Chapter 4

$J^\Delta$ Transit Function

4.1 Introduction

In this chapter, we study the triangle induced path transit function $J^\Delta$. We mainly discuss the betweenness axioms and monotone axiom. As in the case of the induced path transit function $J$, here also we identify the graphs for which the $J^\Delta$-transit function satisfies these axioms. We recall the definition of the $J^\Delta$-transit function. For any two vertices $u$ and $v$ of a connected graph $G$,

$$J^\Delta(u,v) = \{x | x \text{ is on some } u-v \text{ induced path or } x \text{ is adjacent to two adjacent vertices on some induced path between } u \text{ and } v \}.$$ 

We write $J = J^0\Delta$ and $J^k\Delta = (J^{(k-1)\Delta})^\Delta$, for $k \geq 1$. In general $J^\Delta \subset J^{2\Delta} \subset J^{3\Delta} \subset \ldots \subset J^{n\Delta}$. It is to be noted that $J^\Delta = J^n\Delta$ for $n \in N$ if and only if $G$ has no induced subgraph isomorphic to $K_4 - x$. Though $J^\Delta$ need not be equal to $J^n\Delta$ for $n > 1$, the convexity induced by $J^\Delta$ and $J^n\Delta$ on any connected graph coincide each other. The convexity properties of $J^\Delta$ will be discussed in the next chapter. For the coming discussions, we need the following definition.

$t$-neighbors: let $x$ be an interior vertex on an $a-b$ triangle induced path $P$ then the vertices $x_1$ and $x_2$ are called the $t$ neighbors of $x$ on $a \rightarrow P \rightarrow b$ if i) $x_1 \in a \rightarrow P \rightarrow (x)$, $x_2 \in (x) \rightarrow P \rightarrow b$. ii) $x$ is adjacent to both $x_1$ and $x_2$. iii) $x_1$ is adjacent to $x_2$ if and only if $x$ is not on $P\ell$. The special graphs that we
come across in this chapter are the $G_i$, $i=3,4,5,6$ and 7. Recall the definition of $G_i$ from Chapter 1. The graphs are shown below.

In section 2, we study the betweenness axioms and in section 3, the monotone and peano axioms.

4.2 Betweenness

It is trivial to note that $J^\Delta$ transit function satisfies the betweenness axiom $b1$ only for triangle free graphs in which case $J^\Delta$ transit function coincides with the induced path transit function $J$. $J^\Delta$ satisfies $b1$ axiom if and only if $G$ is free from domino and cycles except $C_4$. We can easily verify that $J^\Delta$ satisfies the betweenness axiom $b2$ if $G$ is $HHD$, $K_4 - x$- free. First let us consider a long cycle. Let $a, x$ and $y$ be 3 vertices of the long cycle so that $x$ and $y$ are adjacent vertices and $a$ is not adjacent to $x$ or $y$. Let $z$ be a vertex on the $a - y$ segment of the long cycle not containing $x$. Let $b$ be an extra vertex which is
adjacent only to \( z \) and \( x \). Then \( x \in J^\Delta(a,b) \), \( y \in J^\Delta(a,x) \), but \( y \notin J^\Delta(a,b) \).

For a house and a domino the extra vertex \( b \) and other vertices \( a, x \) and \( y \) are as shown below.

For a \( K_4 - x \), choose the vertices \( a, b, x \) and \( y \) so that \( b \) and \( y \) are non adjacent vertices. In all these cases we can see that \( x \in J^\Delta(a,b) \), \( y \in J^\Delta(a,x) \), but \( y \notin J^\Delta(a,b) \).

In the next theorem we analyse the \( b_2 \) axiom.

**Theorem 12** Let \( G \) be a connected graph. Then the triangle induced path transit function \( J^\Delta \) satisfies the betweenness axiom \( b_2 \): \( J^\Delta(a,w) \subseteq J^\Delta(a,b) \) for \( w \in J^\Delta(a,b) \) if and only if \( G \) is HHD and \( K_4 - x \) free.

**Proof.** It was already observed that if the house, the domino, the long cycle and \( K_4 - x \) are not forbidden, then \( b_2 \) is not satisfied. So to complete the theorem it suffices to show that if \( J^\Delta \) does not satisfy \( b_2 \), then \( G \) has an induced subgraph isomorphic to a house, domino, long cycle or \( K_4 - x \). First we show that for any neighbor \( w \) of \( b \) in \( J^\Delta(a,b) \), we have \( J^\Delta(a,w) \subseteq J^\Delta(a,b) \). Assume
that this is not true and let \( y \) be a vertex in \( J^\Delta(a, w) \) and not in \( J^\Delta(a, b) \) with \( w \) a neighbor of \( b \) in \( J^\Delta(a, b) \). Let \( P \) be an \( a - b \) triangle induced path containing \( w \) and \( Q \) be an \( a - w \) triangle induced path containing \( y \). Let \( y_1 \) and \( y_2 \) be the \( t \)-neighbors of \( y \) on \( a \to Q \to w \). By the choice of \( w \), one of its \( t \) neighbors is \( b \). Let \( z \) be the other \( t \)-neighbor of \( w \) on \( a \to P \to b \). Then we have the following observations.

i) If \( Q \) contains \( b \), then \( G \) has an induced \( K_4 - x \).

Assume the contrary. Since \( y \notin J^\Delta(a, b) \), \( y \) cannot be a vertex on \( a \to Q \to b \). Hence \( y \in (b) \to Q \to (w) \). But \( b \) is adjacent to \( w \). So \( y \) is adjacent to both \( b \) and \( w \). If \( w \in P \), then \( a \to P \to w \to y \to b \) is an \( a - b \) triangle induced path containing \( y \), a contradiction. If \( w \notin P \), then \( w \) is adjacent to both \( z \) and \( b \). Suppose \( z \) is not adjacent to \( y \), then the subgraph induced by the vertices \( w, y, z \) and \( b \) is isomorphic to \( K_4 - x \). Suppose \( z \) is adjacent to \( y \), then the subgraph induced by \( a \to P \to z \to y \to b \) is an \( a - b \) triangle induced path containing \( y \), again a contradiction.

ii) If \( a \) is adjacent to \( b \), then the existence an induced \( K_4 - x \) follows.

Since \( a \) is adjacent to \( b \), \( w \) is adjacent to both \( a \) and \( b \). Since \( y \in J^\Delta(a, w) \), \( y \) is adjacent to both \( a \) and \( w \). Again \( y \notin J^\Delta(a, b) \) and \( y \) is adjacent to \( a \), so
y cannot be adjacent to $b$. Hence the subgraph induced by the vertices $a$, $b$, $w$ and $y$ is isomorphic to $K_4 - x$.

\[ \text{iii) If } a \text{ is adjacent to } y, \text{ then again } G \text{ contains an induced } K_4 - x. \]

Now $a \to Ql \to w \to b$ is an $a - b$ path containing either $y$ or the edge $y_1y_2$. So it cannot be an induced path. To make the path a non-induced path, the possible chord is $ab$ and which is not possible by observation (ii). So let us proceed the proof under the assumption that $a$ is not adjacent to $b$ or $y$.

Now $a \to Ql \to w \to b$ is an $a - b$ path containing $y$ or the edge $y_1y_2$. So there must be a chord from \((a) \to Ql \to y_1\) to $b$. Let $cb$ be the chord from \((a) \to Ql \to y_1\) to $b$. Choose $a$ to be the first vertex common to $P_l$ and $Ql$ after $c$, as we move along $Ql$ from $w$, so that $a \to Ql \to c \to b$ and $a \to P_l \to b$ have no internal vertices in common. Also there is no chord from $a \to Ql \to (c)$ to $w$ or $b$. Let $C$ be the cycle $a \to Ql \to c \to b \to P_l \to a$. Since it has already been proved that the induced path transit function $J$ on $G$ satisfies $b2$ if and only if $G$ has no induced house, domino or long cycles, to complete the proof it is enough to prove the following two cases only.

Case 1: $w \notin P_l$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4_4.png}
\caption{Figure 4.4:}
\end{figure}
In this case $C$ is of length $\geq 4$. Let $uv$ be the chord from $c \to Qi \to (a)$ to $z \to Pi \to a$. Then $u \to v \to Pi \to b \to c \to u$ is an induced cycle. To avoid a long cycle, either $u = c$ or $v = z$.

Case 1.1: $c$ is not adjacent to $w$.

In this case the subgraph induced by $b, c, u, v, z$ and $w$ is isomorphic to a $K_4 - x$; when $u = c$ and $y = z$. Otherwise it is isomorphic to a house.

Case 1.2: $c$ is adjacent to $w$.

Here, if $v \neq z$ or $u \neq c$, then the subgraph induced by the vertices $b, c, w$ and $z$ is isomorphic to a $K_4 - x$. Suppose $v = z$ and $u = c$. Let $pq$ be the chord, different from $cz$, from $c \to Qi \to (a)$ to $z \to Pi \to a$. Then $c \to Qi \to p \to q \to Pi \to z \to c$ is an induced cycle of length $\geq 4$. So if it is not a long cycle, then the subgraph induced by the vertices $c, z, p, q$ and $b$ is isomorphic to $K_4 - x$ or a house, according as the length of the cycle is 3 or 4.

Case 2: $w \in Pi$ and $y \notin Qi$.

The main difference between cases 1 and 2 is about the length of the path $a \to Pi \to w$. In this case the length of this path is at least two and hence the length of the cycle $C$ is at least five. To avoid an induced long cycle there must
exist a chord from \( c \rightarrow Ql \rightarrow (a) \) to \( w \rightarrow Pl \rightarrow a \). Let us take it as \( uv \). Then either \( u = c \) or \( v = w \). Suppose \( u = c \), then \( v \neq w \), since \( c \) cannot be adjacent to \( w \). Hence \( v = z \). Now consider the cycle \( a \rightarrow Pl \rightarrow z \rightarrow c \rightarrow Ql \rightarrow a \). In order to avoid a long cycle, there must be a chord \( pq \), different from \( cz \), from \( c \rightarrow Ql \rightarrow (a) \) to \( (z) \rightarrow Pl \rightarrow a \). Then \( c \rightarrow Ql \rightarrow p \rightarrow q \rightarrow Pl \rightarrow z \rightarrow c \) is an induced cycle. If it is not a long cycle, then its vertices together with \( w \) and \( b \) form either an induced house or domino. Similarly we can prove the existence of an induced house, domino or long cycle when \( v = w \) and \( u \neq c \). As the existence of an induced house, domino, long cycle or a \( K_4 - x \) is impossible, it follows that \( J^\Delta (a, w) \subseteq J^\Delta (a, b) \), for any neighbor \( w \) of \( G \) in \( J^\Delta (a, b) \). Now let \( y \) be any vertex in \( J^\Delta (a, b) \). Choose a triangle induced \( a - b \) path \( P \) containing \( y \), (say) \( P = a \rightarrow P \rightarrow y \rightarrow y_1 \rightarrow y_2 \rightarrow \ldots \rightarrow y_k \rightarrow b \). Then \( y_k \) is a neighbor of \( b \) in \( J^\Delta (a, b) \), so that by the previous argument we have \( J^\Delta (a, y_k) \subseteq J^\Delta (a, b) \). Similarly, we infer that \( J^\Delta (a, y) \subseteq J^\Delta (a, y_1) \subseteq \ldots \subseteq J^\Delta (a, y_k) \subseteq J^\Delta (a, b) \). This concludes the proof that \( J^\Delta \) satisfies \( b2 \) axiom on \( G \). 

4.3 Monotone axiom

In this section we study the monotone axiom. We can prove that \( J^\Delta \)-convexity is \( JHC \) (the proof will be given in the next chapter) and hence the peano axiom and monotone axiom are equivalent, using Corollary 1 of chapter 1. Therefore we follow our discussion similar to that of the induced path transit function \( J \).
Let $J^\Delta$ be the triangle induced path transit function on a $G_i$, $i = 1, 2, ..., 7$. We can prove that $J^\Delta$ does not satisfy the Peano axiom on $G_i$.

First let us consider a $G_i$, $i = 1, 2, ..., 6$. In this case $v \in J^\Delta(s, t)$, $f \in J^\Delta(v, u)$, $J^\Delta(s, u) = \{s, u\}$, $J^\Delta(u, t) = \{u, t\}$. Since $f \notin J^\Delta(s, t)$, we get $f \notin J^\Delta(z, t)$, for all $z \in J^\Delta(s, t)$.

For a $G_7$, take $v = s$. Then we have $J^\Delta(s, t) = \{s, t, u\}$ and $J^\Delta(v, u) = \{v, u, f, t\}$. Hence $u \in J^\Delta(s, t)$, $f \in J^\Delta(u, v)$. But $f \notin J^\Delta(z, t)$, for all $z \in J^\Delta(s, v)$, since $J^\Delta(s, v) = \{s\}$.

**Theorem 13** The transit function $J^\Delta$ on a connected graph $G$ containing at least four vertices satisfies the Peano axiom only if $G$ is $G_i$-free, $i = 3, 4, 5, 6, 7$.

**Proof.** Suppose the transit function $J^\Delta$ on the connected graph $G$ does not satisfy the Peano axiom. Then there exist five vertices $a, b, c, x,$ and $y$, such that $x \in J^\Delta(a, b)$, $y \in J^\Delta(c, x)$ but $y \notin J^\Delta(z, b)$ for all $z \in J^\Delta(a, c)$. In particular $y \notin J^\Delta(a, b) \cup J^\Delta(b, c) \cup J^\Delta(c, a)$. Hence $x \neq y$; $x, y \neq a, b, c$ and $b \neq c$. Since $x \in J^\Delta(a, b)$, there exists an $a - b$ triangle induced path $P$ containing $x$. Let $x_1$ and $x_2$ be the $t$-neighbors of $x$ on $a \rightarrow P \rightarrow b$. Since $y \in J^\Delta(c, x)$, there exists a $c - x$ triangle induced path $Q$ containing $y$ so that $Q'$ is an $x - c$ induced path. Let $y_1$ and $y_2$ be the $t$-neighbors of $y$ on $x \rightarrow Q \rightarrow c$. Then at most one of the vertices $x_1$ or $x_2$ can be adjacent to a vertex on $(x) \rightarrow Q' \rightarrow (y)$. Without loss of generality we can choose $P$ and $Q$ so that

(a): The paths $P$ and $Q$ coincides with their corresponding induced paths if $x \in P'$ and $y \in Q'$ respectively. In other words if $P$ and $P'$ are different, then they differ only by the vertex $x$ and the edges $xx_1$ and $xx_2$.

and

(b): no vertex on $y_1 \rightarrow Q \rightarrow (x)$ belongs to $J^\Delta(a, b)$. 
Hence no vertex on $P$ other than $x$, $x_1$ and $x_2$ can be adjacent to a vertex on $(y) \rightarrow Q \rightarrow (x)$. Moreover, at most one of the vertices $x_1$ or $x_2$ can be adjacent to a vertex on $(y) \rightarrow Q \rightarrow (x)$. If $x$ is on some $a - b$ induced path and $y$ is on some $x - c$ induced path of $G$, then by Theorem 7 of chapter 3, the existence of an induced $G_3$ follows. So we shall assume that either $x$ is not on any $a - b$ induced path or $y$ is not on any $c - x$ induced path. Let us prove the theorem in two cases depending on whether $c \rightarrow Q \rightarrow y_2$ has vertices in common with $P$ or not.

Case 1: $c \rightarrow Q \rightarrow y_2$ has no vertex in common with $P$.

Now $a \rightarrow P \rightarrow x_1 \rightarrow x \rightarrow Q \rightarrow y \rightarrow Q \rightarrow c$ is an $a - c$ path containing $y$. So there exists a chord from $y_2 \rightarrow Q \rightarrow c$ to $a \rightarrow P \rightarrow x_1$. Let $y_4x_4$ be the chord from $y_2 \rightarrow Q \rightarrow c$ to $a \rightarrow P \rightarrow x_1$. Similarly $b \rightarrow P \rightarrow x_2 \rightarrow x \rightarrow Q \rightarrow y \rightarrow Q \rightarrow c$ is a $b - c$ path containing $y$. Hence there exists a chord from $y_2 \rightarrow Q \rightarrow c$ to $b \rightarrow P \rightarrow x_2$. Let $y_3x_3$ be the chord from $y_2 \rightarrow Q \rightarrow c$ to $b \rightarrow P \rightarrow x_2$. Then, either $y_3 \in y_4 \rightarrow P \rightarrow c$ or $y_4 \in y_3 \rightarrow P \rightarrow c$. Let us assume that $y_3 \in y_4 \rightarrow P \rightarrow c$. 

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Figure 4.7:
Then $a \to P \to x_4 \to y_4 \to Q \to y \to Q \to x \to x_2 \to P \to b$ is an $a-b$ path containing $y$. Since $y$ is not an element of $J^\Delta(a, b)$, we have either $y_3 = y_4$ or $y_3 \neq y_4$ and $x_4 = x_1$. Choose $x_4'$ as the first vertex on $x_1 \to P \to a$ which is adjacent to $y_4$ and $x_3'$ as the first vertex on $x_2 \to P \to b$ which is adjacent to $y_3$. By the choice of the chords $y_3x_3$, $y_4x_4$ and the paths $P$ and $Q$, at least one of the cycles $x_1 \to P \to x_4' \to y_4 \to Q \to x \to x_1$ or $x_2 \to P \to x_3' \to y_3 \to Q \to x \to x_2$ is an induced cycle, assume that the last one is an induced cycle.

Case 1.1: $y_3 = y_4$.

Then we have the following three cases.

i. $x$ is not a vertex on $P'$ and $y$ is not a vertex on $Q'$.

ii. $x$ is not a vertex on $P'$ but $y$ is a vertex on $Q'$. 

iii. $x$ is a vertex on $P'$ but $y$ is not a vertex on $Q'$.

Consider the case $i$. Suppose $x = y_1$. Since $y \notin J^\Delta(a, b)$, the only vertices on $P$ which can be adjacent to $y$ are $x_1$ or $x_2$. Further, both of them cannot be adjacent to $y$.

So, if $x_1$ is adjacent to $y$, then the subgraph induced by the vertices $x_1$, $x_2$, $x$ and $y$ is isomorphic to $G_7$. Similarly, if $x_2$ is adjacent to $y$, then the subgraph induced by $x_1$, $x_2$, $x$ and $y$ is isomorphic to $G_7$.

If $y$ is not adjacent to $x_1$ or $x_2$, then the subgraph induced by $x_4' \to P \to x_3'$, $x \to Q \to y_3$ is isomorphic to $G_6$, with $s = x_4'$, $t = x_3'$, $v = x$, $u = y_3$, $f = y$.

Suppose $x \neq y_1$. If $x_1$ is adjacent to the neighbour of $x$ on $(x) \to Q \to (y_4)$, then the existence of an induced $G_7$ follows. Otherwise, the subgraph induced by $x_4' \to P \to x_3$, $x \to Q' \to y_4$ is isomorphic to $G_4$, with $s = x_4'$, $t = x_3'$, $v = x$, $u = y_3$, the central axis being $(u) \to Q' \to (v)$. Thus the existence of a $G_i$ follows in $i$.

ii. $x$ is not a vertex on $P'$ but $y$ is a vertex on $Q'$.

If $x_1$ is adjacent to the neighbour of $x$ on $(x) \to Q \to y_3$, then the subgraph induced by this neighbour together with the vertices $x_1$, $x_2$ and $x$ is isomorphic to $G_7$.

If $x_1$ is not adjacent to the neighbour of $x$ on $(x) \to Q \to y_3$, then the subgraph induced by $x_4' \to P \to x_3'$, $x \to Q' \to y_3$ is isomorphic to $G_4$ with $u = y_3$, $v = x$, $s = x_4'$, $t = x_3'$, central axis being $(x) \to Q' \to (y_3)$. Thus the theorem follows in $ii$.

iii. $x$ is a vertex on $P'$ but $y$ is not a vertex on $Q'$. 

Since no vertex on $(x) \to Q \to y_1$ belongs to $J^\Delta(a, b)$, $x_1$ cannot be adjacent to the neighbour of $x$ on $(x) \to Q \to y_1$.

Suppose $x = y_1$. 

Since \( y \notin J^\Delta(a,b) \), the only vertex on \( P \) which can be adjacent to \( y \) is \( x \). Hence the subgraph induced by \( x_1' \to P \to x_3' \), \( x \to Q \to y_3 \) is isomorphic to \( G_5 \) with \( v = x, u = y_3, s = x_4', t = x_3', f = y \).

Suppose \( x \neq y_1 \). In this case the subgraph induced by \( x_1' \to P \to x_3' \), \( x \to Q' \to y_3 \) is isomorphic to \( G_3 \) with \( s = x_4', t = x_3', u = y_3, v = x \) and the central axis is \( (x) \to Q' \to y_3 \).

In particular, when \( x = y_1, y_2 = y_3, x_4 = x_1 \), the subgraph induced by the vertices \( x, x_1, y \) and \( y_3 \) is isomorphic to \( G_7 \).

Case 1.2: \( y_3 \neq y_4 \).

Since \( y_3 \neq y_4 \), \( x_1 = x_4 \). Therefore \( x_4' = x_4 \).

Here also we have the following three cases.

i. \( x \) is not a vertex on \( P' \) and \( y \) is not a vertex on \( Q' \).

ii. \( x \) is not a vertex on \( P' \) but \( y \) is a vertex on \( Q' \).

iii. \( x \) is on \( P' \) but \( y \) is not a vertex on \( Q' \).

Consider the first case i when \( x \) is not a vertex on \( P' \) and \( y \) is not a vertex on \( Q' \).

Suppose \( x = y_1 \).

If \( x_1 \) is adjacent to \( y \), then \( x_2 \) is not adjacent to \( y \) and vice versa. Hence the subgraph induced by the vertices \( x_1, x_2, x \) and \( y \) is isomorphic to \( G_7 \).

Suppose \( x_1 \) and \( x_2 \) are not adjacent to \( y \). Here \( y_2 \) is the neighbour of \( x \) on \( (x) \to Q \to y_4 \). If \( x_1 \) is adjacent to \( y_2 \), then the subgraph induced by \( x_1, x_2, x \) and \( y_2 \) is isomorphic to \( G_7 \). If \( x_1 \) is not adjacent to \( y_2 \), then the subgraph induced by \( x_1 \to P \to x_3', (x) \to Q \to y_3 \) is isomorphic to \( G_6 \), with \( s = x_4, v = x, f = y \). Here \( u \) is the last vertex on \( y_4 \to Q \to y_3 \) which is adjacent to \( x_4 \) and \( t \) is the first vertex on \( u \to Q \to y_3 \) which is not adjacent to \( x_1 \), if \( u \neq y_3 \); otherwise \( t = x_3' \).
Suppose \( x \neq y_1 \).

If \( x_1 \) is adjacent to the neighbour of \( x \) on \( (x) \to Q \to y_4 \), then the subgraph induced by this neighbour together with \( x, x_1 \) and \( x_2 \) is isomorphic to \( G_7 \).

If \( x_1 \) is not adjacent to the neighbor of \( x \) on \( (x) \to Q \to y_4 \), then the subgraph induced by \( x_4 \to P \to x_3', x \to Q' \to y_3 \) is isomorphic to \( G_4 \), with \( v = x, s = x_1 \). Here also, \( u \) is the last vertex on \( y_4 \to Q \to y_3 \) which is adjacent to \( x_4 \) and \( t \) is the first vertex on \( u \to Q \to y_3 \) which is not adjacent to \( x_1 \), if \( u \neq y_3 \); otherwise \( t = x_3' \). The central axis is \((u) \to Q' \to (v)\).

\( ii. \) \( x \) is not on \( P' \) but \( y \) is on \( Q' \).

If \( x_1 \) is adjacent to the neighbor of \( x \) on \( (x) \to Q \to y_4 \), then the subgraph induced by \( x_1, x_2, x \) together with this neighbor of \( x \) is isomorphic to \( G_7 \).

If \( x_1 \) is not adjacent to the neighbor of \( x \) on \( (x) \to Q \to y_4 \), then the subgraph induced by \( x_4 \to P \to x_3', x \to Q \to y_3 \) is isomorphic to \( G_3 \) with \( v = x, s = x_1 \) and \( u, t \) and the central axis are defined as in the above case.
iii. $x$ is on $P'$ but $y$ is not on $Q'$.

Suppose $x = y_1$.

Since $y \notin J^\Delta(a, b)$, $y$ is not adjacent to $x_4$. Hence we have, if $y_2$ is adjacent to $x_4$, then the subgraph induced by $x, x_4, y$ and $y_2$ is isomorphic to $G_7$ and if $y_2$ is not adjacent to $x_4$, then the subgraph induced by $x_3' \rightarrow P \rightarrow x_4$, $x \rightarrow Q' \rightarrow y_3$ is isomorphic to $G_4$, with $s = x_1, v = x$, and $u, t$ and the central axis are defined as above.

In particular when $y_2 = y_4$, the subgraph induced by $x, x_1, y$ and $y_4$ is isomorphic to $G_7$.

Suppose $x \neq y_1$.

Since $x_1$ cannot be adjacent to the neighbor of $(x) \rightarrow Q \rightarrow y_4$, the subgraph induced by $x_4 \rightarrow P \rightarrow x_3', x \rightarrow Q' \rightarrow y_3$ is isomorphic to $G_3$, with $s = x_1, v = x, u$ and $t$ are as defined above. The central axis is $(u) \rightarrow Q' \rightarrow (v)$.

Similarly we can prove the existence of an induced $G_i$, $i = 3, 4, 5, 6$ or 7 when $C_1$ is an induced cycle.

Case 2: $c \rightarrow Q \rightarrow y_2$ has vertices in common with $P$.

Without loss of generality, we can choose $w$ as the last vertex common to $P$ and $Q$ before $y$ and $x$ as the first vertex common to $P$ and $Q$ after $y$, as we traverse from $c$ along $Q$. Again we can choose $w$, so that no vertex on $(w) \rightarrow Q \rightarrow (y)$ lies on an $a - b$ induced path containing $x$. Now $a \rightarrow P \rightarrow w \rightarrow Q \rightarrow x \rightarrow x_2 \rightarrow P \rightarrow b$ is an $a - b$ path containing $y$. So there exists a chord from $y_2 \rightarrow Q \rightarrow (w)$ to $x_2 \rightarrow P \rightarrow b$; let it be $y_3x_3$. Assume that $y_4x_4$ is the chord from $y_2 \rightarrow Q \rightarrow (w)$ to $a \rightarrow P \rightarrow x_1$. Since $w$ is a common vertex of $P$ and $Q$, the existence of such a chord follows.

Now $a \rightarrow P \rightarrow x_1 \rightarrow x \rightarrow Q \rightarrow y \rightarrow Q \rightarrow y_3 \rightarrow x_3 \rightarrow P \rightarrow b$ is an $a - b$ path containing $y$. Since $y \notin J^\Delta(a, b)$, either $y_4 \notin y_3 \rightarrow P \rightarrow y_2$ or $x_2 = x_3$. 
Figure 4.10:

Figure 4.11:
Let us consider the possibilities when \( y_3 \neq y_4 \). Since \( y_3 \neq y_4 \), either \( y_4 \notin y_3 \to Q \to y_2 \) or \( y_4 \in (y_3) \to Q \to y_2 \). In the former case, we get \( x_2 = x_3 \) and in the latter case \( a \to P \to x_4 \to y_4 \to Q \to y \to Q \to x \to x_2 \to P \to b \) is an \( a - b \) path containing \( y \). Since \( y \notin J^\Delta(a, b) \), we get \( x_4 = x_1 \).

Then we have the following subcases.

Subcase 2.1 \( y_3 = y_4 \).

Subcase 2.2 \( y_3 \neq y_4 \), \( y_4 \notin (y_3) \to Q \to y_2 \) and \( x_1 = x_4 \).

Subcase 2.3 \( y_3 \neq y_4 \), \( y_4 \notin y_3 \to Q \to y_2 \) and \( x_2 = x_3 \).
Choose $x_3'$ as the first vertex on $x_2 \to P \to b$ which is adjacent to $y_3$ and $x_4'$ as the first vertex on $x_1 \to P \to a$ which is adjacent to $y_4$. Then by the choice of the chords $y_3x_3$, $y_4x_4$ and the paths $P$ and $Q$, one of the cycles $x_1 \to P \to x_4' \to y_4 \to Q! \to x \to x_1$ or $x_2 \to P \to x_3' \to y_3 \to Q! \to x \to x_2$ is an induced cycle. Without loss of generality let us assume that $x_2 \to P \to x_3' \to y_3 \to Q! \to x \to x_2$ is an induced cycle. So $x_2$ cannot be adjacent to any vertex on $(x) \to Q! \to y$. But $x_1$ can be adjacent to the vertices in $(x) \to Q! \to y$. We can easily see that subcase 2.1 is the same as subcase 1.1 and subcase 2.2 is the same as subcase 1.2 and subcase 2.3 is similar to subcase 2.1. This completes the proof. 

**Remark 6** In the above proof, $y_3x_3'$ is the chord from $y_2 \to Q \to c$ to $x \to P \to b$ and $y_4x_4'$ is the chord from $y_2 \to Q \to c$ to $x \to P \to a$. In the remaining part of the discussion, let us denote the vertex $x_3'$ by $x_3$ and $x_4'$ by $x_4$.

**Remark 7** In the proof of the above theorem we have assumed that if $y_3 \neq y_4$, then $y_4 \in y_2 \to Q \to (y_3)$, except in the subcase 2.3, where $y_3 \in y_2 \to Q \to (y_4)$. Another important assumption in the proof is that the cycle $C_2 : x_2 \to P \to x_3 \to y_3 \to Q! \to x \to x_2$ is taken as an induced cycle. Under these two assumptions, we note the following observations which are very useful to the proof of the next theorem. For $G_i$, $i=3,4,5$ and 6, $y_3x_3$ is the chord from...
Let us take the neighbour \( w \) of \( v \) on the central axis as the first vertex, so that by the choice of the paths \( w \notin J^\Delta(a, b) \). Hence \( f \notin J^\Delta(a, b) \) and no vertex on \( P \), except \( x \), can be adjacent to \( f \) for all \( i = 3, 4, 5 \) and 6. When \( y_3 = y_4 \), the cycle of \( G_i \) is \( x_4 \rightarrow y_4 \rightarrow x_3 \rightarrow P \rightarrow x_4 \) and when \( y_3 \neq y_4 \), the cycle of \( G_i \) is \( x_4 \rightarrow y_4 \rightarrow Q \rightarrow y_3 \rightarrow x_3 \rightarrow P \rightarrow x_4 \). It should be noted that \( y_3 \neq y_4 \) implies \( x_3 = x_4 \).

**Notation 2** Suppose \( G \) has an induced subgraph isomorphic to a \( G_i \), so that there is no triangle induced path in \( G \) connecting the \( s \) and \( t \) vertices of \( G_i \) and containing a central vertex \( f \), then we say \( G \) has an induced \( G_i(s, t, f) \).

Our immediate observation is that if, \( G \) has an induced \( G_7 \) with vertices \( s, t, u \) and \( f \), \( s \) not adjacent to \( f \), then it turns out an induced \( G_7(s, \hat{f}, t) \). Assume that there is an \( s - t \) triangle induced path containing \( f \). Since \( f \neq s, t \) and \( s \) adjacent to \( t \), we get the contradiction that \( s \) is adjacent to \( f \). This shows that there is no \( s - t \) triangle induced path in \( G \) containing \( f \). Hence \( G \) has an induced \( G_7(s, \hat{f}, t) \).

We now use the above remarks, notation and techniques to prove the next theorem.

**Theorem 14** The transit function \( J^\Delta \) on a connected planar graph \( G \), satisfies the peano axiom if and only if \( G \) is \( G_i(s, \hat{f}, t) \)-free, for \( i = 3, 4, 5, 6 \) or 7.

**Proof.** If \( G \) has an induced \( G_i(s, \hat{f}, t) \) for some \( i \), and for some central vertex \( f \), then evidently \( J^\Delta \) does not satisfy the peano axiom. So to complete the proof, we have to prove the sufficiency part alone. For that, let us assume
that \( J^\Delta \) does not satisfy the **Peano axiom**. Hence by **Theorem 13**, we have the following:

(i). there exist 5 vertices \( a, b, c, x \) and \( y \) of \( G \) with \( a \neq b; x \neq y; x, y \neq a, b, c \); two triangle induced paths \( P \) and \( Q \) connecting \( a \) to \( b \) and \( c \) to \( x \) respectively so that \( x \) is on \( P \) and \( y \) is on \( Q \). Without loss of generality assume that the paths \( P \) and \( Q \) coincides with their corresponding induced paths if \( x \in P \) and \( y \in Q \) respectively.

(ii). there exists another set of vertices \( x_1, x_2, x_3 \) and \( x_4 \) on \( P \) and \( y_1, y_2, y_3 \) and \( y_4 \) on \( Q \); so that \( x_1 \) and \( x_2 \) form the \( t \)-neighbours of \( x \) on \( a \rightarrow P \rightarrow b \); \( y_1 \) and \( y_2 \) form the \( t \)-neighbours of \( y \) on \( x \rightarrow Q \rightarrow c \); \( x_3 x_3 \) forms the chord from \( y_2 \rightarrow Q \rightarrow c \rightarrow x_2 \rightarrow P \rightarrow b \); \( y_4 x_4 \) forms the chord from \( y_2 \rightarrow Q \rightarrow c \rightarrow x_1 \rightarrow P \rightarrow a \). In this case, either \( y_3 \in c \rightarrow Q \rightarrow y_4 \) or \( y_4 \in c \rightarrow Q \rightarrow y_3 \). Without loss of generality, let us assume that \( y_3 \in c \rightarrow Q \rightarrow y_4 \). Also let us assume that \( x_2 \rightarrow P \rightarrow x_3 \rightarrow y_3 \rightarrow Q \rightarrow x_2 \) is an induced cycle. By the remark, it follows that \( y_3 \neq y_4 \rightarrow x_4 = x_1 \). By **Theorem 13**, we get that, the subgraph induced by the vertices of \( P \) and \( Q \) has an induced subgraph \( G^* \) isomorphic to \( G'_i \); \( i = 3, 4, 5, 6 \) or \( 7 \). For \( i = 3, 4, 5, 6 \) the \( s, t, u, v \) and \( f \) vertices are as follows. For all \( i \), \( s = x_4, v = x \). For \( i = 3, 4, t = w \), where \( w \) is the neighbour of \( x \) on \( Q \); and for \( i = 5, 6 \); \( f = y \). Then the only vertex on \( P \) which can be adjacent to \( f \) is \( x \) and \( f \notin J^\Delta(a, b) \). When \( t = x_3 \), \( y_3 \) is adjacent to \( x_4 \), \( u = y_3 \) and the cycle \( C \) of \( G_i \), \( i = 3, 4, 5, 6 \) is \( x_4 \rightarrow y_4 \rightarrow x_3 \rightarrow P \rightarrow x_4 \). When \( t \neq x_3 \), \( u \) is the last vertex on \( y_4 \rightarrow Q \rightarrow y_3 \), which is adjacent to \( x_4 \), \( t \)-vertex is the first vertex on \( u \rightarrow Q \rightarrow y_3 \) which is not adjacent to \( x_4 \) and the cycle of \( G_i \) is \( C : x_4 \rightarrow y_4 \rightarrow Q \rightarrow y_3 \rightarrow x_3 \rightarrow P \rightarrow x_4 \).

We have already seen that an induced \( G_7 \) of \( G \) turns out an induced \( G_7(s, \hat{f}, t) \). So let us assume that \( G \) has no induced subgraph isomorphic to \( G_7 \).
Assume that $G$ is embedded in the plane. Now we can prove that the existence of an induced $G_i, i = 3, 4, 5, 6$ gives rise to an induced $G_i(s, f, t)$. For that we can prove that there is no $s-t$ triangle induced path in $G$ containing $f$. Assume that there exists an $s-t$ triangle induced path $\mu$ containing the $f$ vertex of $G_i$. Without loss of generality, let us assume that the path $\mu$ coincides with $\mu f$ if $f \in \mu$. Let $f_1$ and $f_2$ be the $t$-neighbours of $f$ on $s \to \mu \to t$.

We now define four vertices $p_1, p_2, q_1$, and $q_2$, on $\mu$ as follows.

Assume that we are traversing from $s$ to $t$ along $\mu$. Let $p_1$ be the last vertex before $f$ and $p_2$ be the first vertex after $f$ and common to $C$ and $\mu$. Let $q_1$ be the first vertex after $p_1$ and common to $(u) \to Q \to (v)$ and $\mu$, and $q_2$ be the last vertex before $p_2$ and common to $(u) \to Q \to (v)$ and $\mu$. Then $p_1 \neq p_2, t; p_1, p_2 \neq q_1, q_2; q_1, q_2 \neq s, t$ and $p_2 \neq s$.

Let us observe some other properties of $p_1, p_2, q_1$ and $q_2$ which are needed in the proof.
\((\alpha_1)\): \(p_2\) cannot be adjacent to \(s = x_4\).

Suppose not, then the subpath \(s \rightarrow \mu \rightarrow f \rightarrow \mu \rightarrow p_2\) of \(\mu\) gives the contradiction that \(f\) is adjacent to \(s\). Hence \((\alpha_1)\) follows.

\((\alpha_2)\): \(p_1\) cannot be adjacent to \(t\).

Suppose \(p_1\) is adjacent to \(t\), then from the subpath \(p_1 \rightarrow \mu \rightarrow q_1 \rightarrow \mu \rightarrow f \rightarrow \mu \rightarrow q_2 \rightarrow \mu \rightarrow t\) of \(\mu\), we get \(q_1 = q_2 = f\) and \(p_1f t\) is a \(C_3\). The only vertex on \(P\) which can be adjacent to \(f\) is \(x\) and \(x\) cannot coincide with \(t\). Hence \((\alpha_1)\) follows.

\((\alpha_3)\): \(p_1\) and \(p_2\) are not adjacent.

Suppose not, then the subpath \(p_1 \rightarrow \mu \rightarrow q_1 \rightarrow \mu \rightarrow f \rightarrow \mu \rightarrow q_2 \rightarrow p_2\) of \(\mu\) gives \(q_1 = q_2 = f\) and \(p_1fp_2\) form a \(C_3\). Since \(p_1\) and \(p_2\) are adjacent to \(f\), both of them cannot be vertices on \(P\). Suppose \(p_1 \in P\). Then, \(p_2 \notin P\). Therefore, \(p_2 \in y_3 \rightarrow Q \rightarrow y_4\). Since \(p_1\) is adjacent to \(f\), we get \(p_1 = x\). Therefore \(p_2 = y_4\), a contradiction by \((\alpha_1)\). Suppose \(p_2 \in P\), then \(p_1 \notin P\). Therefore \(p_1 \in y_3 \rightarrow Q \rightarrow y_4\). Since \(p_2\) is adjacent to \(f\), \(p_2 = x\). Therefore \(p_1 = y_4\). By \((\alpha_2)\), \(p_1 \neq y_3\). Therefore \(y_3 \neq y_4\), so \(x_4 = x_1\). Hence the subgraph induced by
Figure 4.16:

$x_1, x, y_4$ and $y$ is isomorphic to a $G_7$, which is a contradiction. Hence both $p_1, p_2 \in y_4 \rightarrow Q \rightarrow y_3$. Then either $p_1 \in (y_3) \rightarrow Q \rightarrow p_2$ or $p_2 \in y_3 \rightarrow Q \rightarrow p_1$.

In both cases as in $(\alpha_2)$, we can find an $a-b$ triangle induced path containing $f$. This contradiction proves $(\alpha_3)$.

$(\alpha_4)$: when $q_1 \neq q_2$ and $p_1, p_2 \in x_4 \rightarrow P \rightarrow x$; $p_1 \in x_4 \rightarrow P \rightarrow p_2 \implies q_1 \in y_4 \rightarrow Q \rightarrow q_2$ and $p_2 \in x_4 \rightarrow P \rightarrow p_1 \implies q_2 \in y_4 \rightarrow Q \rightarrow q_1$.

Now $p_1, p_2 \in x_4 \rightarrow P \rightarrow x$. By $(\alpha_3)$, $p_1$ is not adjacent to $p_2$. Therefore $x_4$ is not adjacent to $x$. Therefore $x_4 \neq x_1$ and hence $y_3 = y_4$.

Let us prove $p_1 \in x_4 \rightarrow P \rightarrow p_2 \implies q_1 \in y_4 \rightarrow Q \rightarrow q_2$ in two cases:

Case 1. $p_2 = x$ and Case 2 :$p_2 \neq x$.

Case 1. $p_2 = x$. 

Since $x$ is always adjacent to $f$, if we consider the path $f \rightarrow \mu \rightarrow x_1 \rightarrow \mu \rightarrow p_2$, we get either $q_1 = f$ or $q_2 = f$. Assume that $q_1 = f$. By the definition of $q_1, q_1 \in y_4 \rightarrow Q \rightarrow y_3 \rightarrow f$. Therefore $q_1 \in y_4 \rightarrow Q \rightarrow q_2$.

Also we can prove that $q_1 \in y_4 \rightarrow Q \rightarrow q_2$. Assume the contrary that $q_1 \neq q_2$. Then $q_1 \in y_4 \rightarrow Q \rightarrow q_2$. This contradicts $(\alpha_3)$. Hence $q_1 \neq q_2$.
Since $x$ is always adjacent to $f$, if we consider the subpath $f \rightarrow \mu \rightarrow q_2 \rightarrow \mu \rightarrow p_2$, we get either $q_2 = f$ or $fp_2q_2$ is a $C_3$. Assume that $q_2 = f$. By the definition of $q_1$, $q_1 \in y_4 \rightarrow Q \rightarrow y_2 \rightarrow f$. Therefore $q_1 \in y_4 \rightarrow Q \rightarrow q_2$.

Assume that $fp_2q_2$ is a $C_3$. We claim that $q_1 = f$. Suppose $q_1 \neq f$. In this case there is no chord from $(p_1) \rightarrow \mu \rightarrow (q_1)$ to $p_2$, since $p_2$ is a vertex on $\mu$. Also by $(\alpha_3)$, $p_1$ is not adjacent to $p_2$. Hence $a \rightarrow P \rightarrow p_1 \rightarrow \mu \rightarrow p_2 \rightarrow P \rightarrow b$ is an $a-b$ triangle path containing $f$. This contradiction proves our claim.

Since $q_1 \neq f$, $q_1 \in y_4 \rightarrow Q \rightarrow q_2$.

**Case 2: $p_2 \neq x$.**

Here also we can prove that $q_1 \in y_4 \rightarrow Q \rightarrow q_2$. Assume the contrary that $q_1 \notin y_4 \rightarrow Q \rightarrow q_2$. Therefore $q_2 \in (q_1) \rightarrow Q \rightarrow y_4$. Let $C_3$ be the cycle $x_4 \rightarrow \mu \rightarrow f \rightarrow x \rightarrow P \rightarrow x_3 \rightarrow y_3 \rightarrow Q \rightarrow y_4 \rightarrow x_4$.

Since $p_2 \neq x$, we can easily see that either $p_2$ is an interior vertex of $C_3$ and $q_2$ is its exterior vertex or vice versa. Since $S_1 : p_2 \rightarrow \mu \rightarrow q_2$ is a subpath of $\mu$ and $q_2 \neq f$, $S_1$ cannot have common vertices with the subpath $x_4 \rightarrow \mu \rightarrow f$ of $C_3$. Since $p_2 \notin x \rightarrow P \rightarrow x_3$, $S_1$ cannot have common vertices with the subpath $x \rightarrow P \rightarrow x_3$ of $C_3$. Our next claim is that $y_4$ is not a vertex on $S_1$. Suppose
not. Since $f \in x_4 \rightarrow \mu \rightarrow q_2$ and $x_4$ is adjacent to $y_4$, we get the contradiction that $f$ is adjacent to $x_4$. Hence $y_4$ is not a vertex on $S_1$. Hence an edge of $S_1$ must cross some edge of $C_3$. This conflicts with the planarity structure of $G$ which proves $(\alpha_4)$.

$(\alpha_5)$; when $p_1, p_2 \in y_4 \rightarrow Q \rightarrow y_3; p_1 \in y_4 \rightarrow Q \rightarrow p_2 \implies q_1 \in y_4 \rightarrow Q \rightarrow q_2$.

$p_2 \in y_4 \rightarrow Q \rightarrow p_1 \implies q_2 \in y_4 \rightarrow Q \rightarrow q_1$.

Assume that $p_1 \in y_4 \rightarrow Q \rightarrow p_2$. Let us prove that $q_1 \in y_4 \rightarrow Q \rightarrow q_2$. Suppose $q_1 \notin y_4 \rightarrow Q \rightarrow q_2$. Therefore, $q_1 \in (q_2) \rightarrow Q \rightarrow x$.

Consider the cycle $C_4 : x_4 \rightarrow P \rightarrow x \rightarrow Q \rightarrow q_1 \rightarrow \mu \rightarrow x_4$ and the subpath $S_2 : p_2 \rightarrow \mu \rightarrow q_2$ of $\mu$. Since $q_1 \neq q_2$, we can easily see that either
is no chord from \( a \rightarrow P \rightarrow (p_1) \) to \( f_2 \rightarrow \mu \rightarrow (q_2) \). Since \( \mu \rightarrow (p_2) \) lies interior to the cycle \( p_2 \rightarrow \mu \rightarrow f \rightarrow Q \rightarrow y_3 \rightarrow x_3 \rightarrow P \rightarrow p_1 \). Therefore there is no chord from \( a \rightarrow P \rightarrow (p_2) \) to \( q_2 \rightarrow \mu \rightarrow (q_2) \). By \( (a_i) \) and \( p_2 \) are not adjacent. So we get the contradiction that \( a \rightarrow P \rightarrow b \) is an \( a \rightarrow b \) triangle.

Subcase 1.2. \( q_1 \neq q_2 \)

Now there arise two possibilities:

(i) Both \( p_1 \) and \( p_2 \) belong to one of the subpaths \( s \rightarrow P \rightarrow s \) or \( x \rightarrow P \rightarrow x \).

(ii) One of them (say) \( p_1 \) \( \in s \rightarrow P \rightarrow s \) and the other \( p_2 \) \( \in x \rightarrow P \rightarrow x \).

Figure 4.19:

\( p_2 \) is an interior vertex of \( C_4 \) and \( q_2 \) is its exterior vertex or vice versa. Since \( S_2 \) is a subpath of \( \mu \), \( S_2 \) cannot have common vertices with \( x_4 \rightarrow \mu \rightarrow p_2 \). By the choice of \( p_2 \), \( S_2 \) cannot contain the adjacent vertices \( x_4 \) or \( x \). Hence an edge of \( S_2 \) must cross an edge of \( C_2 \). This contradicts the planarity of \( G \) which completes the proof of \((\alpha_5)\). Now, let us prove the theorem by using \((\alpha_i)\), \( i = 1, 2, 3, 4, 5 \) in two cases.

Case 1. \( t = x_3 \).

In this case, \( y_3 \) is adjacent to \( y_4 \), \( s = x_4 \) and \( t = x_3 \). If \( \mu \) contains the vertex \( y_3 \), then either \( f \in x_4 \rightarrow \mu \rightarrow y_4 \) or \( f \in x_3 \rightarrow \mu \rightarrow y_3 \). In the first case we get the contradiction that \( f \) is adjacent to \( x_4 \) and in the second case we get the contradiction that \( f \) is adjacent to \( x_3 \). Hence \( \mu \) does not contain \( y_3 \). Therefore \( p_1, p_2 \in s \rightarrow P \rightarrow t \). Without loss of generality, assume that \( p_1 \in s \rightarrow P \rightarrow p_2 \).

Subcase 1.1. \( q_1 = q_2 \).

Since, \( q_1 = q_2 \), both coincides with \( f \). Here the subpath \( f_2 \rightarrow \mu \rightarrow (p_2) \) lies interior to the cycle \( p_1 \rightarrow \mu \rightarrow f \rightarrow Q \rightarrow y_3 \rightarrow x_3 \rightarrow P \rightarrow p_1 \). Therefore there
is no chord from $a \to P \to (p_1)$ to $f_2 \to \mu \to (p_2)$. Similarly $f_1 \to \mu \to (p_1)$ lies interior to the cycle $p_2 \to \mu \to f \to Q \to y_4 \to x_4 \to P \to p_2$. Therefore there is no chord from $b \to P \to (p_2)$ to $f_1 \to \mu \to (p_1)$. By $(\alpha_1)$, $p_1$ and $p_2$ are not adjacent. So we get the contradiction that $a \to P \to p_1 \to \mu \to p_2 \to P \to b$ is an $a - b$ triangle path containing $f$.

Subcase 1.2. $q_1 \neq q_2$

Now there arise two possibilities;

(i). Both $p_1$ and $p_2$ belong to one of the subpaths $x_4 \to P \to x$ or $x \to P \to x_3$.

(ii). One of them (say) $p_1 \in x_4 \to P \to x$ and the other $p_2 \in x \to P \to x_3$.

(i). Assume that $p_1, p_2 \in x_4 \to P \to x$.

Without loss of generality, we can assume that $p_1 \in s \to P \to p_2$. Hence by $(\alpha_4)$, $q_1 \in y_4 \to Q \to q_2$.

Now consider the subpath $P_1 : a \to P \to p_1 \to \mu \to q_1 \to Q \to x \to P \to b$ containing $f$. Each interior vertex of $p_1 \to \mu \to q_1$ is an interior vertex of the cycle $x_4 \to P \to p_2 \to \mu \to q_2 \to Q \to y_4 \to x_4$. Hence there is no chord from $(p_1) \to q_1 \to (q_1)$ to $x \to P \to b$. Suppose there exists a chord $q_1 \mu p_1$ from $q_1 \to Q \to x$ to $b \to P \to x$. Then by the choice of $y_4$ and $P$, $q_1 \mu \in y_1 \to Q \to x$ and $p_1\mu = x_2$. But in this case, we get the contradiction that $q_1 \in J^\Delta(a, b)$. Hence there is no chord from $(p_1) \to \mu \to (q_1)$ to $x \to P \to b$. Therefore $P_1$ is an $a - b$ triangle path containing $f$ and which is a contradiction. We can derive a similar contradiction when $p_1, p_2 \in x \to P \to x_3$.

(ii). $p_1 \in x_4 \to P \to x$ and $p_2 \in x \to P \to x_3$. 
Here each vertex of \((p_1) \to \mu \to (p_2)\) lies interior to the cycle \(C_1\) and each vertex of \((q_2) \to \mu \to (p_2)\) lies interior to \(C_2\).

If this is the case, we can easily see that, without losing the planarity of \(G\), it is impossible to have chords to \(x\) and \(y\). Hence, in this case also we get the contradiction \(f \notin J^\Delta(a, b)\). Similarly, we can derive the contradiction \(f \notin J^\Delta(a, b)\) and \(p_1 \in Q \to P \to x_2\).

Case 2. \(f \neq x_3\).

Therefore \(y_3\) is not adjacent to \(y_4\) and so \(y_3 = y_4\), hence the \(u\)-vertex is the last vertex on \(y_4 \to Q \to y_3\) which is adjacent to \(y_3\) and \(b\)-vertex is the neighbour of \(u\) on \(u \to Q \to y_3\). Hence \(f \neq y_4\) is adjacent to \(y_4\). Here we have to consider three cases.

1. Both \(p_1\) and \(p_2\) are vertices of \(x_2 \to P \to x_3\).
2. Both \(p_1\) and \(p_2\) belong to \(y_3 \to Q \to u\).
3. One of the vertices (say), \(p_1 \in x_2 \to P \to x_3\) and the other vertex \(p_2 \in y_3 \to Q \to u\).

In the first case we can derive the contradiction \(f \notin J^\Delta(a, b)\). The proof is similar to the proof of the subcase (ii) of \((a)\).

So let us consider the other two cases.

Let \(x'\) be the adjacent of \(x\) on \(x \to P \to y_4\).

2. Both \(p_1\) and \(p_2\) belong to \(y_3 \to Q \to u\).

Let us assume that \(p_1 \in y_3 \to Q \to p_2\). Then \((y_3, p_1) = (y_4, \mu) \to (p_2)\), hence each vertex of \((p_2) \to \mu \to (q_2)\) lies interior to \(C_1\), and each vertex of \((q_2) \to \mu \to (p_2)\) lies interior to \(C_2\). So there is no chord from \((p_2) \to \mu \to (q_2)\) to \(x \to P \to b\). Then \(a \to P \to y_4 \to x_2 \to Q \to p_2 \to \mu \to y_3 \to Q \to x \to P \to b\) is an \(a - b\) triangle path containing \(f\), again a contradiction.

Similarly we can derive a contradiction if \(p_1 \in y_3 \to Q \to y_4\).
Here each vertex of \((p_1) \to \mu \to (p_2)\) lies interior to the cycle \(C_1\) and each vertex of \((q_2) \to \mu \to (p_2)\) lies interior to \(C_2\).

If this is the case, we can easily see that, without disturbing the planarity of \(G\), it is impossible to have chords to avoid \(f\) from \(J^\Delta(a,b)\). Hence, in this case also we get the contradiction \(f \in J^\Delta(a,b)\). Similarly we can derive the contradiction \(f \in J^\Delta(a,b)\) when \(p_2 \in x_4 \to P \to x\) and \(p_1 \in x \to P \to x_3\).

Case 2. \(t \neq x_3\).

Therefore \(y_3\) is not adjacent to \(y_4\) and so \(x_1 = x_4\). Here the \(u\)-vertex is the last vertex on \(y_4 \to Q \to y_3\) which is adjacent to \(x_4\) and \(t\)-vertex is the neighbour of \(u\) on \(u \to Q \to y_3\). Hence \(t\) is not adjacent to \(y_4\). Here we have to consider three cases.

1. Both \(p_1\) and \(p_2\) are vertices of \(x_4 \to P \to x_3\).

2. Both \(p_1\) and \(p_2\) belong to \(y_3 \to Q \to u\).

3. One of the vertices (say), \(p_1 \in x_4 \to P \to x_3\) and the other vertex \(p_2 \in y_3 \to Q \to u\).

In the first case we can derive the contradiction \(f \in J^\Delta(a,b)\). The proof is similar to the proof of the subcase 1.2(i)and 1.2 (ii).

So let us prove the cases 2 and 3.

Let \(x\, y\, y\, x\) be the chord from \(a \to P \to x_4\) to \(p_2 \to Q \to y_4\).

2. both \(p_1\) and \(p_2\) belong to \(y_3 \to Q \to u\).

Let us assume that \(p_1 \in y_3 \to Q \to p_2\). Then by \((\alpha_3), q_2 \in y_4 \to Q \to q_1\).

Hence each vertex of \((p_2) \to \mu \to (q_2)\) lies interior to the cycle \(p_1 \to Q \to q_1 \to \mu \to p_1\). So there is no chord from \((p_2) \to \mu \to (q_2)\) to \(x \to P \to b\). Then \(a \to P \to y \to Q \to p_2 \to \mu \to q_2 \to Q \to x \to P \to b\) is an \(a-b\) triangle path containing \(f\), again a contradiction.

Similarly we can derive a contradiction when \(p_2 \in y_3 \to Q \to p_1\).
Here we have to consider two possibilities namely, \( P_1 = x_4 \) and \( P_1 \in x \rightarrow P \rightarrow x_3 \). Suppose \( P_1 = x_4 \). Then \( a \rightarrow P \rightarrow x_4 \rightarrow y_2 \rightarrow y_3 \rightarrow \mu \rightarrow f \rightarrow x \rightarrow P \rightarrow b \) is an \( a-b \) path containing \( f \). Hence there must exist a chord (say) \( p_2 \) from \( (p_2) \rightarrow \mu \rightarrow (f) \) to \( x \rightarrow P \rightarrow x_3 \). But in this case there is the \( a-b \) path \( a \rightarrow P \rightarrow x \rightarrow Q \rightarrow f \rightarrow \mu \rightarrow p \rightarrow q \rightarrow P \rightarrow b \). Evidently it is an \( a-b \) triangle path containing \( f \) or it gives such a path. This gives us the contradiction that \( f \in J^\Delta(a,b) \). Similarly we can derive the same contradiction when \( P_1 \in x \rightarrow P \rightarrow x_3 \). Thus in all cases we have derived contradictions and which shows that there is no \( a-b \) triangle path in \( G \) containing \( f \). Hence \( G_i \) is actually \( G_i(s, \hat{f}, t) \), for \( i= 1,2,3,4,5 \) and 6. This completes the proof.