CHAPTER -03

Certain Connections of Randers Conformal of Finsler Space with Anisotropic Cosmological Models

3.1: Introduction

In this chapter we study an anisotropic model of space – time with Finslerian metric. Here the observed anisotropy of the microwave background radiation is incorporated in the Finslerian metric of space time inspired with the recent work published in physical letter [36-37] under which the author has considered that in Finslerian manifold there exists a unique linear connection known as Chern connection, it is torsion firmness and metric compatibility, we are not agree with this result, because there are well known result published by H. Rund that in Finsler geometry there exits and infinite number of linear connection defined by the same metric structure and that the Chern and Bernwald connection are not metric compatible. It is also noticed that many researchers are interested to investigate something for new insite of physics beyond the standard model related to Finsler geometry with curved space-time and quantum gravity and looking for possible application in modern cosmology [37 - 39].

We are fully agreed with the theory expressed in the above publication as it is based on tangent bundle on space time manifold which are positively with local Lorentz violations this may be related with dark energy and dark matter models with variable $\Lambda$ in cosmology. Certainly this may be important connections between Finsler geometry and cosmology.

Now in these days many researchers are constructing suitable cosmological models with variable Lambda term including our own research group [21-28]. In the above
communication we have presented several cosmological models with different assumptions and exact solutions to Einstein’s field equations for spatially homogeneous and anisotropic Bianchi type I-II, models with string fluid as a source of matter and variable deceleration parameter (DP) have been established. Under suitable assumptions author has shown the accelerating models and the models with transition, from the early decelerated phase to present accelerating phase. This result is in good agreement with the recent supernovae observations. With this motivation we have decided to investigate anisotropic model of General Theory of Relativity based on the framework of Finsler geometry.

3.2: Basic Assumptions & Consideration

In 1984, Shibata [40] introduce the concept of $\beta$ -change and defined as:

$$L(x, y) = \bar{L}(x, y) + \beta(x, y) ,$$

(3.1)

where, $\beta$ is a one-form metric and $L$ is any Finsler metric, He has obtained very interesting result in this paper. The conformal theory of Finsler spaces has been initiated by M. S. Knebelman [41] in 1929 and has been investigated in detail by many authors [42-45]. This change may be defined as

$$L' = e^{\sigma(x)}L(x, y) ,$$

(3.2)

where $\sigma(x)$ is a function of position only and known as conformal factor.

3.3: Generalization of Result with Different Assumption

In this chapter we tried to generalize the above changes and defined a Randers conformal metric with new input as:

$$L = e^{\sigma(x)}\alpha + \beta ,$$

(3.3)
where, $\alpha = \sqrt{a_{ij}(x^i)} y^i y^j$ is a Riemannian metric, $\beta = b_i(x) y^i$ is a one-form and $\sigma(x)$ is the conformal factor.

If $\sigma(x) = 0$, then this change is reduces to simple Randers metric and for this metric Stavrinos and Diakogiannis [46] obtained the relationship between the anisotropic cosmological models of space time and Randers Finslerian metric. The purpose of the present study is to obtain the relationship between the anisotropic cosmological models of space time with above generalized Finslerian metric.

For this purpose let us consider an $n$- dimensional Finsler space $(M^n, L)$ and an adaptable 1-form $\beta = b_i(x) dx^i$ on $M^n$. We shall use a Lagrangian function on $M^n$, given by the equation:

$$L = e^{\sigma(x)} \alpha + \varphi(x) \hat{b}_i y^i,$$

where $b_i(x) = \varphi(x) \hat{b}_i$, the vector $\hat{b}_i$ represents the observed anisotropic of the microwave background radiation. A coordinate transformation on the total space $TM$ may be expressed as

$$\tilde{x}^i = \tilde{x}^i(x^0, x^1, x^2, x^3), \quad \det \left| \frac{\partial \tilde{x}^i}{\partial x^j} \right| = 0, \quad \tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j} y^j, \quad y^i = \delta^i_j y^j$$

As per investigation done by Wells & Jordan [47, 48], Finsler metric on $M$ is a function $F: TM \rightarrow R$ having the properties:

1. The restriction of $F$ to $TM$ is positively homogeneous of degree 1 with respect to $y^a$.

$$F(x, ky) = kF(x, y), k \in R_+^*$$

2. The quadratic form on $R^n$ with the coefficients

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F}{\partial y^i \partial y^j}$$

Defined on $TM$, is non degenerate $\det \frac{\partial \tilde{x}^i}{\partial x^j} \neq 0$, with rank $g_{ij} = 4$.

The Carton torsion coefficients $C_{ijk}$ are given by
\[ C_{ik} = \frac{1}{2} \partial_k g_{ij} \] (3.7)

The torsions and curvatures used as suggested by several researchers [49-50]:

\[ P_{ijk} = C_{ijk,l}y^l \] (3.8)

\[ S_{jikh} = C_{iks} C_{jk}^s - C_{iks} C_{jk}^s \] (3.9)

\[ P_{iklj} = C_{jik|l} - C_{jik} + C_{jk}^s C_{rik|l} y^l - C_{jlk}^s C_{rkh} y^l \] (3.10)

\[ S_{ikl}^l = g^{jl} S_{jikh} \] (3.11)

\[ P_{ik}^l = g^{jl} P_{jikh} \] (3.12)

Differentiating equation (3.4) with respect to \( y^i \), the normalized supporting elements

\[ l^i = \frac{\partial}{\partial y^i} L \]

is given by

\[ l^i = \frac{\partial}{\partial y^i} L = e^{\sigma(x)} \alpha \partial^i \alpha + \phi(x) \hat{b}_i \] (3.13)

\[ \partial^i \alpha = \frac{1}{2 \alpha} \left[ a_y (x) y^i + a_y (x) y^j \right] \]

\[ \partial^i \alpha = \frac{y_i}{\alpha} \quad \text{as} \quad y_i = a_y y^j \]

\[ \partial^i \alpha = \frac{y_i}{\alpha} \] (3.14)

From equations (3.13) and (3.14), we have

\[ l^i = \frac{\partial}{\partial y^i} L = e^{\sigma(x)} \frac{y_i}{\alpha} + \phi(x) \hat{b}_i \] , (3.15)

where \( l^i = \frac{\partial}{\partial y^i} L \) is the normalized supporting elements in the Finsler space \( F^n \).

Differenting equation (3.15) with respect to \( y^j \), the normalized supporting element

\[ h_{ij} = L \hat{\partial}_j L = \frac{Le^{\sigma(x)}}{\alpha} \left[ a_y - \frac{y_i y_j}{\alpha^2} \right] , \] (3.16)
\[ \dot{\varphi}_j \dot{\varphi}_j L = \frac{e^{\sigma_i(x)}}{\alpha} \left[ a_{ij} \frac{y_i y_j}{\alpha^2} \right] \],

where \( h_{ij} = L \dot{\varphi}_i \dot{\varphi}_j L \) is angular metric tensor in the Finsler space \( F^n \).

### 3.4: Anisotropic Cosmological Model with Randers Conformal of Finsler Space

We may express The Lagrangian function on \( M^n \), with the expression:

\[ L = e^{\sigma(x)} \alpha + \phi(x) \tilde{b}_i y^i, \]

and

\[ \alpha = \sqrt{a_{ij} (x^i y^j)}. \]

Here \( a_{ij} \) is the Riemannian metric with signature \((-, +, +, +)\). Because of the anisotropy, we must insert an additional term to the Riemannian line element. This term fulfill the following requirements:

(i) It must give absolute maximum contribution for the direction of movement parallel to the anisotropy axis.

(ii) The new line element must coincide with the cubic metric one for the direction vertical to the anisotropy axis.

(iii) It must not be symmetric with respect to replacement of \( y^i \) to \(- y^i\).

(iv) We see that a term which satisfies the above conditions is \( \beta = \phi(x) \tilde{b}_i \)

where \( b_i(x) \) reveals this anisotropic axis.

Let \( b_i(x) = \phi(x) \tilde{b}_i \) where \( \tilde{b}_i \) the unit vector in the direction is \( b_i(x) \). Then \( \phi(x) \) plays the Role of “length” of the vector \( b_i(x), \phi(x) \epsilon \). \( \beta \) is the Finslerian line element and \( \alpha \) is Riemannian one. In order for the Finslerian metric to be physically consistent with General theory of Relativity, it must have the same signature with Riemannian metric \((-, +, +, +)\).

We have

\[ \alpha = c d \tau = c \mu d t = \mu d (ct) = \mu dx^0 \]
where $\mu = \sqrt{1 - \frac{v^2}{c^2}}$ and $v$: 3 – velocity in Riemannian space-time. One possible explanation of the anisotropy axis could be that it represents the resultant of spin densities of the angular momenta of galaxies in a restricted area of space [ $b_i(x)$ space like ]. It is assumed that the mass is anisotropically distributed for regions of space with radius $\leq 10^8$ light years [45].

Therefore The Finsler metric tensor $g_{ij}$ may be expressed as:

$$g_{ij} = h_{ij} + l_j l_j,$$

(3.21)

$$g_{ij} = \frac{Le^{\sigma(x)}}{a}a_{ij} + [\varphi(x)]^2 \hat{b}_i \hat{b}_j + \frac{e^{\sigma(x)}\varphi(x)}{a}\{\hat{b}_i y_j + \hat{b}_j y_i\} - \frac{\beta e^{\sigma(x)}}{a^3} y_i y_j,$$

(3.22)

where $\gamma = \frac{Le^{\sigma(x)}}{\alpha}$.

(3.23)

where we put $y^i = a_{ij} y^j$ and $a_{ij}$ is the fundamental tensor for the Finsler space $F^n$. It will be easy to see that the determinant $\|g_{ij}\|$ does not vanish, and the reciprocal tensor with components $g^{ij}$ is given by

$$g^{ij} = \gamma^{-1} a^{ij} + y^i y^j - \alpha^{-1} \gamma^{-2} \{y^i \hat{b}^j + y^j \hat{b}^i\},$$

(3.24)

$$\varphi = e^{-2\sigma(x)}\{\alpha e^{\sigma(x)} b^2 + \beta\}L^3$$

(3.25)

where $g^{ij}$ is the reciprocal tensor of $g_{ij}$ and $b^2 = \hat{b}_i \hat{b}^i$, $\hat{b}^i = a^{ij} \hat{b}_j$ and $a^{ij}$ is the inverse matrix of $a_{ij}$ as it may be verified by direct calculation, where $m = \hat{b}_i \hat{b}^i = 0, \pm 1$ according whether $\hat{b}_i$ is null, space like or time like.

It is interesting to observe that, that if $y^i$ represents the velocity of a particle (time like) then $\hat{b}^i$ is bound to be space like. This follows from the fact that one possible value of $\hat{b}^i y^i$ is zero. Therefore we have decided to calculate Carton covariant tensor $C$. 

The Carton covariant tensor \( C \) with the components \( C_{ijk} \) is obtained as follows:

\[
C_{ijk} = \frac{1}{2} \partial_k g_{ij},
\]

\[
C_{ijk} = \frac{1}{2} \left[ \partial_k \left\{ \frac{Le^{\sigma(x)}}{\alpha} a_i + \partial_i \left\{ \frac{e^{\sigma(x)}}{\alpha} \left\{ \delta y_j + y_j \delta_j \right\} \right\} \right] + \frac{e^{\sigma(x)}}{\alpha} \left\{ \partial_k y_j + \partial_j y_k \delta_j \right\} - \partial_k \left( \frac{\beta e^{\sigma(x)}}{\alpha^3} \right) y_j y_j + \frac{\beta e^{\sigma(x)}}{\alpha^3} \left\{ \partial_k y_j \right\} y_j + \left\{ \partial_k y_j \right\} y_j, \]

\[
C_{ijk} = \frac{\gamma}{2L} \left\{ h_i m_k + h_j m_i + h_k m_j \right\}, \quad (3.26)
\]

where \( m_i = \delta_i - \beta \alpha^{-2} y_i \) \quad (3.27)

The Covariant indices \( j \) is replaced by \( k \) and \( k \) by \( s \), we have

\[
C_{iks} = \frac{\gamma}{2L} \left\{ h_i m_k + h_k m_i + h_k m_k \right\}, \quad (3.28)
\]

\[
C_{ihs} g^{hs} = C_{ik} = \frac{1}{2L} \left\{ h_i m_k + h_k m_i + h_k m_k \right\} - \frac{\gamma^{-1}}{2\alpha L} \left\{ 2m_i m_k + m^2 h_k \right\} y^k, \quad (3.29)
\]

where \( m^2 = m_i m^i \) \quad (3.30)

Replacing the covariant indices \( h \) by \( s \), \( i \) by \( j \) and \( k \) by \( h \) in equation (3.29), we have

\[
C_{ijs} = \frac{1}{2L} \left\{ h^i m_h + h_i m_j + h_j m^i \right\} - \frac{\gamma^{-1}}{2\alpha L} \left\{ 2m_i m_h + m^2 h_i \right\} y^i, \quad (3.31)
\]

\[
S_{jikh} = C_{iks} C_{jhs} - C_{ihk} C_{jks}, \quad (3.32)
\]

\[
C_{iks} C_{jhs} = \frac{\gamma}{2L} \left[ h_i m_k + h_k m_i + h_k m_k \left\{ \frac{1}{2L} \left\{ h_i m_k + h_k m_j + h_j m^i \right\} - \frac{\gamma^{-1}}{2\alpha L} \left\{ 2m_i m_k + m^2 h_i \right\} y^i \right] \right],
\]
C_{ik} C_{jk}^i = \frac{\gamma}{4L^2} \left\{ 2\gamma h_i h_j m_i m_j + m^2 h_i h_j + h_i h_j m_i m_j + h_k h_i m_i m_j \right\}, \quad (3.33)

Replacing k by h and h by k in equation (3.33), we get

C_{ik} C_{jk}^i = \frac{\gamma}{4L^2} \left\{ 2\gamma h_i h_j m_i m_j + m^2 h_i h_j + h_i h_j m_i m_j + h_k h_i m_i m_j \right\}, \quad (3.34)

Thus, equation (3.32), with the help of Equations (3.33) and (3.34) yields

S_{jk} = \frac{\gamma}{4L^2} \left[ m^2 \{ h_i h_j - h_j h_i \} + (2\gamma - 1)m_i h_j (2\gamma - 1)m_j h_i + (2\gamma - 1)m_i m_j h_i h_j + (2\gamma - 1)m_i m_j h_i h_j \right], \quad (3.35)

P_{i\ell k} = C_{ijk} - C_{jik} + C_{i\ell k} - C_{i\ell k} \quad (3.36)

Now Differenting equation (3.26) with respect to covariant h, we get

C_{jk,h} = \left( \frac{\gamma}{2L} \right) \left\{ h_i m_k + h_j m_i + h_j m_i \right\} + \frac{\gamma}{2L} \left\{ h_i m_k + h_j m_i + h_j m_i \right\},

Since \ h_i j k = 0, \ L_{h} = 0, \ i_{h} = 0, \ \alpha_{h} = 0

C_{ijk,h} = \frac{1}{2} e_{\delta}^{\alpha} \left\{ h_i m_k + h_j m_i + h_j m_i \right\} + \frac{\gamma}{2L} \left\{ h_i m_k + h_j m_i + h_j m_i \right\}, \quad (3.37)

where \ \left( \frac{\gamma}{2L} \right)_{h} = \frac{e_{\delta}^{\alpha}}{2L}, \quad \left( e_{\delta}^{\alpha} \right)_{h} = \frac{1}{2} \left[ \alpha_{h}^{\alpha} - e_{\delta}^{\alpha} \right],

\left( \frac{\gamma}{2L} \right)_{h} = \frac{e_{\delta}^{\alpha}}{\alpha}, \quad \therefore \alpha_{h} = 0

Replacing i by h and h by i in equation (3.37), we get

C_{ijk,i} = \frac{1}{2} e_{\delta}^{\alpha} \left\{ h_i m_k + h_j m_i + h_j m_i \right\} + \frac{\gamma}{2L} \left\{ h_i m_k + h_j m_i + h_j m_i \right\}. \quad (3.38)
Combining equations (3.37) and (3.38) reduces to

\[ C_{y|k} - C_{y|k} = \theta_{(ih)} \left[ \frac{1}{2} \frac{e^0}{\sigma} \left\{ h_y m_k + h_y m_i + h_{ik} m_j \right\} + \frac{\gamma}{2L} \left\{ h_y m_{k|l} + h_y m_{i|l} + h_{ik} m_{j|l} \right\} \right], \]  

(3.39)

where \( \theta_{(ih)} \) is interchanging the indices and substitution.

From equation (3.37), we have

\[ C_{y|k} = C_{y|k} + \frac{1}{2} \frac{e^0}{\alpha} \left\{ h_y m_k + h_y m_i + h_{ik} m_j \right\} + \frac{\gamma}{2L} \left\{ h_y m_{k|l} + h_y m_{i|l} + h_{ik} m_{j|l} \right\}. \]  

(3.40)

With the help of equations (3.31) and (3.40), we write

\[ C_{y|k} C_{y|k} = \frac{e^0}{4\alpha L} \left\{ h_y m_k + h_y m_i + h_{ik} m_j + 2h_{ik} m_k m_i + h_{ik} m_j m_k + h_{ik} m_j m_i + 2h_{ik} m_j m_k + m^2 h_{ik} m_j \right\} \]

\[ + \frac{\gamma}{4L} \left\{ h_y m_k m_{k|l} + h_y m_i m_{i|l} + h_{ik} m_j m_{k|l} + h_{ik} m_j m_{i|l} + h_{ik} m_j m_{k|l} + h_{ik} m_j m_{i|l} \right\} \]  

(3.41)

where

\[ m = m^2 m_{i|l} \]

\[ C_{y|k} C_{y|k} - C_{y|k} C_{y|k} = \frac{e^0}{4\alpha L} \theta_{(ih)} \left\{ h_y m_k + h_y m_i + h_{ik} m_j + h_{ik} m_j + m^2 h_{ik} h_{jl} \right\} \]

\[ + \frac{\gamma}{4L} \theta_{(ih)} \left\{ h_{ik} m_k m_{i|l} + h_{ik} m_i m_{i|l} + h_{ik} m_j m_{i|l} + h_{ik} m_j m_{i|l} \right\} \]  

(3.42)

The equation (3.36), with equations (3.39) and (3.42) yields
\[ P_{\alpha\beta\gamma} = \theta_{(\alpha)} \left[ \frac{1}{2} e^{\alpha}_{\mu} \left\{ h_{\mu} m_4 + h_{\mu} m_5 + h_{\mu} m_6 \right\} + \frac{\gamma}{2L} \left\{ h_{\mu} m_4 + h_{\mu} m_5 + h_{\mu} m_6 \right\} \right] + \]

\[ \frac{e^{\alpha}_{\mu}}{4\alpha L} \theta_{(\mu)} \left\{ h_{\nu} m_4 m_4 + h_{\nu} m_5 m_5 + h_{\nu} m_6 m_6 + m^2 h_{\omega} h_{\omega} \right\} + \]

\[ \frac{\gamma}{4L^2} \theta_{(\mu)} \left\{ h_{\nu} m_4 m_4 + h_{\nu} m_5 m_5 + h_{\nu} m_6 m_6 + m^2 h_{\omega} h_{\omega} \right\} \]  

(3.43)

3.5: Concluding Remarks

As discussed in the previous sections particular attention over the last decade has been paid on the so-called Finsler-Randers cosmological model [50]. It has been established that in general metrical extensions, Riemann geometry may provide a Finslerian geometrical structure in a manifold which leads to generalized gravitational field theories. During the last decade there is a rapid development of applications of Finsler geometry in its F-R context, mainly in the topics of General Theory of Relativity, astrophysics and cosmology [51–60]. Keeping this in mind we have considered the Randers conformal metric to the general relativity & calculated the measure of anisotropy of the matter. From the above section it is clear that if \( \sigma(x) = 0 \), then this change is reduces to simple Randers metric and GTR is well connected with Randers metric. For this metric we have also obtained the relationship between the anisotropic cosmological models of space time with above generalized Finslerian metric which is shown in the equations derived in section 3.3. Now we wish to investigate the connections by taking cubic metric as suggested by M. Matsumoto in the year 1979 [62].