Chapter 5

$J^\Delta$-Convexity

5.1 Introduction

In this chapter we mainly study the convexity induced by the $J^\Delta$-transit function. We first characterise the $J^\Delta$-convex hull similar to the $J$-convex hull due to Dutchet [17]. The chapter consists of four sections. In the first section we classify the graphs according to the number of non-trivial convex sets similar to the $I$-convexity and $J$-convexity. In section two we have made an attempt for the evaluation of $J^\Delta$-convexity invariants like Caratheodory, Helley and Radon type partition numbers. Section three deals mainly with the concept of $J$-gated sets similar to the $I$-gated sets. We have proved that for $HHD$-free graphs the $J$-gated sets are precisely the $I^\Delta$ convex sets and hence the family of $J$-gated sets form a convexity for the $HHD$-free graphs.

Let $G = (V, E)$ be a connected graph. A $V$ subset $S$ separates two others $A$ and $B$ (say), when every path joining a vertex of $A$ to a vertex of $B$ encounters $A$. The subset $S$ separates a set $A$ and a vertex $v \notin A$ if it separates $A$ and $\{v\}$. A complete subgraph of $G$ is called a clique and a clique containing more than one vertex and different from $V$ is called a nontrivial clique. A clique separating, two sets or a set and a vertex is called a clique separator. A path connecting a vertex $v \notin A$, a subset of $V(G)$, to a vertex $a \in A$ is called a $v - A$ path if $a$ is the only vertex of $A$ lying on the path.
Lemma 9 Let $G = (V, E)$ be a graph and $A \subseteq V$. Then a vertex $v \in V(G)$ is not an element of $< J^\Delta(A) >$ if and only if there exists a clique separator $C$ separating $v$ and $A$ such that every $v - C$ path connecting two distinct vertices of $C$ encloses an induced cycle of length greater than or equal to four.

Proof. Suppose $v \notin < J^\Delta(A) >$, then there exists a triangle induced path $\mu$ connecting $v$ to a vertex $a$ in $A$ and a vertex $x$ on $\mu$ such that $x \in< J^\Delta(A) >$, but the successor of $x$ on $\mu$, as we traverse from $a$ to $v$ along $\mu$, is not in $< J^\Delta(A) >$. Let $C$ be the set formed by the vertices like $x$. Evidently every $v - A$ path encounters $C$, hence $C$ is a separator; separating $v$ and $A$. If $C$ contains only one vertex, we are done. So let us assume that $C$ contains at least two vertices. Now take any two vertices $x$ and $y$ in $C$. We claim that $x$ is adjacent to $y$. Suppose not, since $x$ and $y$ are elements of $C$, there exist two triangle induced paths $\mu_x$ and $\mu_y$ (say) connecting $v$ to the two vertices of $A$, so that $x$ is a vertex on $\mu_x$ and $y$ is on $\mu_y$. Let $xtyl$ be the chord from $x \rightarrow \mu_x \rightarrow v$ to $y \rightarrow \mu_y \rightarrow v$. Since $x$ is not adjacent to $y$, either $xt$ is different from $x$ or $yl$ is different from $y$. In this case $xt$ and $yl$ are vertices of the $x - y$ induced-path formed by the $x - xt$ subpath of $\mu_x$, $y - yl$ subpath of $\mu_y$ and the edge $xtyl$. Hence $xt, yl \in< J^\Delta(A) >$, a contradiction. So $C$ forms a clique. Let $\mu_1$ and $\mu_2$ be two $v - C$ paths connecting two distinct vertices $x$ and $y$ of $C$ respectively, to $v$. Let $xt$ be the immediate successor of $x$ on $\mu_1$ and $yl$ be the immediate successor of $y$ on $\mu_2$. Evidently $xt$ and $yl$ are different. Hence $\mu_1$ and $\mu_2$ encloses an induced cycle of length greater than three. Conversely, let there exists a clique separator $C$, separating $v$ and $A$ as stated in the lemma. Let $B$ be the component of $G \setminus C$ containing $v$. Then $V \setminus B$ is $J^\Delta$-convex. For, if not we can find two vertices $x$ and $y$ in $V \setminus B$ and a vertex $z$ on an $x - y$ triangle induced path $P$, so that $z \notin V \setminus B$. Therefore $z \in B$ and which implies $z \notin C$. Consider the $x - z$ subpath $\mu_x$ and the $y - z$ subpath $\mu_y$ of $P$. Since $B$ is the component of $G \setminus C$, $\mu_x$ and $\mu_y$ must intersect $C$ at some vertices $u$ and $w$ (say) respectively. If $u \neq w$, then $uzw$ forms a triangle, since $P$ is a triangle induced path and $u, w \in C$. Now consider the two paths connecting $v$ to the vertices $u$ and $w$ obtained by extending $\mu_x$ and $\mu_y$ to $v$ through some
convexity

v – z path. Clearly these two paths do not enclose an induced cycle of length greater than three, a contradiction. If \( u = w \), then \( C = \{ u \} \). Hence \( P \) is not a path, again a contradiction. So \( V \setminus B \) is \( J^\Delta \)-convex. Now, \( A \subseteq V \setminus B \) and therefore \( < J^\Delta(A) > \subseteq V \setminus B \). Hence \( v \notin < J^\Delta(A) > \). ■

Remark 8 If \( v \in V \) is not an element of \( < J^\Delta(A) > \), \( A \subseteq V \), it is possible to find a clique separator \( C \) separating \( v \) and \( A \) such that \( C \subseteq < J^\Delta(A) > \) and every vertex of \( C \) is the end vertex of some \( v - C \) path.

Remark 9 In the first paragraph of the proof, we have chosen \( x \) as a vertex on an \( A - v \) triangle induced path so that \( x \in < J^\Delta(A) > \) and its immediate successor is not in \( < J^\Delta(A) > \). But in this case \( x \) lies on an \( A - v \) induced path. Hence \( C \) can be redefined as the set of all vertices like \( x \) lying on some \( A - v \) induced path so that its immediate successor on the path is not in \( < J^\Delta(A) > \).

5.2 k-convex graphs

Classification of graphs according to the number of nontrivial convex sets has been attempted by various authors. In the case of \( I \)-convexity considerable study has been made by Hebbare, Acharya, Varthak, Rao, Batten, Parvathy etc. [1],[22],[23], [40] and [6]. A similar attempt for \( J \)-convexity has been done by Parvathy [39]. It has been proved that for a \( k \)-convex, triangle free and two connected graph, there is a positive integer \( n \) such that \((n - 1)(n + 2)/2 \leq k \leq 2^n - 2\). For the \( I \) and \( J \) convexities the cliques are trivially convex, so the classification of graphs according to the nontrivial convex sets is discussed with respect to the clique number \( \omega \). In \( J^\Delta \)-convexity, there is no role for the clique number \( \omega \), since the cliques need not be \( J^\Delta \)-convex. In \( J^\Delta \) convexity instead of the size of the clique, the number of non trivial clique separators play an important role. So we define a non trivial \( J^\Delta \)- convex set as one which is different from the empty set, singletons and \( V \). Mainly there are two types of \( k \)-convex graphs. One is the nested \( k \)-convex graphs, that is graphs in which the \( k \) nontrivial convex sets form a nested sequence. The other is the minimal
$J^\Delta$-Convexity

$k$-convex graphs. A nontrivial $J^\Delta$-convex subset is called minimal if it does not contain any nontrivial convex set properly. A graph which contains exactly $k$-minimal convex subsets is called a minimal $k$-convex graph. Both nested and minimal $k$-convex graphs have nice characterizations. They have been characterised by non trivial clique separators. There are also graphs belonging to neither of the two classes. A $J^\Delta$ convex set different from the empty set, singleton and $V$ is called a nontrivial $J^\Delta$ convex set or simply a nontrivial convex set. A graph $G$ is called a $k$-convex graph if it has exactly $k$ nontrivial $J^\Delta$-convex sets. A $k$-convex graph with nontrivial $J^\Delta$ convex sets $C_1, C_2, \ldots, C_k$ is called a nested $k$-convex graph if $C_1 \subseteq C_2 \subseteq \ldots \subseteq C_k$. In other words a $k$-convex graph $G$ is said to be nested if any two nontrivial convex sets of $G$ are comparable. So a nested $k$-convex graph cannot have cut vertices. For graphs with number of vertices up to six, one can easily produce nontrivial noncomparable convex sets. But the graph shown below is nested 1-convex.

Hence for a nested $k$-convex graph, the minimum number of vertices is seven. Nested 1-convex graph is otherwise known as uniconvex graph. A $J^\Delta$ convex set is said to be minimal if it does not contain any non trivial convex set as its subset. A $k$-convex graph is called a minimal $k$-convex graph if all of its $k$ non-trivial convex sets are minimal convex. Every minimal 1-convex graph is the same as a uniconvex graph. The smallest minimal 2-convex graph is $P_3$. This is the only minimal $k$-convex graph with three vertices. There is no minimal 3-convex graph with 4 or 5 vertices. The smallest minimal 3-convex graph is isomorphic to the graph obtained by deleting two consecutive leaves from $W_6$. Hence to have a minimal $k$-convex graph with $k \geq 3$, the minimum number of vertices is six.

For any positive integer $k \geq 4$ there is the minimal $k$-convex graph $C_k$.

Theorem 15 Let $G$ be a connected graph with at least seven vertices. Then $G$ is a nested $k$-convex graph if and only if $G$ has exactly $k$ non-trivial cliques $C_i, i = 1, 2, \ldots, k$ such that corresponding to each $C_i$, there is a non trivial component $D_i$ of $G \setminus C_i$ satisfying (i). $D_{i+1} \subseteq D_i, i = 1, 2, \ldots, k - 1$. (ii). $< J^\Delta(C_i), D_i >$ is partition of $V(G)$, for $i = 1, 2, \ldots, k$. (iii) For any three
Proof. Let us prove the necessary part. Assume that $G$ is a nested $k$-convex graph. Hence $G$ has exactly $k$ nontrivial convex sets $B_i$, $i = 1, 2, \ldots, k$ with the property that $B_1 \subset B_2 \subset \ldots \subset B_k$. For each $i = 1, 2, \ldots, k$, we can prove the existence of a nontrivial clique $C_i$ and a unique component $D_i$ of $G \setminus C_i$ satisfying

$(\alpha_1)$: $< J^\Delta(C_i) > = B_i$, for $i = 1, 2, 3, \ldots, k$.

$(\alpha_2)$: $(V(B_i), V(D_i))$ forms a partition of $V(G)$, for $i = 1, 2, \ldots, k$.

Let us start with the smallest nontrivial convex set $B_1$ of $G$ and a vertex $d_1 \notin V(B_1)$. Since $d_1 \notin V(B_1)$, by the Lemma 9, there exists a clique separator $C_1 \subseteq B_1$, separating $d_1$ and $B_1$, such that

$(a)$: every vertex of $C_1$ is an end vertex of some $C_1 - d_1$ path.
(b): any two $C_1 - d_1$ paths connecting two distinct vertices of $C_1$ contains an induced cycle of length greater than or equal to four.

Since $G$ has no cutvertex $< J^\Delta(C_1) >$ is nontrivial and is contained in $B_1$. Hence $< J^\Delta(C_1) > = B_1$. Let $D_1$ be the component of $G \setminus C_1$, containing $d_1$. If possible, let $F_1$ be any other component of $G \setminus C_1$, containing a vertex $f_1 \notin V(B_1)$. Since $f_1 \notin V(B_1)$, by the Lemma 9, there exists a clique separator $K_1 \subseteq B_1$, separating $f_1$ and $B_1$ and satisfying conditions similar to (a) and (b). Every path connecting $f_1$ to a vertex of $D_1$ encounters $C_1$ and $C_1 \subseteq B_1$. Hence every path connecting $f_1$ to $C_1 \cup D_1$ encounters $K_1$. Hence $K_1$ acts as a clique separator separating $f_1$ and $C_1 \cup D_1$ and satisfying conditions similar to (a) and (b). So $f_1 \notin < J^\Delta(C_1 \cup D_1)$. Similarly, we can prove that the clique $C_1$ separates $d_1$ and $C_1 \cup F_1$ and satisfies conditions similar to (a) and (b). So $d_1 \notin < J^\Delta(C_1 \cup F_1)$. Thus we have two nontrivial noncomparable convex sets $< J^\Delta(C_1 \cup D_1) >$ and $< J^\Delta(C_1 \cup F_1) >$. This contradiction proves that all vertices of $G \setminus C_1$ which are not in $B_1$ lie in $D_1$. Also no vertex of $D_1$ lies in $B_1$. Hence $(V(B_1), V(D_1))$ forms a partition of $V(G)$.

Now consider any $i \geq 2$. We can find a vertex $d_i \in V(B_{i+1})$ so that $d_i \notin V(B_i)$. Here we take $V(B_{k+1}) = V(G)$. Since $d_i \notin V(B_i)$, there exists a clique separator $C_i \subseteq B_i$ separating $d_i$ and $B_i$ and satisfying conditions similar to (a) and (b). We can prove that $C_i$ has vertices in common with $V(B_i) \setminus V(B_{i-1})$. Suppose not. Then $C_i \subseteq B_j$ for some $j \leq i - 1$. Take any vertex $u$ in $V(B_i) \setminus V(B_{i-1})$. Then every path connecting $u$ to a vertex in $B_{i+1}$ encounters $C_i$. Moreover $u \notin < J^\Delta(C_i) >$, if not we get the contradiction $u \in < J^\Delta(C_i) > \subseteq B_j \subseteq B_{i-1}$. Hence by the Lemma 9, $u \notin B_{i+1}$ and which conflicts with the condition $B_i \subseteq B_{i+1}$. So $C_i$ has a vertex in common with $V(B_i) \setminus V(B_{i-1})$. This shows that $< J^\Delta(C_i) >$ is not a subset of $B_{i-1}$.

Hence

$< J^\Delta(C_i) >= B_i - I$.
Let $D_i$ be the component of $V \setminus C_i$ containing $d_i$. If possible, let $F_i$ be another component of $V \setminus C_i$ containing a vertex $f_i \notin V(B_i)$. As in the case of $D_1$, we can easily produce the nontrivial noncomparable convex sets $< J^\Delta(C_i \cup D_i)$. $< J^\Delta(C_i \cup F_i)$. This contradiction shows that

$$(V(B_i), V(D_i))$$ forms a partition of $V(G)$—II.

Since $B_1 \subsetneq B_2 \subsetneq \ldots \subsetneq B_k$, I and II together proves conditions (i) and (ii) of the theorem. Now let us prove (iii).

Take any three vertices $x, y$ and $z$ of $G$. Suppose $z \notin < J^\Delta(x, y) >$. Hence $< J^\Delta(x, y) >$ is a nontrivial convex set. Therefore $< J^\Delta(x, y) > \subsetneq B_i$ for some $i$. So by (ii), $z \in D_i$. Evidently, by (i) and (ii), $< J^\Delta(C_1) > \subsetneq < J^\Delta(C_2) > \subsetneq \ldots \subsetneq < J^\Delta(C_k) >$. We can prove that $\{ < J^\Delta(C_i) > \}, i = 1, 2, \ldots, k$ are the only nontrivial convex subsets of $G$. Let $C$ be any nontrivial convex set. Our first finding is that $C$ cannot have a common vertex with $D_k$. Suppose $C$ has a vertex (say) $x$ in common with $D_k$. Since $C$ is nontrivial, we can find another vertex $y$ of $C$ different from $x$. Since $x, y \in C$, we have $< J^\Delta(x, y) > \subsetneq C$. Since $x \in D_k$, $x \notin < J^\Delta(C_i) >$ for all $i = 1, 2, 3, \ldots, k$ by (ii). Hence by (iii), for all any $z \in V$, $z \in < J^\Delta(x, y) >$. So this gives the contradiction that $C = V$. Hence $C$ has no vertex in common with $D_k$. Let $j$ be the smallest integer so that $D_j$ has no vertex in common with $C$. Hence

$$C \subsetneq < J^\Delta(C_j) >$$—III.

Now we can prove the other way of inclusion. By the choice of $j$, $C$ has a vertex (say) $u$ in common with $D_{j-1}$. Since $u \in D_{j-1}$,

$$u \notin < J^\Delta(C_{j-1}) >$$—IV

Take any vertex $v \in C$. Hence by III, $v \in < J^\Delta(C_j) >$. Since $u, v \in C$,

$$< J^\Delta(u, v) > \subsetneq C$$—VI.

Take
$z \in \langle J^\wedge(C_j) \rangle \quad \text{VII}$

Therefore

$z \notin D_j \quad \text{VIII}$

$IV$ gives $u \notin \langle J^\wedge(C_{j-1}) \rangle, \langle J^\wedge(C_{j-2}) \rangle, \ldots$

$VIII$ gives $z \notin D_j, D_{j+1}, \ldots$ Hence by (iii),

$z \in \langle J^\wedge(u,v) \rangle \quad \text{IX}$

Therefore by $VII$ and $IX$, $\langle J^\wedge(C_j) \rangle \subseteq \langle J^\wedge(u,v) \rangle$. Therefore by $VI$,

$\langle J^\wedge(C_j) \rangle \subseteq C \quad \text{X}$

Therefore by $IV$ and $X$, $C = \langle J^\wedge(C_j) \rangle$. This completes the sufficiency part of the theorem.

**Theorem 16** For every integer $k \geq 1$, there exists a nested $k$-convex graph $H_k$.

**Proof.** We use induction on $k$. Let $D$ be the graph $H_1$ with two specified vertices $u$ and $v$ as shown in the figure 5.1(B).

Evidently the $J^\wedge$-convexity on $D$ is trivial. Consider a $K_2$ with vertices $u_1$ and $v_1$. Let $H_1$ be the graph obtained by identifying the vertex pairs $(u,u_1)$ and $(v,v_1)$, so that $K_2$ and $D$ remain as two edge disjoint subgraphs of $H_1$. Moreover $V(K_2) \cap V(D) = \{u_1,v_1\}$ and $V(D) \cup V(K_2) = V(H_1)$. Hence the graph $H_1$ which proves the proposition when $k = 1$.

Now suppose that $k \geq 2$ and $H_k$ is a nested $k$ convex graph. Hence it has exactly $k$ non trivial convex sets $C_i, i = 1, 2, \ldots, k$ with $C_1 \subseteq C_2 \subseteq \ldots \subseteq C_k \subseteq V(H_k) \ldots \ldots \ldots \ldots \ldots \ldots$ (*).

Choose the vertices $u_k$ and $v_k$ of $H_k$ so that $u_k \in V(C_k)$ and $v_k \in V(H_k) \setminus V(C_k)$. Since $H_k$ is connected, without loss of generality we can assume that $u_k$ is adjacent to $v_k$ in $H_k$. Let $H_{k+1}$ be the graph obtained by identifying the vertex pairs $(u,u_k)$ and $(v,v_k)$ so that $H_k$ and $D$ are two edge disjoint subgraphs of $H_{k+1}$. Moreover $V(H_k) \cap V(D) = \{u_k,v_k\}$ and $V(H_k) \cup V(D) = V(H_{k+1})$. 


Claim I: Every subset of $H_k$ which is convex in $H_k$ is so in $H_{k+1}$. Let $C$ be any convex subset of $H_k$.

Take any two vertices $x, y \in C$ and any triangle induced path $P$, connecting $x$ and $y$ in $H_{k+1}$. Suppose there is a vertex $z$ on $P$ such that $z \notin C$. Then $z \notin V(H_k)$, since $C$ is a convex subset of $H_k$. Therefore $z \in V(D) \setminus \{u_k, v_k\}$. By the construction of $H_{k+1}$, $P$ contains both $u_k$ and $v_k$. But $u_k$ is adjacent to $v_k$. Hence $z$ is adjacent to both $u_k$ and $v_k$. But no vertex in $D$ is adjacent to both $u_k$ and $v_k$. Hence $z \notin C$. So $C$ is a convex subset of $H_{k+1}$. In particular $V(H_k)$ is a convex subset of $H_{k+1}$. If we consider $V(H_k)$ and any vertex $w \in V(D) \setminus \{u_k, v_k\}$, then by the construction, edge $u_kv_k$ act as clique separator forbidding $w$ to be an element of $V(H_k)$. So $V(H_k)$ is a nontrivial convex subset of $H_{k+1}$. Thus $H_{k+1}$ has $k+1$ nontrivial convex subsets, $C_1 \subseteq C_2 \subseteq \cdots \subseteq C_k \subseteq V(H_k)$.

Claim II: The extension of the trivial convexity on $D$ to $H_{k+1}$ is the trivial convexity on $H_{k+1}$.

Let $< J^{\Delta}(A) >_G$ denote the $J^{\Delta}$-convex hull of the subset $A$ on the graph $G$. Evidently for any $x, y \in V(D)$, $V(D) \subseteq < J^{\Delta}(x, y) >_{H_{k+1}}$. Therefore $u_k, v_k \in < J^{\Delta}(x, y) >_{H_{k+1}}$. Therefore

$$< J^{\Delta}(u_k, v_k) >_{H_{k+1}} \subseteq < J^{\Delta}(x, y) >_{H_{k+1}} \quad (I).$$

Now $u_k, v_k \in V(H_k)$. Therefore $< J^{\Delta}(u_k, v_k) >_{H_k} \subseteq < J^{\Delta}(V(H_k)) >_{H_k}$. That is, $< J^{\Delta}(u_k, v_k) >_{H_k} \subseteq V(H_k)$, since $V(H_k)$ is convex in $H_k$. But $v_k \notin C_k$. Therefore $C_k \subseteq < J^{\Delta}(u_k, v_k) >_{H_k}$. Therefore $C_i \subseteq < J^{\Delta}(u_k, v_k) >_{H_k}$, for all $i = 1, 2, \ldots, k$. Therefore $< J^{\Delta}(u_k, v_k) >_{H_k} = V(H_k)$ since $H_k$ is a nested $k$-convex graph. That is,

$$< J^{\Delta}(u_k, v_k) >_{H_{k+1}} = V(H_k) \quad (II).$$

From (I) and (II), we get $V(H_k) \subsetneq < J^{\Delta}(x, y) >_{H_{k+1}}$. Also $V(D) \subsetneq < J^{\Delta}(x, y) >_{H_k}$. Therefore $V(H_k) \cup V(D) \subsetneq < J^{\Delta}(x, y) >_{H_k}$. That is, $V(H_{k+1}) \subsetneq < J^{\Delta}(x, y) >_{H_{k+1}}$. Therefore $V(H_{k+1}) = < J^{\Delta}(x, y) >_{H_{k+1}}$. Hence the claim.
Claim III: Every nontrivial convex subset of $H_{k+1}$ is a convex subset of $H_k$.

Let $C'$ be a nontrivial convex subset of $H_{k+1}$. Suppose it has a vertex $a$ in common with $D \setminus \{u_k, v_k\}$. Since $C'$ is non trivial, it has another vertex $b$ different from $a$. If $b$ lies in $D \setminus \{u_k, v_k\}$, then by claim II, $C' = V(H_{k+1})$, a contradiction. If $b$ is not a vertex of $D \setminus \{u_k, v_k\}$, then $C'$ is connected. Without loss of generality we can assume that either $b$ is $u_k$ or $v_k$. Hence in this case also, by claim II, we can arrive at the same contradiction. Thus the only nontrivial convex subsets of $H_{k+1}$ are $C_1 \subset C_2 \subset \ldots \subset C_k \subset (V(H_k))$.

Remark 10 In the construction of $H_{k+1}$, it is always possible to find $u_k, v_k \in V(H_k) \setminus V(C_k)$.

Theorem 17 If $G$ is a minimal $k$-convex graph with $k \geq 3$, then $G$ has no cut vertex.

Proof. Let $G$ be a minimal $k$-convex graph with $k \geq 3$. Suppose $G$ has a cut vertex $v$. Hence $v$ has two neighbours $u$ and $w$ in $G$ such that every $u - w$ path contains $v$. Hence by the Lemma 9, $u \not< J^\wedge(v, w)$ and $w \not< J^\wedge(u, v)$. Thus $< J^\wedge(v, w)>$ and $< J^\wedge(u, v)>$ are two distinct non trivial convex sets. Hence they are minimal convex subsets. Since $k \geq 3$, $G$ has another minimal nontrivial convex set $C$ different from $< J^\wedge(v, w)>$ and $< J^\wedge(u, v)>$. Now take any vertex $x \in C$. Then either every $x - u$ path contains $v$ or every $x - w$ path contains $v$. Let us assume that every $x - u$ path contains $v$. Let $B = C \cup \{u, w\}$. Then every $u - B$ path contains $v$. Hence by the Lemma , $u \not< J^\wedge(B)$. So $< J^\wedge(B)>$ is a non trivial convex set. But it is not a minimal convex set as it contains $C$ and $< J^\wedge(v, w)>$ properly, which is a contradiction. Hence the theorem. □

Theorem 18 Let $G$ be a graph with at least six vertices. Then $G$ is a minimal $k$-convex graph ($k \geq 3$) if and only if it has exactly $k$ nontrivial cliques $C_i; i = 1, 2, \ldots, k$ with the properties (i) $i \neq j \rightarrow < J^\wedge(C_i) > < J^\wedge(C_j) >$. (ii) for $x, y, z \in V$, $z \not< J^\wedge(x, y) \Rightarrow x, y \in < J^\wedge(C_i) >$ and $z \not< J^\wedge(C_i) >$ for some $i$. 
Proof. Let \( G \) be a minimal \( k \)-convex graph with \( k \geq 3 \). So \( G \) has exactly \( k \)-non trivial convex sets \( B_i \); \( i = 1, 2, \ldots, k \) such that the only convex sets properly contained in each \( B_i \) are singletons or emptyset. Since \( B_i \neq V \), there is a \( v \in V \setminus B_i \). Hence by the Lemma 9, there exists a clique separator \( C_i \subseteq B_i \) separating \( v \) and \( B_i \) such that any two \( v - C_i \) paths connecting two distinct vertices of \( C_i \) contains an induced cycle of length greater than or equal to four. Since \( k \geq 3 \), \( G \) has no cut vertex. Hence \( C_i \) contains more than one vertex. Therefore \( \langle J^A(C_i) \rangle = B_i \), for \( i = 1, 2, \ldots, k \). Since \( B_i \)'s are different, (i) follows. Now consider \( x, y, z \in V(G) \) and assume that \( z \notin \langle J^A(x, y) \rangle \). Hence \( \langle J^A(x, y) \rangle \) is a non trivial convex set. So by assumption \( \langle J^A(x, y) \rangle = B_i \) for some \( i \) and which proves (ii).

Conversely, assume that \( G \) contains \( k \) nontrivial cliques \( C_i \); \( i = 1, 2, \ldots, k \) satisfying (i): \( i \neq j \Rightarrow \langle J^A(C_i) \rangle \neq \langle J^A(C_j) \rangle \); (ii): For any \( x, y, z \in V(G) \), \( z \notin \langle J^A(x, y) \rangle \Rightarrow x, y \in \langle J^A(C_i) \rangle \), \( z \notin \langle J^A(C_i) \rangle \) for some \( i \). Then by (ii), it is evident that each \( \langle J^A(C_i) \rangle \) is a non trivial minimal convex subset of \( G \). Hence to complete the proof, it is enough to prove that if \( C \) is any nontrivial convex subset of \( G \), then \( C = \langle J^A(C_i) \rangle \), for some \( i \). For that our first claim is that \( C \cap V(G) \setminus \cup\{ \langle J^A(C_i) \rangle \mid i = 1, 2, \ldots, k \} = \emptyset \). Suppose not, then we can find \( x \in C \cap V(G) \setminus \cup\{ \langle J^A(C_i) \rangle \mid i = 1, 2, \ldots, k \} \) and which implies that \( x \notin \langle J^A(C_i) \rangle \) for all \( i \). Since \( C \) is nontrivial, \( C \) contains a vertex \( y \) different from \( x \). Hence by (ii) \( \langle J^A(x, y) \rangle = V \), a contradiction.

Now we can prove that \( C = \langle J^A(C_i) \rangle \) for some \( i \).

First we prove that for any two distinct vertices \( u \) and \( v \) of \( C \); \( u, v \in \langle J^A(C_i) \rangle \) for a unique \( i \).

Let \( u \) and \( v \) be any two distinct vertices of \( C \). Assume the contrary that for no \( i \), \( u \) and \( v \) belong to the same \( \langle J^A(C_i) \rangle \). Take any vertex \( w \in V \), then by (ii), \( w \in \langle J^A(u, v) \rangle \). Hence \( \langle J^A(u, v) \rangle = V \) and which gives the contradiction \( C = V \). So both \( u \) and \( v \) are vertices of \( \langle J^A(C_i) \rangle \), for some \( i \). Suppose \( u, v \in \langle J^A(C_j) \rangle \) for some \( j \neq i \). Then by the minimality of the nontrivial convex sets, we get \( \langle J^A(C_i) \rangle = \langle J^A(x, y) \rangle = \langle J^A(C_j) \rangle \). This contradicts (i). Hence any two vertices \( u \) and \( v \) of \( C \) lie in a unique...
< J^\Delta(C_i) >$. Therefore $< J^\Delta(u, v) > \subseteq < J^\Delta(C_i) >$. Hence by the minimality of the nontrivial convex set $< J^\Delta(C_i) >$ we get $< J^\Delta(u, v) > = C$. Hence $C = < J^\Delta(C_i) >$. This completes the sufficiency part.

5.3 Invariants

In general the $I$-convexity has no nice structure, so the invariants behave quite arbitrarily. For particular classes of graphs like Dismantlable graphs, some of the $I$-convexity invariants have been studied by Bandelt and Mulder [4], Bandelt [2] etc. In this section we prove that for $J^\Delta$ convexity the combinatorial parameters $c$, $h$ and $r$ are universal, in the sense that for any connected graph $G$, $c = 2$, $h = 2$ and $r \leq 4$. The Tverberg type partition number $r_m$ for the $J^\Delta$ convexity has been computed by Changat et.al [11]. We also quote the result.

We start with the definition of the invariants for an abstract convexity $C$. The Carathéodory number $c$ of the convexity space $C$ is the smallest integer (if exists) such that for any finite subset $S$ of $V$, $< S >_C = \bigcup\{ < F >_C | F \subseteq S, |F| \leq c \}$.

The Helly number $h$ of $C$ is the smallest integer (if exists) such that every family of convex sets with an empty intersection contains a subfamily of at most $h$ members with an empty intersection. Equivalently, $h$ is the smallest natural number such that $\bigcap_{s \in S} (S \setminus \{s\})_C \neq \emptyset$ for every $(h+1)$-element subset $S$ of $V$.

The Radon number $r$ of $C$ is the smallest integer (if exists) such that every $r$-element set $A \subseteq V$ admits a Radon partition, that is, a partition $A = A_1 \cup A_2$, $(A_1 \cap A_2 = \emptyset)$ with $< A_1 >_C \cap < A_2 >_C \neq \emptyset$.

The $m^{th}$ Radon number, denoted by $r_m$, is the smallest number (if exists) such that every $r_m$-element set $A \subseteq V$ admits a Radon $m$-partition, that is a partition of $A$ into $m$ pair wise disjoint subsets $A_1, A_2, \ldots, A_m$ such that $< A_1 >_C \cap < A_2 >_C \cap \ldots \cap < A_m >_C \neq \emptyset$.

For a graph $G = (V, E)$, a $V$-subset $A$ is said to be $J^\Delta$-convexly independent, if $a \not\in < J^\Delta(A \setminus a) >$ for every $a \in A$ and $A$ is $J^\Delta$-convexly dependent otherwise.
Theorem 19 Let $G = (V, E)$ be a connected graph. Then the $J^\Delta$ convexity has the Caratheodory number 2.

Proof. Let $A$ be any subset of $V$, let $x \in <J^\Delta(A)>$; if $x \in <J(A)>$, then clearly $x \in <J(B)>$, for some $B \subseteq A$ with $|B| \leq 2$, since the induced path convexity in $G$ has Caratheodory number less than or equal to 2 ([17]). If $x \notin <J(A)>$, then without loss of generality, we can assume that $x$ is a vertex lying on a $J^\Delta$ path joining two adjacent vertices of $<J(A)>$, (say) $u$ and $v$. Then $u \in J(u_1, u_2)$ and $v \in J(v_1, v_2)$ where $u_1, u_2, v_1, v_2 \in A$. Suppose there exists an induced $u - u_1$ or $u - u_2$ path such that no vertex on it is adjacent to the one on an induced $v - v_1$, or $v - v_2$ path, then $u, v$ belong to $<J(u_1, v_1)>$ or $<J(u_1, v_2)>$ or $<J(v_1, v_2)>$ or $<J(u_2, v_1)>$ or $<J(u_2, v_2)>$ as the case may be and hence we are done. Suppose there exist vertices on every induced $u - u_1$ and $u - u_2$ paths adjacent to vertices on every induced $v - v_1$ and $v - v_2$ paths. Take the first adjacent pairs of vertices $w_1, w_1', w_2, w_2'$ on an induced $u - u_1$, $v - v_1$, $u - u_2$ and $v - v_2$ paths as we traverse from $u_1, v, u_2$ and $v$, respectively. Then the path formed by the union of the induced paths $u_1 - w_1, w_1' - v_1, w_1' - v, v - w_2', w_2' - v_2, w_2 - u_2$ is an induced $u_1 - u_2$ path containing $x$. Therefore, $u, v \in <J(u_1, u_2)>$ and hence $x \in <J^\Delta(u_1, u_2)>$. ■

Since an abstract convexity with Carathéodory number 2 satisfies the JHC property, We have the important corollary which we used to prove the Theorem 13 of Chapter 4.

Corollary 2 The $J^\Delta$ convexity satisfies the JHC property.

Theorem 20 Let $G = (V, E)$ be a connected graph with at least two vertices. Then the Helly number $h$ and the Radon number $r$ of the $J^\Delta$ convexity of $G$ are given by $h = 2$ and $3 \leq r \leq 4$.

Proof.

First we prove $r \leq 4$. We will show that any 4-point set $A = \{u_1, u_2, u_3, u_4\} \subseteq V$ has a Radon partition and that there are connected graphs $G$ in which there
J\textsuperscript{\triangle} - Convexity

are 3-point sets with no Radon partition. Assume that \( A \) is J\textsuperscript{\triangle} - convexly independent (if \( a \) is J\textsuperscript{\triangle} - convexly dependent then \( A \) has a Radon partition). There are two cases.

Case 1: \(< J\textsuperscript{\triangle}(A) >\) does not contain any triangle.

In this case, every J\textsuperscript{\triangle} path between two vertices in \( A \) is an induced path between the vertices. Therefore, \(< J\textsuperscript{\triangle}(B) > = < J(B) >\), for every subset \( B \) of \( A \). By Radon’s theorem for the induced path convexity, \( A \) has a Radon partition for the induced path convexity and hence \( A \) has a Random partition for the J\textsuperscript{\triangle} - convexity.

Case II: \(< J\textsuperscript{\triangle}(A) >\) contains at least one triangle.

In this case, at most one vertex of \( A \) can be a vertex of a triangle in \(< J\textsuperscript{\triangle}(A) >\), since if at least two vertices of \( A \) can form the vertices of a triangle in \(< J\textsuperscript{\triangle}(A) >\), then \( A \) is J\textsuperscript{\triangle} - convexly dependent. Thus no two of the four vertices \( u_1, u_2, u_3, u_4 \) form an edge of a triangle in \(< J\textsuperscript{\triangle}(A) >\). If \( A \) has no Radon partition then the shortest \( u_1, u_2, u_2 - u_3, u_3 - u_4, u_4 - u_1 \) paths induce a chordless cycle of length at least 4 contained in \(< J\textsuperscript{\triangle}(A) >\), which is a contradiction and hence \( A \) has a Random partition. To prove that \( h = 2 \), we have \( h \leq 3 \), by Levi’s inequality, \( (h \leq r - 1) [28] \). Clearly \( h \geq 2 \). Let \( A = \{u_1, u_2, u_3\} \) be any 3-point subset of \( V \). If \( A \) is J\textsuperscript{\triangle} - convexly dependent, then clearly \( \cap\{< J\textsuperscript{\triangle}(A \setminus a) > | a \in A \} \neq \emptyset \). So assume that \( A \) is J\textsuperscript{\triangle} - convexly independent. Then as in the proof of the first part, at most one vertex of \( A \) can be a vertex of a triangle in \(< J\textsuperscript{\triangle}(A) >\). Now, if \( \cap\{< J\textsuperscript{\triangle}(A \setminus a) > | a \in A \} = \emptyset \), then the shortest \( u_1 - u_2, u_2 - u_3, u_3 - u_1 \) paths induce a chordless cycle of length at least 4 and hence we get that \( A \) is convexly dependent, which is a contradiction. Thus it follows that \( h = 2 \), completing the proof. \( \blacksquare \)
5.4 J-gated sets

In this section we focus our attention on J-gated sets. The notion of I-gated sets in graphs has been studied by Mulder [35], Bandelt [2] etc. It has been proved that the family of I-gated sets form a convexity in any connected graph and is finer than the I-convexity [35]. In fact it has been shown in [35] that the family of I-gated sets form a convexity generated by the transit function

\[ F(u, v) = \{ z \in V | I(u, z) \cap I(z, v) = \{ z \} \}. \]

The notion of R-gated sets for any transit function \( R \) is introduced by Mulder [35]. Since our main investigation in the thesis is about J-transit functions, we have made an attempt to study the J-gated sets. We can easily prove that the family of J-gated sets need not form a convexity. A natural question is, for which graphs the family of J-gated sets form a convexity. So it is interesting to study the family of J-gated sets for particular classes of graphs. As we have considered \( HHD \)-free graphs quite often in this thesis, we examine whether the J-gated sets for \( HHD \)-free graphs form a convexity. The answer is in the affirmative and further we have proved that the J-gated sets are precisely the \( I^\Delta \)-convex sets for the \( HHD \)-free graphs. The characterisation of graphs for which the J-gated sets form a convexity still remains as an unsolved problem.

\textit{J-gated sets:} Let \( W \) be a subset of \( V(G) \) and \( z \) be a vertex of \( G \) not in \( W \). A vertex \( x \) in \( W \) is a \textit{J-gate} for \( z \) in \( W \) if \( x \) lies in \( J(z, w) \) for all \( w \) in \( W \). The set \( W \) is called \textit{J-gated} if every vertex \( z \) not in \( W \) has a unique \textit{J-gate} in \( W \). \( V \) and \( \emptyset \) are \textit{J-gated} trivially. Singleton subset \( \{ v \} \) of \( V \) is \textit{J-gated} since \( v \) itself is the unique gate of every \( w \neq v \).

We can give other examples.

Example 1. Consider the cycle \( C_n(n \geq 5) \) with vertices \( 1, 2, \ldots n \). Then the subset \( \{ 3, 4, \ldots n \} \) is J-gated since 3 is the gate of 2 and \( n \) is the gate of 1.
J^\Delta-Convexity

Example 2. Consider $C_n(n \geq 5)$ as in the previous example. Then the subset $\{4, 5, 6, \ldots, n\}$ is not J-gated since 2 has two gates, namely, 4 and $n$. Hence 2 has no unique gate.

Example 3. Consider a $C_3$ with vertices $x, y$ and $z$. Then the set $\{y, z\}$ is not J-gated since $x$ has no gate in the set.

Remark 11 The family of J-gated sets of a graph need not form a convexity. For, consider the subsets $\{3, 4, \ldots, n\}$ and $\{1, 4, 5, \ldots, n\}$ in $C_n$ of Example 1. They are J-gated sets of $C_n$, but their intersection $\{4, 5, 6, \ldots, n\}$ is not J-gated.

Theorem 21 Let $G$ be a HHD-free graph. Then the J-gated sets are precisely the $I^\Delta$-convex sets.

Proof. Let $C$ be a J-gated subset of $G$. We claim that $C$ is J-convex. Suppose not, then there exists two distinct vertices $x$ and $y$ in $C$ and a vertex $z$ on some $x-y$ induced path $P$, such that $z \notin C$. Choose $x$ and $y$ so that $x$ and $y$ are the only vertices of $P$ lying in $C$. So, without loss of generality, we can assume that $z$ is the neighbour of $x$ on $P$. Then $x$ is the J-gate of $z$ in $C$. Therefore $x \in J(z, y)$. Since $G$ is HHD-free, $J$ satisfies the two betweenness axioms and which leads to the contradiction $z \notin J(x, y)$, hence the claim. Since J-convex sets are I-convex, $C$ is I-convex. Now we can prove that $C$ is $I^\Delta$-convex. Let $xt$ and $yt$ be any two vertices of $C$ and $zt$ be any vertex on any $x-y$ triangle shortest path $Qt$. Let $Qut$ be the corresponding $x-y$ geodesic. If $z$ is on $Qut$ then $z \in C$, since $C$ is I-convex. If $z$ is not on $Qut$, then we can find two neighbours $xt$ and $yt$ of $z$ on $Qut$ such that $z$ is adjacent to both $xt$ and $yt$. Then $xt$, $yt$ $\in C$. So if $z \notin C$, it has no J-gate in $C$, which is false. Hence $C$ is $I^\Delta$ convex. Now let us prove the converse. Assume that $C$ is $I^\Delta$-convex. Let $w$ be any vertex not in $C$. Define $L = \{c \in C | J(w, c) \cap C = \{c\}\}$. Take any vertex $c \in C$. Suppose $C \cap J(c, w) \neq \{c\}$. Then we can find a vertex $c_1 \in C$ and different from $c$. So that $c_1 \in C \cap J(c, w)$. Since $G$ is HHD free, $c_1 \in J(c, w) \implies c \notin J(c_1, w)$ and
$J(c_1, w) \subseteq J(c, w)$. Hence $|J(c_1, w)| < |J(c, w)|$. Now consider $J(c_1, w) \cap C$. If it contains vertices other than $c_1$, we can find a vertex $c_2 \neq c_1$, such that $c_2 \in J(c_1, w) \cap C$. Then by the same argument given above $|J(c_2, w)| < |J(c_1, w)|$. Now continuing this argument, we get a sequence $c_1, c_2 \ldots$ of vertices of $C$ satisfying $|J(c_1, w)| > |J(c_2, w)| > \ldots$. Since $C$ is finite, eventually we get a vertex $c_m$. So that $|J(c_m, w)| = 1$. Then $c_m \in L$. Therefore $L$ is non-empty.

We claim $|L| = 1$.

Suppose not, then we can find two distinct vertices $c_1, c_2 \in L$. Since $c_1 \in L$, $J(w, c_1) \cap C = \{c_1\}$. Hence $c_2$ is not on any $w-c_1$ induced path. Similarly $c_1$ is not on any $w-c_2$ induced path. Let $P$ be a $c_1c_2$ geodesic, then all the vertices of $P$ are in $C$, since $C$ is $I$-convex. Let $P_1$ be a $c_1-w$ induced path and $P_2$ be a $c_2-w$ induced path. Then $c_1$ is not on $P_2$ and $c_2$ is not on $P_1$. Let us assume that $w_1$ is the first vertex common to $P_1$ and $P_2$ as we traverse from $c_1$ to $w$ along $P_1$. Now $P_1 \cup P$ is a $w-c_2$ path containing $c_1$. Hence $P_1 \cup P$ cannot be an induced path. So there exists a chord $uv$ from $c_2 \rightarrow P \rightarrow c_1$ to $w \rightarrow P_1 \rightarrow c_1$; $u, v \neq c_1$. Then $c_2 \rightarrow P \rightarrow u \rightarrow v \rightarrow P_1 \rightarrow w$ is a $c_2-w$ induced path containing $u \in C$.

Since $J(w, c_2) \cap C = \{c_2\}$, $u = c_2$. Since $v \in P_1$, $v \notin C$. Now consider the cycle $c_2 \rightarrow P \rightarrow c_1 \rightarrow P_1 \rightarrow v \rightarrow c_2$. To avoid a long cycle, there exists a chord $u_1v_1$ from $c_1 \rightarrow P \rightarrow c_2$ to $c_1 \rightarrow P_1 \rightarrow v$. Then $C_I: c_1 \rightarrow P \rightarrow u_1 \rightarrow v_1 \rightarrow P_1 \rightarrow c_1$ is an induced cycle. Hence it is of length 3 or 4. But, it cannot be of length 3, since if it is of length 3, then we get $v_1 \in I^2(c_1, u_1) \subseteq C$ which is a contradiction as $v_1$ is vertex on $P_1$. Therefore $C_I$ is of length 4. Now consider the cycle $u_1 \rightarrow P \rightarrow c_2 \rightarrow v \rightarrow P_1 \rightarrow v_1 \rightarrow u_1$. Since this cycle cannot be a long cycle, there exists a chord $u_2v_2$ from $u_1 \rightarrow P \rightarrow c_2$ to $v_1 \rightarrow P_1 \rightarrow v$. Then $C_{II}: u_1 \rightarrow P \rightarrow u_2 \rightarrow v_2 \rightarrow P_1 \rightarrow v_1 \rightarrow u_1$ is an induced cycle. Hence it is of length 3 or 4. Hence the subgraph induced by the vertices of $C_I$ together with $C_{II}$ is either a house or a domino, a contradiction. Hence the claim.
Let $L = \{c\}$. Now we can prove that $c$ is the gate of $w$. Let $x$ be any vertex in $C$. Assume that $c \notin J(w, x)$. Let $P$ be an $x - c$ geodesic, then as we have already stated, all the vertices of $P$ are in $C$. Let $P_I$ be a $w - c$ induced path. Then $P \cup P_I$ is an $x - w$ path containing $c$. Hence there exists a chord $uv$ from $c \rightarrow P \rightarrow x$ to $c \rightarrow P_I \rightarrow w$. Since $v \notin C$, the induced cycle $c \rightarrow P \rightarrow u \rightarrow v \rightarrow P_I \rightarrow c$ is of length four. If $d(c, u) = 2$, then $v \in I(c, u) \subseteq C$, a contradiction. If $d(c, u) = 1$, then $w \rightarrow P_I \rightarrow v \rightarrow u \rightarrow c$ is an induced path. Hence $u \in J(w, c)$ and $u \in C$, a contradiction. So $c$ is the gate of $w$, which completes the proof. ■