each method usually is designed to fit a specific problem type rather than a variety of applications.

1.3.9.3 Set Theoretic Approach in solving a class of Multi Stage Flow Shop Problems

In 1980, Maggu and Dass introduced operator \( O_{i,w} \) named as Idle waiting time operator as follows:

Definition: Let the symbol \( O_{i,w} \) designate idle/ waiting-time operator.

Let \( \mathbb{R}^+ \) be the set of non negative real numbers. Let \( D = \mathbb{R}^+ \times \mathbb{R}^+ \) then \( O_{i,w} \) is defined as a map from \( D \times D \rightarrow D \) given by

\[
O_{i,w} ((x_1,y_1), (x_2,y_2)) = (x_1,y_1) \cdot O_{i,w}(x_2,y_2)
\]

\[
= (x_1 + \max \{x_2 - y_1, 0\}, y_2 + \max \{y_1 - x_2, 0\})
\]

Where \( x_1, x_2, y_1, y_2, y \in \mathbb{R}^+ \)

As will be observed that the operator \( O_{i,w} \) defined as above a set theoretic approach in solving a class of multistage flow shop scheduling.
problems. This concept has been used in dealing with problems on scheduling in the thesis.

1.3.10 Metaheuristic

Metaheuristic is a general solution method that provides both a general structure and strategy guidelines for developing a specific heuristic method to fit a particular kind of problem. Metaheuristic have become one of the most important techniques in the toolkit of O.R. practitioners.

1.3.10.1 Tabu Search

Tabu search is a widely used Metaheuristic that uses some common-sense ideas to enable the search process to escape from a local optimum. In fact, Tabu search only provide a general structure and strategy guidelines for enveloping a specific heuristic method to fit a specific situation. The selection of no parameters is a key part of developing a successful heuristic method.

1.3.10.2 Simulated Annealing

Simulated Annealing is another widely used metaheuristic that enable the search process to escape from a local optimum.
1.3.10.3 Genetic Algorithms

Genetic algorithms provide a third type of metaheuristic that is quite different than the first two. This type tends to be particularly effective at exploring various parts of the feasible region and gradually evolving towards the best feasible solutions.

Just as simulated annealing is based on an analogy to a natural phenomenon (the physical annealing process), genetic algorithms are greatly influenced by another form of a natural phenomenon. In this case, the analogy is to be biological theory of evolution formulated by Charles Darwin in the mid 19th Century. Each species of plants and animals has great individual variation. Darwin observed that those individuals with variations that impart a survival advantage through improved adaptation to the environment are most likely to survive to the next generation. This phenomenon has since been referred to as survival of the fittest.

1.3.11 Polynomial Time Approximation Scheme

A scheme for generating heuristic algorithms of any desired accuracy and when the algorithm runs in polynomial time, as per the specified accuracy, is called polynomial time approximation scheme. In other words, the approximation scheme that produces algorithm that has polynomial time complexity, i.e. polynomial both in the problem size
and the desired accuracy, is called polynomial time approximation scheme.

1.3.12 The Classes P and NP

Two classes P and NP are used to distinguish between the two types of algorithms: those with the polynomial time complexity and other without polynomial time complexity. The class P consists of all problems for which algorithms with polynomial time behaviour have been found. The class NP is essentially the set of problems for which algorithms with exponentially behaviors have been found. Clearly P is contained in NP. If one has a polynomial time algorithm for a problem it can always be inflated inefficiently so that it takes exponential time. Also, occasionally, a problem originally in NP but not in P, as someone with flush of height discovers a polynomial type algorithm.

The static sequencing problems are solved by the following three approaches:

(i). **Combinatorial Approach**: Combinatorial approach deal with the changing of the one permutation to another switching around of jobs that specifies a given criterion. Johnson, Mitten & Smith have discussed the combinatorial approach.
(ii). **General Mathematical Programming Approach:** The static sequencing problems have been solved by mathematical programming techniques which are linear, dynamic, convex and quadratic programming, integer programming, network of flows and Langragians methods. There are many articles published on the subject by authors like Bellman & Wagner.

(iii). **Reliable Heuristic approach:** Reliable heuristic approach is also called as ‘combinatorial programming’ or ‘controlled examination’.

This approach is based on the basis of two main concepts –

(a) The use of controlled enumeration method for obtaining all potential solutions and

(b) The elimination from explicitly consideration of particular potential situation which are known from dominance, bounding and flexibility consideration to be unacceptable.

The reliable heuristic approach studied by Mc Mohan and Burton. The whole theory of sequencing to date has been developed with a highly restrictive discipline which can be said to define a simple job-shop process. Additional restrictions are made on the definition of job set and machines as well as on the manner in which a schedule may be constructed.
1.3.13 Literature Survey in Scheduling

For the past three decades a significant amount of work has been done on production scheduling Problem focusing mainly on the development of deterministic model where the problem data is assumed to be known in advance. Johnson made a successful attempt to find optional solution for two and three stage scheduling using heuristic technique. Convey etal formulate the Integer programming model for scheduling. Ignall, E. and Scharge, I. applied branch and bound technique in flow shop. Maggu and Dass established equivalent job block theorem is scheduling. The work was further extended by Singh T. P. taking into account the various parameters, as concept of the transportation time, arbitrary time, breakdown interval and associated probabilities with processing time or set up time depending upon the realistic situations. Narain L. discussed some special cases of flow shop models with bi-objective scheduling problem using branch and bound technique.

1.3.13.1 Johnson’s Algorithm for n Jobs, 2 Machines

The Johnson’s iterative procedure for determining the optimal sequence for an n job 2 machines sequencing problem can be outlined as follows:

**Step 1.** Check the $A_i$’s and $B_i$’s for $i=1,2,3\ldots,n$ and find out $\text{Min } [A_i,B_i]$
Step 2. (i) If this minimum be $A_k$ for some $i = k$, the $k^{th}$ job first of all.

(ii) If this minimum of $B_r$ for some $i = r$ the $r^{th}$ job last of all.

Step 3. (i) If there is a tie for minima $A = B$ process the $k^{th}$ job first of all and $r^{th}$ job in the last

(ii) If the tie for the minimum occurs among the $A_i$'s select the job corresponding to the minimum of $B_i$'s and processes it first of all.

(iii) If the tie for minimum occurs among the $B_i$'s select the job corresponding to the minimum of $A_i$'s and processes it in the last. Go to next step.

Step 4. Cross out the job already assigned and repeat steps 1-3 arranging the jobs next to first or next to last until all the jobs have been assigned.

1.3.13.2 Restrictive case n jobs 3 Machines

The earlier method adopted by Johnson can be extended to three machines the special cases where either one or both conditions hold.

(i) The minimum time on machine $A \geq$ the maximum time on machine $B$.

(ii) The minimum time on machine $C \geq$ the maximum time on machine $B$.

The procedure is explained above to replace the problem with an equivalent problem involving $n$ jobs and two fictitious machine denoted by $G$ and $H$, and the corresponding time $G_i$ and $H_i$ are defined by

$$G_i = A_i + B_i, \quad H_i = B_i + C_i.$$
1.3.13.3 Processing n jobs, M machines

Let each of the n jobs be processed through m machines say M_1,M_2 ,........M_m , in the M_1,M_2 ,........M_m , and T_{ij} denote the time taken by the i^{th} machine, to complete the j^{th} job.

Step 1. First find,

(i) \( \min (T_{ij}) \)

(ii) \( \min (T_{mj}) \) and

(iii) \( \max (T_{2j}, T_{3j}, \ldots, T_{(m-1)j}) \) for \( j = 1, 2, 3, \ldots \)

Step 2. Check whether

(1) \( \min (T_{1j}) \geq \max (T_{ij}) \) for \( i = 1, 2, 3, \ldots, m-1 \) or

(2) \( \min (T_{mj}) \geq \max (T_{ij}) \) for \( i = 1, 2, 3, \ldots, m-1 \)

Step 3. If inequality of step 2 is not satisfied then go to the next step.

Step 4. Convert the m-machine problem into 2-machine problem considering two fictitious machine G and H so that \( T_{Gj} = T_{2j} + T_{3j} + \ldots + T_{(m-1)j} \) & \( T_{Hj} = T_{2j} + T_{3j} + \ldots + T_{mj} \).

Now determine the optimal sequence of n jobs, 2 machines by using the optimal sequence algorithm.