CHAPTER 4

NEIGHBORHOOD CONNECTED EDGE DOMINATION IN GRAPHS

4.1 INTRODUCTION

As an analogy to vertex domination, the concept of edge domination was introduced by Mitchell and Hedetniemi (1977). A set $X \subseteq E$ is said to be an edge dominating set if every edge in $E - X$ is adjacent to some edge in $X$. The edge domination number of $G$ is the cardinality of a smallest edge dominating set of $G$ and is denoted by $\gamma'(G)$. Arumugam and Velammal (2009) introduced the concept of connected edge domination of a connected graph. An edge dominating set $X$ of a connected graph $G$ is called a connected edge dominating set if the edge induced subgraph $\langle X \rangle$ is connected. The minimum cardinality of a connected edge dominating set of $G$ is called connected edge domination number and is denoted by $\gamma'_c(G)$.

In this paper we study the edge analogue of neighborhood connected domination number. Through out this chapter we consider connected graphs.

In section 4.2 we introduce the concept of neighborhood connected edge domination and neighborhood connected edge domination number $\gamma'_{nc}(G)$. We determine $\gamma'_{nc}$ for standard graphs.
such as paths, cycles, complete graphs and complete bipartite graphs. We also discuss the relation between $\gamma'_{nc}$ and other domination parameters such as $\gamma'$ and $\gamma'_c$. Also we obtain a characterization of minimal neighborhood connected edge dominating set.

In section 4.3 we present some bounds for neighborhood connected edge domination number $\gamma'_{nc}$. We also obtain a characterization of trees and unicyclic graphs for which $\gamma'_{nc} = m - \Delta'$.

In section 4.4 we present some of the open problems that arise naturally.

4.2 NEIGHBORHOOD CONNECTED EDGE DOMINATION IN GRAPHS

Definition 4.2.1. An edge dominating set $X$ of a connected graph $G$ is called the neighborhood connected edge dominating set (nced-set) if the edge induced subgraph $\langle N(X) \rangle$ of $G$ is connected. The minimum cardinality of a nced-set is called the neighborhood connected edge domination number (nced-number) and is denoted by $\gamma'_{nc}(G)$.

Example 4.2.2. Consider the following graph $G$

\[ G \]

Figure 4.1: $\gamma'(G) = 2 \quad \gamma'_c(G) = 4 \quad \gamma'_{nc}(G) = 3$
Remark 4.2.3. (i) Clearly $\gamma'_nc(G) \geq \gamma'(G)$. Further if $X$ is a connected edge dominating set with $|X| > 1$ then $N(X) = E$ and hence $\gamma'_nc(G) \leq \gamma'_c(G)$.

(ii) For any connected graph $G$ that is not a star $\gamma'_nc(G) = 1$ if and only if there exists a non cut edge $e$ such that $\deg e = m - 1$. That is $G$ contains two adjacent vertices $u$ and $v$ such that all other vertices are mutually non adjacent, adjacent to either $u$ or $v$, and at least one vertex is adjacent to both $u$ and $v$.

Theorem 4.2.4. For any graph $G$, $\gamma'(G) \leq \gamma'_nc(G) \leq 2\gamma'(G)$. Further given two positive integers $a$ and $b$ with $a \leq b \leq 2a$, there exists a graph $G$ with $\gamma'(G) = a$ and $\gamma'_nc(G) = b$.

Proof. Let $G$ be a connected graph and let $X$ be an edge dominating set of $G$. Obviously pairing each $e \in X$ with a private neighbor forms a nced-set of cardinality $2\gamma'(G)$.

Now, let $a$ and $b$ be two positive integers with $a \leq b \leq 2a$. Let $b = a + k, 0 \leq k \leq a$. Consider the galaxy with stars $G_1, G_2, \cdots, G_a$ with $|V(G_i)| \geq 3, \ 1 \leq i \leq a - 1$. Join the maximum degree vertices of $G_i$ and $G_{i+1}$ by an edge $e_i, 1 \leq i \leq a - 1$. Let $H$ be the graph obtained from the above graph by subdividing exactly once the edges $e_i$ where $1 \leq i \leq a - 1$. Clearly $\gamma'(H) = \gamma'_nc(H) = a$. Let $G$ be the graph obtained from $H$ by subdividing an edge of $G_i$ exactly once where $1 \leq i \leq k$. Then $\gamma'(G) = a$ and $\gamma'_nc(G) = a + k = b$. \qed
Example 4.2.5. For the graph $G$ given in figure 4.2, $\gamma'(G) = 5$ and $\gamma'_{nc}(G) = 8$

Figure 4.2: A graph with $\gamma' = 5$ and $\gamma'_{nc} = 8$

Theorem 4.2.6. For the path $P_n, n \geq 2, \gamma'_{nc}(P_n) = \left\lceil \frac{n-1}{2} \right\rceil$.

Proof. Let $P_n = (v_1, v_2, \cdots, v_n)$ and let $e_i = v_iv_{i+1}$. If $n$ is odd, then $X = \{e_j : j = 2k \text{ or } 2k + 1 \text{ and } k \text{ is odd}\}$ is a need-set of $P_n$ and if $n$ is even then $X_1 = X \cup \{e_{n-1}\}$ is a need-set of $P_n$. Hence $\gamma'_{nc}(P_n) \leq \left\lceil \frac{n-1}{2} \right\rceil$. Further, if $X$ is any $\gamma'_{nc}$-set of $P_n$ then $N(X)$ contains all the internal edges of $P_n$. Since we need at least two edges to dominate with connected neighborhood set, every sub path of length 4, we have $|X| \geq \left\lceil \frac{n-1}{2} \right\rceil$. Thus $\gamma'_{nc}(P_n) = \left\lceil \frac{n-1}{2} \right\rceil$. \qed

Corollary 4.2.7. For any non-trivial path $P_n$,

(i) $\gamma'_{nc}(P_n) = \gamma'(P_n)$ if and only if $n = 3$ or 5.

(ii) $\gamma'_{nc}(P_n) = \gamma'_c(P_n)$ if and only if $n = 2, 3, 5$ or 6.

Proof. Since $\gamma'(P_n) = \left\lceil \frac{n-1}{3} \right\rceil$ and $\gamma'_c(P_n) = n - 3$ the corollary follows. \qed
Theorem 4.2.8. For the cycle $C_n$ on $n$ vertices

$$\gamma_{nc}'(C_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil & \text{if } n \not\equiv 3 \pmod{4} \\ \left\lfloor \frac{n}{2} \right\rfloor & \text{if } n \equiv 3 \pmod{4} \end{cases}$$

Proof. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$ and $n = 4k + r$ where $0 \leq r \leq 3$ and $e_i = v_iv_{i+1}$. Let $X = \{e_i : i = 2j, 2j + 1, j \text{ is odd and } 1 \leq j \leq 2k - 1\}$.

Let $X_1 = \begin{cases} X & \text{if } n \equiv 0 \pmod{4} \\ X \cup \{e_n\} & \text{if } n \equiv 1 \text{ or } 2 \pmod{4} \\ X \cup \{e_{n-1}\} & \text{if } n \equiv 3 \pmod{4} \end{cases}$

Clearly $X_1$ is a nced-set of $C_n$ and hence

$$\gamma_{nc}'(C_n) \leq \begin{cases} \left\lceil \frac{n}{2} \right\rceil & \text{if } n \not\equiv 3 \pmod{4} \\ \left\lfloor \frac{n}{2} \right\rfloor & \text{if } n \equiv 3 \pmod{4} \end{cases}$$

Now, let $X$ be any $\gamma_{nc}'$-set of $C_n$ then $\langle X \rangle$ contains at most one isolated edge and

$$\langle N(X) \rangle = \begin{cases} C_n - \{e\} & \text{if } n \not\equiv 0 \pmod{4} \\ C_n & \text{if } n \equiv 0 \pmod{4} \end{cases}$$

Hence

$$|X| \geq \begin{cases} \left\lceil \frac{n}{2} \right\rceil & \text{if } n \not\equiv 3 \pmod{4} \\ \left\lfloor \frac{n}{2} \right\rfloor & \text{if } n \equiv 3 \pmod{4} \end{cases}$$

and the result follows. \qed
Corollary 4.2.9. (i) $\gamma'_{nc}(C_n) = \gamma'(C_n)$ if and only if $n = 3, 4$ or 7.

(ii) $\gamma'_{nc}(C_n) = \gamma'_c(C_n)$ if and only if $n = 3, 4$ or 5.

Proof. Since $\gamma'(C_n) = \left\lceil \frac{n}{3} \right\rceil$ and $\gamma'_c(C_n) = n - 2$ the result follows.

Theorem 4.2.10. $\gamma'_{nc}(K_n) = \left\lfloor \frac{n}{2} \right\rfloor$, $n \geq 3$.

Proof. Let $X$ be a maximum matching of $K_n$. Hence $X$ is an edge dominating set. Also $\langle N(X) \rangle = K_n - X$ which is connected. Hence $X$ is a nc-epend set which implies $\gamma'_{nc}(K_n) \leq |X| = \left\lfloor \frac{n}{2} \right\rfloor$. Since $\gamma'(K_n) = \left\lfloor \frac{n}{2} \right\rfloor$ the result follows.

Example 4.2.11. Consider $G = K_5$ given in figure 4.3, $S = \{e_1, e_3\}$ is a $\gamma'_{nc}$-set

![Figure 4.3: Complete graph with 5 vertices](image-url)
Theorem 4.2.12. \( \gamma'_{nc}(K_{r,s}) = \min\{r, s\} \).

Proof. Let \( v \) be a vertex such that \( \deg v = \min\{r, s\} \). Let \( X \) be the set of all edges incident with \( v \). It is clear that \( X \) is an edge dominating set. Also \( \langle N(X) \rangle = K_{r,s} \) if \( K_{r,s} \) is not a star and \( \langle N(X) \rangle = K_{1,n-1} \) if \( K_{r,s} \) is a star. Thus \( X \) is a need-set. Hence \( \gamma'_{nc}(K_{r,s}) \leq |X| = \deg v = \min\{r, s\} \). Since \( \gamma'(K_{r,s}) = \min\{r, s\} \) the result follows.

Example 4.2.13. Consider \( G = K_{3,4} \) given in figure 4.4, \( S = \{e_1, e_2, e_3\} \) is a \( \gamma'_{nc} \)-set.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.4.png}
\caption{A complete bipartite graph \( K_{3,4} \)}
\end{figure}

Theorem 4.2.14. For a tree \( T \), \( \gamma'_{nc}(T) = 1 \) if and only if \( T \) is a star.

Proof. Let \( \gamma'_{nc}(T) = 1 \) and let \( X = \{e\} \) be the \( \gamma'_{nc} \)-set of \( G \). Let \( e = uv \) and let \( \deg u \geq 2 \). If \( \deg v > 1 \) then \( \langle N(X) \rangle = T - e \) is disconnected. Hence \( \deg v = 1 \). Thus \( T \) is a star. The converse is obvious.

We now proceed to obtain a characterization of minimal need-sets.
Lemma 4.2.15. A superset of a ncen-set is a ncen-set.

Proof. Let $X$ be a ncen-set of a graph $G$ and let $X_1 = X \cup \{e\}$, where $e \in E - X$. Let $e = uv$. Clearly $e \in N(X)$ and $X_1$ is an edge dominating set of $G$. Now let $x, y \in V(\langle N(X) \rangle)$. If $x, y \in V(\langle N(X) \rangle)$ then any $x - y$ path in $\langle N(X) \rangle$ is an $x - y$ path in $\langle N(X_1) \rangle$. If $x \in V(\langle N(X) \rangle)$ and $y \notin V(\langle N(X) \rangle)$, then without loss of generality we assume $x - u$ path in $\langle N(X) \rangle$, and hence $x - u$ path together with $u - y$ path gives a $x - y$ path in $\langle N(X_1) \rangle$. Also if $x, y \notin V(\langle N(X) \rangle)$ then $(x, u, v, y)$ or $(x, v, u, y)$ or $(x, u, y)$ or $(x, v, y)$ or $(x, y)$ is a $x - y$ path in $\langle N(X_1) \rangle$. Thus $\langle N(X_1) \rangle$ is connected, so that $X_1$ is a ncen-set of $G$. \qed

Theorem 4.2.16. A ncen-set $X$ of a graph $G$ is a minimal ncen-set if and only if for every $e \in X$, one of the following holds,

(i) $\text{pn}[e, X] \neq \emptyset$

(ii) There exists two vertices $x, y \in \langle N(X) \rangle$ such that every $x - y$ path in $\langle N(X) \rangle$ contains at least one edge of $N(X) - N(X - \{e\})$.

Proof. Let $X$ be a minimal ncen-set of $G$. Let $e \in X$ and let $X_1 = X - \{e\}$. Then either $X_1$ is not an edge dominating set of $G$ or $\langle N(X_1) \rangle$ is disconnected. If $X_1$ is not an edge dominating set of $G$, then $\text{pn}[e, X] \neq \emptyset$. If $\langle N(X_1) \rangle$ is disconnected, then there exists two vertices $x, y \in \langle N(X_1) \rangle$ such that there is no $x - y$ path in $\langle N(X_1) \rangle$. Since $\langle N(X) \rangle$ is connected, it follows that every
Let $G$ be a graph with $\Delta' = m - 1$. Then $\gamma_{nc}'(G) = 1$ or 2. Further $\gamma_{nc}'(G) = 2$ if and only if $G$ is a bistar, $B(r, s), r, s \geq 1$.

Proof. Let $e \in E(G)$ with $\deg e = m - 1$. Then $\{e, e_1\}$, where $e_1 \in E - \{e\}$ is a nced-set of $G$ so that $\gamma_{nc}'(G) \leq 2$. Now suppose $\gamma_{nc}'(G) = 2$. Then $\langle N(e) \rangle = G - \{e\}$ is disconnected and hence $e$ is a cut edge of $G$. Let $e = uv$. Since $\deg e = m - 1, N[u, v] - \{u, v\}$ is an independent set. If $\deg u$ or $\deg v$ is equal to 1 than $G$ is a star which is a contradiction to $\gamma_{nc}'(G) = 2$. Thus $\deg u \geq 2$ and $\deg v \geq 2$. Hence $G$ is a bistar $B(r, s), r, s \geq 1$. The converse is obvious.

4.3 BOUNDS FOR $\gamma_{nc}'$

In the following theorems we obtain a bound for $\gamma_{nc}'(G)$.

Theorem 4.3.1. Let $G$ be a graph with $\Delta' < m - 1$. Then $\gamma_{nc}'(G) \leq m - \Delta'$. 

Proof. Let $e \in E(G)$ and $\deg e = \Delta'$. Since $G$ is connected and $\Delta' < m - 1$, there exists two adjacent edges $e_1$ and $e_2$ such that $e_1 \in N(e)$ and $e_2 \notin N[e]$. Now, let $X = (N(e) - \{e_1\}) \cup \{e_2\}$. Clearly $E - X$ is a nced-set of $G$ and hence $\gamma'_{nc}(G) \leq m - \Delta'$.

Theorem 4.3.2. Let $T$ be a tree with $n > 2$. Then $\gamma'_{nc}(T) = m - \Delta'$ if and only if $T$ is one of the following:

(i) Star

(ii) Tree obtained from bistar $B(|X_1|, |X_2|)$ with $e = uv$ be a non-pendant edge and $X_1$ and $X_2$ are set of pendant edges which are incident with $u$ and $v$ respectively, by subdividing at least one edge of $X_1 \cup X_2$ and subdividing at most one edge of $X_1$ or $X_2$ once, or by subdividing exactly one edge of $X_1 \cup X_2$ twice.

Proof. Let $T$ be a tree with $\gamma'_{nc}(T) = m - \Delta'$. Let $e = uv \in E(T)$ and $\deg e = \Delta'$. Let $Y_1 = N(u) - \{v\} = \{v_1, v_2, \ldots, v_r\}$ and $Y_2 = N(v) - \{u\} = \{v_{r+1}, v_{r+2}, \ldots, v_{\Delta'}\}$. If $r = 0$ then $T$ is a star graph. Let us assume $r \geq 1$ and $r < \Delta'$ and $A = V(T) - N[u, v] = \{w_1, w_2, \ldots, w_k\}$ and $T_1 = \langle A \rangle$.

Case 1. $E(T_1) = \emptyset$.

Suppose $\deg v_i \geq 3$ for some $v_i \in Y_1 \cup Y_2$ without loss of generality we assume $v_i \in Y_1$. Let $uw_i, v_iw_1, v_iw_2 \in E(T)$. Then $X = [E(T) - (N(e) \cup \{v_iw_1, v_iw_2\})] \cup \{uw_i\}$ is a nced-set of $T$ and $|X| = m - \Delta' - 1$, which is a contradiction. Hence $\deg v_i \leq 2$. If $\deg v_i = 1$ for all $i$, $1 \leq i \leq \Delta'$ then $T$ is a bistar which is a contradiction. Thus $\deg v_i = 2$ for some $i$. 
Claim. At most one vertex of $Y_1$ or at most one vertex of $Y_2$ has degree 2.

Suppose $v_1, v_2 \in Y_1$ and $v_i, v_j \in Y_2$ with $\deg v_k = 2$, for $k \in \{1, 2, i, j\}$. Let $w_k \in N(v_k) - \{u, v\}$ for $k \in \{1, 2, i, j\}$. Then $X = [E(T) - (N[e] \cup \{v_1w_1, v_2w_2, v_iw_i, v_jw_j\})] \cup \{uv_1, uv_2, vv_i, vv_j\}$ is a nced-set with $|X| = m - \Delta' - 1$ which is a contradiction. Hence at most one vertex of $Y_1$ or at most one vertex of $Y_2$ has degree 2.

Case 2. $E(T_1) \neq \emptyset$.

Let $G_1$ be any non-trivial component of $T_1$ and we may assume without loss of generality that $v_1 \in N[V(G_1)]$. If $G_1$ contains more than one pendant vertex of $T$, then $X = [E(T) - (N(e) \cup E_1)] \cup \{uv_1\}$ where $E_1$ is the set of all pendant edges of $T$ in $G_1$, is a nced-set of $T$ with $|X| < m - \Delta'$ which is a contradiction. Hence $G_1$ is a path. Suppose $G_1 = (x_1, x_2, \cdots, x_k), k \geq 3$ and let $v_1x_1 \in E(T)$. Then $X = [E(T) - [N(e) \cup \{v_1x_1, x_1x_2\}]] \cup \{uv_1\}$ is a nced-set of $T$ with $|X| = m - \Delta' - 1$ which is a contradiction. Thus $G_1 = P_2$. Now, if $T$ has two non-trivial components $G_1 = (x_1, x_2)$ and $G_2 = (y_1, y_2), x_1 \in N(v_i), y_1 \in N(v_j)$ then $X = [E(T) - N(e) \cup \{v_ix_1, v_jy_1\}] \cup \{uv_i\}$ is a nced-set of $T$ which is again a contradiction. Thus $T_1$ has exactly one non-trivial component. Let $X_1 = \{uv_i : 1 \leq i \leq r\}$ and $X_2 = \{vv_j : r + 1 \leq j \leq \Delta'\}$ then the result follows and the converse is obvious.  

Theorem 4.3.3. Let $G$ be a unicyclic graph with cycle $C = (v_1, v_2, \cdots, v_r, v_1)$. Then $\gamma_{nc}'(G) = m - \Delta'$ if and only if $G$ is
isomorphic to $C_3$ or $C_4$ or $C_5$ or one of the graphs $G_i$, $1 \leq i \leq 23$, given in figure 4.5.
Figure 4.5: Unicyclic graph with $\gamma_{nc} = m - \Delta'$
Proof. Let $G$ be a unicyclic graph with cycle $C$ and $\gamma'_{nc}(G) = m - \Delta'$. If $G = C$ then it follows from theorem 4.2.8 that $m \leq 5$ and hence $G$ is isomorphic to $C_3$ or $C_4$ or $C_5$. Suppose $G \neq C$. Let $A$ denote the set of all pendant edges in $G$ and let $|A| = k$. Suppose $k \geq \Delta' + 1$. Since $E(G) - A$ is a nc-set of $G$ we have $\gamma'_{nc}(G) \leq m - \Delta' - 1$ which is a contradiction. Hence $k \leq \Delta'$. Also maximum of two adjacent edges of $e$ are in $C$ we have $\Delta' - 2 \leq k$.

Hence $\Delta' - 2 \leq k \leq \Delta'$. (1)

Let $e = uv$ with $\deg e = \Delta'(G)$. Suppose $d(e, C) \geq 1$, then $k = \Delta'$ or $\Delta' - 1$. Then $X = [E(G) - E(C) \cup A] \cup X_1$ where $X_1$ is nc-set of $C$, is a nc-set of $G$ with $|X| < m - \Delta'$ which is a contradiction. Hence the edge $e$ lies on $C$ or incident with $C$. Let $e$ be incident with $C$ and let $C = (v_1, v_2, \cdots, v_r, v_1)$. Let us assume $u = v_1$.

Claim. $r \leq 4$.

Suppose $r \geq 6$. Then any $\gamma'_{nc}$-set of $C$ does not contain at least 3 edges of $C$. Let $X_1$ be a $\gamma'_{nc}$-set of $C$ which contains an edge adjacent to $e$. Then $X = [E(G) - (E(C) \cup A)] \cup X_1$ is a nc-set of $G$ with $|X| < m - \Delta'$ which is a contradiction. Hence $r \leq 5$. Suppose $r = 5$. Let $C = (v_1, v_2, v_3, v_4, v_5, v_1)$. Then $X = E(G) - [A \cup \{v_1v_2, v_2v_3, v_4v_5\}]$ is a nc-set of $G$ with $|X| < m - \Delta'$ which is a contradiction. Hence $r \leq 4$ and it is clear that every vertex in $V(C) - \{v_1\}$ has degree 2.

Case 1.1. $r = 4$. 
Let $C = (v_1, v_2, v_3, v_4, v_1)$. Suppose there exists a vertex $w \in A$ such that $d(w,e) \geq 2$. Let $d(w,u) = d(w,e)$ and let $(u, w_1, w_2 \cdots, w_k, w)$, $k \geq 1$ be the unique $u - w$ path. Then \( X = [E(G) - [A \cup \{v_1v_2, v_3v_4, v_4v_1, w_1w_2\}]] \cup \{uw\} \) is a nced-set of $G$ with $|X| < m - \Delta'$ which is a contradiction. Similarly we can get a contradiction if $d(w,v) = d(w,e)$. Hence $d(w,e) = 1$ for all $w \in A$. Thus $G$ is isomorphic to $G_1$.

**Case 1.2.** $r = 3$.

Let $C = (v_1, v_2, v_3, v_1)$ and $u = v_1$, suppose there exists a vertex $w \in A$ such that $d(w,e) \geq 3$. Let $d(w,u) = d(w,e)$ and let $(u, w_1, w_2, \cdots, w_k, w), k \geq 2$ be the unique $u - w$ path. Then $X = [E(G) - [A \cup \{v_2v_3, v_3v_1, uw_1, w_1w_2\}]] \cup \{wkw\}$ is a nced-set of $G$ with $|X| < m - \Delta'$ which is a contradiction. Similarly we can get a contradiction if $d(w,v) = d(w,e)$. Hence $d(w,e) \leq 2$ for all $w \in A$. Let $w_1 \in N(u) - [V(C) \cup \{v\}]$ and $\deg w_1 \geq 3$. Then $X = E(G) - [A \cup \{v_2v_3, v_3v_1\}]$ is a nced-set of $G$ with $|X| < m - \Delta'$ which is a contradiction. Similarly we can get a contradiction if $w_1 \in N(v) - \{u\}$. Now, let $w_1, w_2 \in N(u) - [V(C) \cup \{v\}]$ such that $\deg w_1 = \deg w_2 = 2$. Suppose there exist two vertices $w_3, w_4 \in N(v) - \{u\}$ such that $\deg w_3 = \deg w_4 = 2$. Then $X = E(G) - [A \cup \{v_2v_3, v_3v_1, e\}]$ is a nced-set of $G$ with $|X| < m - \Delta'$. Hence at most one vertex of $N(v) - \{u\}$ is of degree 2. Then $G$ is isomorphic to $G_2$ or $G_3$. Let $w_1, w_2 \in N(v) - \{u\}$ with $\deg w_1 = \deg w_2 = 2$. Suppose there exists a vertex $w_3 \in N(u) - [V(C) \cup \{v\}]$ such that $\deg w_3 = 2$. Then $X = E(G) - [A \cup \{v_2v_3, v_3v_1, e\}]$ is a nced-set of $G$ with $|X| < m - \Delta'$ which is a contradiction. Hence $G$ is
isomorphic to \( G_4 \).

Suppose \( e \) lies on \( C \). Let \( C = (v_1, v_2, \ldots, v_r, v_1) \) and \( v_1v_2 = e \).

**Claim 1.** \( \deg w = 1 \) or \( 2 \) for all \( w \in V(G) - V(C) \).

Suppose there exist a vertex \( w \in V(G) - V(C) \) with \( \deg w > 2 \). Then \( k = \Delta' - 1 \) or \( \Delta' \). If \( k = \Delta' - 1 \), then all the vertices of \( V(C) - \{v_1, v_2\} \) have degree 2 and hence \( X = E(G) - [A \cup \{v_2v_3, v_2v_1\}] \) is a ncN-set of \( G \) with \( |X| < m - \Delta' \). If \( k = \Delta' \) then \( X = E(G) - [A \cup \{v_2v_3\}] \) is a ncN-set of \( G \) with \( |X| < m - \Delta' \) which is a contradiction. Hence \( \deg w = 1 \) or \( 2 \) for all \( w \in V(G) - V(C) \).

**Claim 2.** Every vertex of \( V(C) - \{v_1, v_2\} \) has degree 2 or 3.

It follows from (1) that \( \deg v_i \leq 4 \) for all \( i \neq 1, 2 \). If there exists a vertex \( v_i \in V(C) \) with \( \deg v_i = 4 \), then \( k = \Delta' \) and \( X = E(G) - [A \cup \{v_2v_3\}] \) is a ncN-set of \( G \) with \( |X| < m - \Delta' \). This proves claim 2.

**Claim 3.** \( r \leq 5 \).

Suppose \( r \geq 6 \). If \( k = \Delta' \) then \( X = E(G) - [A \cup \{v_2v_3\}] \) is a ncN-set of \( G \) with \( |X| < m - \Delta' \). If \( k = \Delta' - 1 \) then there exists a vertex \( v_i \) such that \( \deg v_i = 2 \). Now \( X = E(G) - [A \cup \{v_{i-1}v_i, v_iv_i+1\}] \) is a ncN-set of \( G \) with \( |X| < m - \Delta' \). If \( k = \Delta' - 2 \) then every vertex of \( V(C) - \{v_1, v_2\} \) has degree 2 and hence \( X = E(G) - [A \cup \{v_2v_3, v_{r-3}v_{r-2}, v_{r-2}v_{r-1}\}] \) is a ncN-set of \( G \) with \( |X| < m - \Delta' \). Thus \( r \leq 5 \).

**Claim 4.** \( d(w, C) \leq 2 \) for all \( w \in A \).
Suppose there exist a pendant vertex \( w_1 \), such that \( d(w_1, C) \geq 3 \). Let \((w_1, w_2, \ldots, w_k, v_i), k \geq 3\) be the unique \( w_1 - v_i \) path. If \( k \neq \Delta - 2 \) then \( X = [E(G) - [A \cup \{v_2v_3, v_3v_4, v_4w_k, w_kw_{k-1}\}]] \cup \{w_2w_1\} \) is a nced-set of \( G \) with \(|X| < m - \Delta'\). If \( k = \Delta - 2 \), then \( X = [E(G) - [A \cup \{v_2v_3, v_3v_4, v_4w_k, w_kw_{k-1}\}]] \cup \{w_2w_1\} \) is a nced-set of \( G \) with \(|X| < m - \Delta'\) which is a contradiction. Hence \( d(w, C) \leq 2 \) for all \( w \in A \).

**Claim 5.** If there are two \( P_3 \) attached with \( v_1 \) then at most one \( P_3 \) is attached to \( v_2 \).

Suppose not, then \( X = E(G) - [A \cup \{v_1v_2, v_2v_3, v_3v_4, v_4v_i\}] \) is a nced-set of \( G \) with \(|X| < m - \Delta'\) which is a contradiction. Hence the Claim 5.

**Case 2.1.** \( k = \Delta' - 2 \).

In this case \( \deg x = 1 \) or \( 2 \) for all \( x \in V(G) - \{v_1, v_2\} \). Now, if \( r = 5 \) and if there exists a vertex \( w \in N(v_i) - V(C), i = 1 \) or \( 2 \), such that \( \deg w = 2 \), then \( X = E(G) - [A \cup \{v_2v_3, v_3v_4, v_4v_5v_1\}] \) is a nced-set of \( G \) with \(|X| < m - \Delta'\). Hence \( \deg w = 1 \) for all \( w \in N(v_i) - V(C) \) and hence \( G \) is isomorphic to \( G_5 \) or \( G_6 \). If \( r \leq 4 \) then \( G \) is isomorphic to \( G_i, 7 \leq i \leq 15 \).

**Case 2.2.** \( k = \Delta' - 1 \).

In this case \( \deg v_i = 3 \) for exactly one vertex \( v_i \neq v_1 \) and \( v_2 \) on \( C \) also \( \deg x = 1 \) or \( 2 \) for all \( x \in V(G) - \{v_1, v_2, v_i\} \). If \( r = 5 \), then \( X = E(G) - [A \cup B] \) where \( B \) is a set of edges in \( C \) not incident with \( v_i \) is a nced-set of \( G \) with \(|X| = m - \Delta' - 1 \) and hence \( r = 3 \) or \( 4 \). Suppose there exists a path \((v_i, x_1, w_1)\) such that \( x_1 \notin V(C) \)
and \( w_2 \in A \), if \( r = 4 \) then \( X = [E(G) - (A \cup B \cup \{v_i x_1\})] \cup \{x_1 w_1\} \)
where \( B \) is \( N[v_i x_1] \cap V(C) \) is a need-set of \( G \) with \(|X| < m - \Delta'\) and if \( r = 3 \), then \( X = E(G) - [A \cup \{v_2 v_3, v_3 v_1, v_3 x_1\}] \cup \{x_1 w_1\} \) is a need-set of \( G \) with \(|X| < m - \Delta'\) and hence \( G \) is isomorphic to \( G_i, 16 \leq i \leq 23 \).

**Case 2.3.** If \( k = \Delta' \).

In this case \( r = 4 \) or 5 and there does not exists a graph with \( \gamma'_{nc}(G) = m - \Delta' \). The converse is obvious. \( \square \)

**Remark 4.3.4.** Since \( \gamma'_{nc}(G) = \gamma_{nc}(L(G)) \) where \( L(G) \) is the line graph of \( G \), it follows from Theorem 1.3.37 that \( \gamma'_{nc}(G) \leq \left\lceil \frac{m}{2} \right\rceil \).

**Theorem 4.3.5.** Let \( G \) be any graph such that both \( G \) and \( \overline{G} \) are connected. Then \( \gamma'_{nc}(G) + \gamma'_{nc}(\overline{G}) \leq m + 1 \).

**Proof.** The proof follows from Remark 4.3.4 \( \square \)

**Remark 4.3.6.** The bounds given in Theorem 4.3.5 is sharp. The graph \( G = C_5 \), \( \gamma'_{nc}(G) + \gamma'_{nc}(\overline{G}) = 6 = m + 1 \).

**Theorem 4.3.7.** For any graph \( G \), \( \gamma'_{nc}(G) \leq \left\lfloor \frac{3m}{4} \right\rfloor \).
Proof. Let $X$ be a maximum matching of the graph $G$. Label the edges of $X$ by $e_1, e_2, \cdots, e_k, e_{k+1}, \cdots, e_r$ such that the edges $e_i$ and $e_{i+1}$, $i$ is odd $1 \leq i \leq k - 1$ are adjacent to common edge $f(e_i)$ with maximum value of $k$. Let $Y = \{f(e_i)/i$ is odd $\}$. It is clear that $|X| \leq \lfloor \frac{n}{2} \rfloor$ and $|Y| = \lfloor \frac{n}{2} \rfloor$. Then $X \cup Y$ is an edge dominating set with $\langle N(X \cup Y) \rangle$ is connected and hence $\gamma'_{nc}(G) \leq |X \cup Y| = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{2} \rfloor \leq \left\lfloor \frac{3n}{4} \right\rfloor$.

Remark 4.3.8. The bound given in Theorem 4.3.7 is sharp. The graph $G = C_5$, $\gamma'_{nc}(G) = 3 = \left\lfloor \frac{3n}{4} \right\rfloor$

4.4 CONCLUSION AND SCOPE

In this chapter we have introduced a new type domination, namely, neighborhood connected edge domination and presented several results on the corresponding domination parameter. The following are some interesting problems for further investigation.

Problem 4.4.1. Characterize the class of graphs for which $\gamma'_{nc}(G) = m - \Delta'$.

Problem 4.4.2. Characterize the class of graphs for which $\gamma'_{nc}(G) + \gamma'_{nc}(\overline{G}) = m + 1$.

Problem 4.4.3. Characterize the class of graphs for which $\gamma'_{nc}(G) = \left\lfloor \frac{3n}{4} \right\rfloor$. 