CHAPTER 2

QUANTIFICATION OF CHAOS IN NONLINEAR DYNAMICS

Detailed description of the method of nonlinear analysis used in the present work is given. Parameters which can quantify the degree of chaos are dealt with in detail.
Simple linear systems are easy to handle and one can develop the basic equations of dynamics without much difficulty. In case of complex systems, it may be difficult to obtain equations of motion, especially when the interactions are nonlinear. However, the dynamics will reflect up on the time dependence of certain easily measurable quantities. The temporal development of such quantities is known as the time series. Time series analysis, which is currently attracting much interest, can give immense insight into the dynamics of the system. In this chapter we describe various methods used in time series analysis. We shall describe in detail the methods suitable for nonlinear systems, after giving a brief description of conventional techniques usually employed in time series analysis.

2.1 FOURIER SPECTRA

Discrete Fourier transform is one of the usual methods used to determine the kind of evolution produced by a dynamical system by studying a time dependent signal $x(t)$, the time series. This will help us to find out various frequencies present in the system under consideration. This method is used to identify the general nature of the system and has recently been successfully applied to the studies of asteroidal belt (Pratap 1977) and Neural system (Hinrichs 1987, Dumermuth and Molinari 1987). In this section we study the method of Fourier analysis of a time series. We shall explain the spectra to be observed for different classes of signals like sinusoidal, non-sinusoidal, quasi periodic etc. The limitations of the method in the analysis of chaotic system shall be discussed.
We assume that the signal $x(t)$ is a continuous function of time. This signal is then sampled such that the experimental results provide a discrete sequence of real numbers $x_j$ which are regularly spaced in time with an interval of $\Delta t$. The number of data is finite, containing $n$ values for a total length of time $t_{\text{max}} = (n-1)\Delta t$. The smallest frequency obtainable from such a time series, $\Delta f = 1/t_{\text{max}}$. We can define Fourier transform of a discrete time series $x_j$ as discrete fourier series $\hat{x}_k$,

$$\hat{x}_k = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} x_j \exp \left(-i \frac{2\pi j k}{n} \right)$$ (2.1)

$k = 1, \ldots, n$, $i = \sqrt{-1}$

The graph representing $|\hat{x}_k|^2$ as a function of the frequency $f (f = k \Delta f)$ is called the power spectrum. The nature of the power spectrum is characteristic of the system.

Let us consider a periodic signal $x(t)$ of period $T$, so that

$$x(t) = x(t+T)$$ (2.2)

If the signal is sinusoidal its power spectrum contains only a single frequency at $1/T$ Hz (Figure 2.1). A nonsinusoidal signal (Figure 2.2a), gives rise to the spectrum containing the frequencies located at $2/T$ Hz, $3/T$ Hz ... etc.

![Fig 2.1 Fourier spectra of pure sinusoidal function.](image-url)
(Figure 2.2b) which are the harmonics of the fundamental frequency $1/T$ Hz. That is, the presence of harmonics in the spectrum shows that the system is nonsinusoidal. The Figure (2.2b) represents a system in which the duration of measurement is an integer multiple of signal period $T$ i.e; $t_{max} = pT$, $p > 1$ (a positive integer).

If we have a situation, where $t_{max}/T$ is noninteger, then

$$\left| \hat{x}_k \right|^2 = n \frac{\sin^2(n\pi\phi_k)}{(n\pi\phi_k)^2}$$

(2.3)

In this case the behaviour of the function is like that of $\sin^2 z/z^2$ as shown in Figure 2.3. The function has a maximum amplitude at $z=0$, with a series of secondary maxima at $\pm(i+1/2)\pi$, whose amplitude decreases as $1/z^2$ where $i$ is a positive integer.
A dynamical system, whose behaviour is due to the superposition of oscillations which differ in amplitude, period and ratio of harmonics, will show a totally different frequency spectrum. This type of system can be represented by a function $y$ of $r$ independent variables $t_1, t_2, \ldots, t_r$, which is said to be periodic with period $2\pi$ and has the property

$$y(t_1, t_2, \ldots, t_j, \ldots, t_r) = y(t_1, t_2, \ldots, t_j + 2\pi, \ldots, t_r)$$

for $j = 1, \ldots, r$. \hspace{1cm} (2.4)

The function $y$ is said to have $r$-periods and it represents a quasiperiodic system with multiple periodicity. An example of a quasiperiodic system can be given in terms of the astronomical position of a point on the surface of the earth. In this system, rotation of the earth about its axis takes 24 hours ($T_1 = 24$ hours), the rotation of the earth around the sun takes 365.242 days ($T_2 = 365.242$ days) and the precession of the earth's axis of rotation takes 25,800 years ($T_3 = 25,800$ years). This system contains three frequencies. Hence it is not possible to describe the trajectory of this in a phase space in terms of limit cycle or fixed point, used for periodic systems. An alternate description can be made for such systems in terms of the new mode of phase space trajectory, viz., 'torus' of dimension $r$ (i.e. $T^r$).

There are two types of quasiperiodic systems. If the quasiperiodic function $x(\omega_1, \ldots, \omega_r)$ is the sum of periodic functions

$$x(\omega_1, \ldots, \omega_r) = \sum_{i=1}^{r} x_i(\omega_i t)$$

then its power spectrum is the sum of $r$ spectra of each of
functions $x(\omega t)$. Then the spectrum contains a set of peaks located at the fundamental frequencies $f_1', f_2' \ldots \ f_r$ and of their harmonics,

$$mf_1', mf_2' \ldots \ mf_r$$

where $m_1', m_2', \ldots \ m_r$ are positive integers. But if the quasiperiodic function includes term like the product of circular function $(\sin(\omega t) \sin(\omega_j t))$, then Fourier spectrum has a complex appearance and contain frequencies $|f_i-f_j|$ and $|f_i+f_j|$ and their harmonics

$$\sin(\omega t) \sin(\omega_j t) = \frac{1}{2} \cos(|f_i-f_j|2\pi t) - \frac{1}{2} \cos(|f_i+f_j|2\pi t)$$

(2.6)

In order to study a quasiperiodic system in detail, consider a bi-periodic case in which each of the nonzero component of the spectrum of the signal $x(\omega_1 t, \omega_2 t)$ is a peak with abscissa $|mf_1'+mf_2'|$. We can classify the systems in terms of the ratio of $f_1'/f_2'$, viz., whether it is rational or irrational. If $f_1'/f_2'$ is rational, then the Fourier spectrum is not dense (Fig.2.4).
\[
\frac{f_1}{f_2} = \frac{n_1}{n_2} \quad (n_1, n_2 \text{ integers}) \quad (2.7)
\]

I.e.; quasiperiodic signal is periodic with period

\[
T = n_1 T_1 = n_2 T_2
\]

So, we can write

\[
x(\omega_1, \omega_2) = x(\omega_1 + 2\pi n_1, \omega_2 + 2\pi n_2)
\]

i.e.,

\[
x(\omega_1, \omega_2) = x(\omega (t_1 + n / f_1), \omega (t_2 + n / f_2))
\]

\[
(2.8)
\]

In this type of system, there is a 'frequency locking' of \( f_1 \) with \( f_2 \). This means that all the lines of spectrum are always separated by same amount of \( 1/T \).

But if \( f_1/f_2 \) is irrational (the system contains incommensurate frequencies), then the power spectrum has a complex appearance. Two peaks are so close together with the appearance of a continuous function. However, usually only for very limited number of frequencies have significant amplitudes. Higher order frequencies have amplitudes too low to be detected. In this case, the higher amplitude lines are presented
by the combinations $|m_1 f_1 + m_2 f_2|$ with $m_1$ and $m_2$ having small values; $0, \pm 1, \pm 2$ (Figure 2.5).

The Fourier spectrum of an aperiodic signal is continuous as shown in Figure (2.6) (Gollub et al 1975, Kurten et al 1986, Brandstater et al 1983). However, we will not be able to conclude that a signal is aperiodic from the appearance of Fourier spectrum alone, since quasiperiodic signals also give a similar looking Fourier spectrum, when the number of frequencies are very high. Moreover, random signals also exhibit similar spectra. It is also known that signals arising from a system exhibiting deterministic chaos have quasi continuous spectra.

2.2 FAST FOURIER TRANSFORM (FFT)

FFT is an algorithm to compute discrete fourier transform from time series (developed by Cooley and Turkey in
1965), and is one of the first techniques usually employed in the identification of deterministic chaos. Deterministic chaos has a Fourier spectrum where a few dominant frequencies are superimposed on a broad band noise floor. FFT technique is useful when the number of data \( n \) is very large with small \( \Delta t \). For example, to calculate discrete Fourier transform with \( n=10^3 \), we have to calculate 1000 sums, each of which contains 1000 terms. This means that the number of operation needed is of the order of \( n^2 \). It will take a large time for the computation of the frequencies using conventional method. However, when \( n \) is a power of two, the FFT algorithm speeds up the calculation of spectrum. For \( n=2^{10} = 1024 \), the gain of computational time is by a factor of 100, while it attains 7000 for \( n=2^{18} \). Hence, the importance of FFT techniques increases as the number of data increases (Berge et al 1984).

2.3 AUTOCORRELATION FUNCTION (ACF)

Like the FFT, Autocorrelation function can also be used to characterize a given system. ACF gives an idea about how predictable the system is. It is useful to estimate the disorder by measuring the resemblance of \( x \) at time \( t \) with itself at a later time \( t+\tau \) or it is the degree of resemblance of signal with itself as time passes. It represents the average of the product of the signal values at a given time and at a time \( m\Delta t \) later.

The ACF of a signal \( x_j \) can be described in the following manner

\[
\psi_m = \frac{1}{n} \sum_{j=1}^{n} x_j x_{j+m} \quad \psi_m = \psi (m\Delta t)
\]

(2.9)

where \( \psi \) is the ACF of the signal \( x_j \).

If we consider the time series \( (V_1', V_2', \ldots V_n') \) (Babloyantz et al 1986), then,
\[
\psi(\tau) = \left( \frac{1}{n} \right) \sum_{i=1}^{n} \left[ V(t_i) - \bar{V} \right] \left[ V(t_i + \tau) - \bar{V} \right] \\
\frac{1}{n} \sum_{i=1}^{N} \left[ V(t_i) - \bar{V} \right]^2
\]

where \( \bar{V} = \frac{1}{n} \sum_{i=1}^{n} V(t_i) \)

According to Wiener-Khintchine theorem,

\[
\psi_m = \frac{1}{n} \sum_{k=1}^{n} |\hat{x}_k|^2 \cos \left( \frac{2\pi mk}{n} \right)
\]

where \( \hat{x}_k \) is a fourier component.

Thus the ACF is, up to a factor of proportionality, the fourier transform of \( |\hat{x}_k|^2 \). This means that the periodic or quasiperiodic signal resembles itself at later times. That is the behaviour of such systems are predictable.

In the chaotic regime, the power spectrum has a continuous floor, so that \( \psi(\tau) \) necessarily tends to zero as \( \tau \) increases (Figure 2.7) (Gollub et al 1975, Babloyantz et al 1986). The resemblance of the signal disappears as time increases. In other words the predictability of the signal looses within finite time. Here also it is difficult to distinguish between aperiodic and random signal.
We found in earlier sections that with both FFT and ACF, it is difficult to distinguish between quasiperiodic, aperiodic and random signals. Hence other methods should be searched for. One such method which gives more information about the behaviour of the system in the phase space was developed by Henri Poincare, and is popularly known as the method of 'Poincare section'.

In the case of a three dimensional system \((x_1, x_2, x_3)\), Poincare plane \(S\) defined by \(x_3=\text{constant}\) and trajectory \(\Gamma\) intersect the plane \(S\) at \(p_0, p_1, \ldots\), where the dynamics are assumed such that \(x_3\) continually crosses from one side of \(S\) to the other. Thus starting from an initial condition, we eventually get a number of points on \(S\), which is called the Poincare section.

The transformation leading from one point on \(S\) to the next is a continuous mapping \(T\) of \(S\) into itself called the Poincare map.

\[
P_{k+1} = T(P_k) = T(T(P_{k-1})) = T^2(P_{k-1}) = \ldots = T^l(P_{(k-l+1)})
\]

\[
= \ldots = T^{k+1}(P_0)
\]

Thus, it is clear that \(P_0\) completely determines \(P_1\), which in turn determines \(P_2\) and so on.

Poincare section and map reflects the property of flow of the system. For example, if the flow is dissipative, its volume in phase space contracts.

In the Poincare method, there is no restriction on the dimension \(n\) of the phase space. It converts an \(n\)-dimensional
flow into \((n-1)\) dimensional difference map. If the flow is three dimensional, then this method maps the flow on to a plane, reducing the number of coordinates by one. Secondly, the time is discretized and time interval between two successive points is not constant. The differential equations are replaced by difference equations, by Poincare map \(P \circ T(P)\), so that it is easy to manipulate the equations to get a sequence of points \(x_1, x_2, \ldots, x_n\) by successive iterates of a difference map

\[ x_{n+1} = f(x_n) \]  

The Poincare sections usually have the following appearance, a point or a number of points are located along a single curve, or distributed on a surface. In the case of a periodic system the Poincare section is a single point in a plane and this point is called fixed point of Poincare map \(T\). This can be represented by

\[ P_0 = T(P_0) = T^2(P_0) = \ldots \]  

In the case of a truly quasiperiodic system, the Poincare section has a number of points which looks like a simple curve (Fig. 2.8a). If the frequencies are commensurate the curve will have only a limited number of points while with incommensurate frequencies it will be densely filled. For aperiodic or chaotic system the points are distributed on a

Fig 2.8 Poincare sections
a) quasiperiodic regime b) chaotic regime
surface as shown in Figure (2.8b). But even with this method, it may not always be helpful to distinguish between an aperiodic system which is strongly contracting and a quasiperiodic system. This is because the contraction of area for a strongly contracting aperiodic system may be too rapid so that Poincare section will look like a simple curve as in the case of quasiperiodic systems.

For example in the Lorenz model, the Poincare section in the $X,Y$ plane with $Z=r-1$ consists of only two line segments as in Figure (2.9). This implies that the trajectories can be inscribed almost on a surface and the attractor has dimension two. But Lorenz attractor has a complex structure consisting of a large number of closely packed sheets. Its Hausdorff-Besicovitch dimension is 2.06. These results prove that the Lorenz attractor is not a simple surface. The reason for the fractal dimension to be very close to two is due to the strong volume contraction.

2.5 LIMITATIONS OF CONVENTIONAL TECHNIQUES

FFT, ACF and the method of Poincare section are three signal processing techniques which are useful for classifying the systems in a general way. However, these give only qualitative ideas about the dynamics of the system. They do not quantify any of the characteristic properties of the system. Even though we are able to say that whether or not the system is chaotic, these methods do not tell us how much chaotic, the system is. A study, therefore, using these techniques will have its own limitations.
FFT and ACF are usually valid only for linear systems which obey Dirichlet's condition regarding the continuity and finite number of finite discontinuities. Any Fourier decomposition of a given dynamical process would imply the existence of a fundamental frequency and other frequencies could be commensurate to this fundamental frequency. However, a nonlinear process can arise from the existence of two or more incommensurate fundamental frequencies in the system, and this will not be revealed by a Fourier decomposition or ACF. Thus for thermodynamically open nonlinear systems like, for example, the neural network, principles like superposition and ergodicity are not valid and hence FFT or ACF methods are inadequate for this analysis. Furthermore, such systems can also exhibit non Markovian characteristics giving, thereby, memory effects. Mathematically speaking, superposability of harmonic functions which is the basic property used in Fourier analysis, would break down if the system is nonlinear.


2.6 NONLINEAR ANALYSIS

Dissipative dynamical systems are characterized by the attraction of all trajectories passing through certain domain of phase space towards a geometrical object called attractor. The attractor is a compact set in phase space which is invariant under the action of the flow or mapping. The set of initial conditions giving rise to trajectories converging towards the attractor is called the basin of attraction.
There are four types of attractors. Let us briefly describe them. The simplest among these is the point attractor. It describes a solution which is independent of time - that is a steady state. This is essentially a fixed point in the phase space. The limit cycle is the second type of attractor, and is basically characterized by its amplitude and period. Its Fourier spectrum contains only a single fundamental frequency and possibly a certain number of harmonics. The solution to the flow can always be expressed as a Fourier series and if the state of the system is known at a given time, one can predict its state at all later times.

A third type of attractor is the torus $T^r (r \geq 2)$ which corresponds to a quasiperiodic regime with $r$ independent fundamental frequencies. Here also the Fourier spectrum is composed of a set of lines, whose frequencies are linear combinations of fundamental frequencies. While the solution to the flow cannot generally be put into the form of an ordinary Fourier series, it is still possible to calculate the state of the system starting from a given initial condition.

The attractor of systems exhibiting chaos are quite different. They are called strange attractors (Ruelle and Takens 1976). To understand the strange behaviour of such attractors, it is necessary to discuss some of the general features exhibited by almost all chaotic systems (Roux et al 1983, Babloyantz et al 1986, Reghunath et al 1987, Nicolis and Nicolis 1986, Albano et al 1985), (i) its power spectrum is continuous or broad band and (ii) the autocorrelation function of the time signal has only finite support, that is, it goes to zero in finite time. The strange attractors have the following properties (Schuster 1984, Atmanspacher et al 1986)

1) phase trajectories are attracted towards it
2) Pairs of neighbouring trajectories diverge on it
3) trajectories are sensitive to initial conditions
4) its dimension $D$ is fractal

We have seen that there are two types of systems viz., Regular systems characterized by simple attractors (equilibrium
point, limit cycle or torus) with integer dimension and chaotic systems characterized by strange attractors which have noninteger dimension. How can we classify these systems? Even though they can be characterized by Fourier analysis, the method does not distinguish between chaos involving small number of degrees of freedom and white noise. Such a distinction can be made with the help of Poincare section. But this method offers only qualitative information and also it is not quite practicable for systems with higher dimensions.

A quantitative characterization can be done using certain characteristics of attractors in phase space. Two of such significant properties of chaotic systems are the Hausdorff dimension of the attractor and Kolmogorov entropy.

Kolmogorov entropy, as we have already seen (section 1.12), is connected with the divergence of trajectories in phase space (Benettin et al 1976) or creation of information. In this connection we can define a whole set of dimension $D_q$ (Hentschel et al 1983) and entropies $K_q$ (Grassberger and Procaccia 1983 b & c), which generalize the concept of Hausdorff dimension and Kolmogorov entropy (Paweizik and Schuster 1987). We shall discuss about these in the latter part of this chapter.

### 2.7 GENERALIZED DIMENSIONS

Trajectories of certain dissipative dynamical systems exhibiting chaotic behaviour shrinks towards an attractor whose dimension is less than the dimension of phase space, and is strange in character (Lorenz 1963, 1984, Ruelle et al 1971, Ott 1981). As indicated earlier, strange attractors can be characterized by their characteristic dimensions. Some of the important dimensions among generalized dimensions which are commonly used to describe nonlinear systems are fractal dimension, information dimension and correlation dimension.
We try to understand the generalized dimension in detail in this section. Let us describe the given dynamical system by a differential equation \( \frac{dX}{dt} = F(X) \), where \( X \) is a \( d \)-dimensional vector obtained from a single time series by using a delay time \( \tau \),

\[
X(t) = \left\{ x(t), x(t+\tau), \ldots, x(t+(d-1)\tau) \right\}
\]

(2.15)

Now consider a \( d \)-dimensional phase space which is uniformly partitioned into boxes of size \( \varepsilon \), and \( N \) points \( \{X_i\}_{i=1}^{N} \) in a time sequence which are given by sampling the signal \( X(t) \). One can estimate the invariant probability measure \( p_i \) associated with box \( i \) by \( \frac{N_i}{N} \), where \( N_i \) is the number of points falling within the box \( i \), provided \( N \) is large enough. In general, it has been shown that generalized dimension \( D_q \), of order \( q \) (Hentschel and Procaccia 1983, Sato et al 1987) can be defined as

\[
D_q = \lim_{\varepsilon \to 0} \frac{\ln \left( \sum p_i^q \right)}{\ln \varepsilon}
\]

(2.16)

The order of parameter \( q \) can take all real values between \(-\infty\) and \(+\infty\). For \( q = -\infty \), it characterizes the rarer regions, while for \( q = \infty \), describes the denser regions of the set. Thus in (2.16) an infinite hierarchy of dimensions are implied.

In (2.16) \( p_i(\varepsilon) \) is the probability that the trajectory \( X_1, X_2, \ldots, X_N \) on the strange attractor visits the box \( i \). Since \( \sum p_i^q \) can be written in terms of the natural probability measure \( \mu^{(x)} \) on the attractor as

\[
\sum p_i^q = \int d\mu^{(x)} \left[ \mu \left( B_{\varepsilon}(x) \right) \right]^{q-1}
\]

(2.17)
where \( B_\varepsilon(x) \) denotes a ball of radius \( \varepsilon \) around \( x \) and

\[
\sum_i p_i^q = \frac{1}{N} \sum_{j=1}^{N} \tilde{p}_j(\varepsilon)
\]  
(2.18)

where \( \tilde{p}_j(\varepsilon) \) is the probability to find a point of the trajectory within the ball of radius \( \varepsilon \) around a point \( X_j \) of the trajectory. The change \( q \) to \( (q-1) \) in the exponents in equation (2.18) is due to the fact that we switch from \( p_i(\varepsilon) \) to \( \tilde{p}_j(\varepsilon) \) (Pawelzik and Schuster 1987). That is, we are switching from the probability to find the trajectory in one of the homogeneously distributed boxes introduced above, to the probability to find the trajectory within a ball around one of the inhomogeneously distributed points of the trajectory.

\[
\tilde{p}_j(\varepsilon) = \frac{1}{N} \sum_i \Theta(\varepsilon - |X_i - X_j|)
\]  
(2.19)

By combining equation (2.16)–(2.19)

\[
D_q = \lim_{\varepsilon \to 0} \frac{1}{\ln(\varepsilon)} \ln C_q(\varepsilon)
\]  
(2.20)

where

\[
C_d(\varepsilon) = \left[ \frac{1}{N} \sum_i \left[ \frac{1}{N} \sum_j \Theta(\varepsilon - |X_i - X_j|) \right]^{q-1} \right]^{1/(q-1)}
\]  
(2.21)

and
\[ C_d (\varepsilon) = \left[ \frac{1}{N} \sum_i \left[ \frac{1}{N} \sum_j (\varepsilon - \left[ \sum_{m=0}^{d-1} (X_{i+m} - X_{j+m})^2 \right]^{1/2} \right]^{q-1} \right]^{1/(q-1)} \] (2.22)

Among \( D_q \), the important ones are \( D_0, D_1 \) and \( D_2 \) with \( D_0 \geq D_1 \geq D_2 \). It has been shown that \( D_0 \) corresponds to \( D_f \), the fractal dimension and \( D_1 \) is identical to \( D_I \), the information dimension (Hentschel and Procaccia 1983) and \( D_2 \) is known as the correlation dimension. For a homogeneous system, all these dimensions are equal to fractal dimension, i.e., \( D_0 = D_1 = D_2 = \ldots = D_\infty \). The inhomogeneity of the system is reflected in the inequalities of \( D_q \) for different \( q \)'s (Schuster 1984).

2.8 BOX-COUNTING ALGORITHM

Fractal dimension \( D_0 \) (\( D_f \)) is usually used to describe the dynamics of the system quantitatively. Consider a set of points in a \( d \)-dimensional space, and if \( N(\varepsilon) \) is the smallest number of cubes necessary to cover this set, then \( D_0 \) is defined as

\[ D_0 = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} \] (2.23)

Box-counting by Taken's algorithm (Takens 1981) is used to evaluate this fractal dimension, which does not take into account of the probabilities. This algorithm counts the number of boxes necessary to cover the set. The time dependence of the system with finite capacity can be described by finite-dimensional deterministic mathematical model. Nonintegral capacity represent
a chaotic system. Infinite capacity implies that an infinite number of degrees of freedom are needed to describe the dynamical system, which is not possible in terms of both Landau's theory and theory of strange attractors.

If the set has volume $V$, the number of boxes of side $\varepsilon$ needed to cover the set is

$$N(\varepsilon) \approx V \varepsilon^{-D_0}$$

for small $\varepsilon$.

Then,

$$\ln N(\varepsilon) = D_0 \ln (1/\varepsilon) + \ln V \quad (2.24)$$

Equation (2.24) is more suitable than equation (2.23) especially for processing experimental data because it has a slowly vanishing correction to the capacity $V/\ln(1/\varepsilon)$.

Takens (Takens 1981) showed that it is possible to compute $D_0$ from a single time series, represented by the infinite sequence of real numbers $\{a_i\}_{i=1}^{\infty}$. For this purpose, he constructed a phase space with infinite set of $D=n+1$ dimensional vectors,

$$S_D = \left\{ \langle a_i, \ldots, a_{i+n} \rangle \right\}_{i=1}^{\infty} \quad (2.25)$$

Capacity calculated using Taken's method in the case of 2/3 Cantor set, Quadratic return map and Henon map are in good
agreement with other methods (Greenside et al 1982).

2.9 IMPRACTICABILITY OF BOX-COUNTING ALGORITHM

Box counting algorithm is very slowly converging even for low dimensional attractors \( D_0 < 2 \). Also, it has severe computational difficulties in calculation for any set whose capacity is greater than two (Mizrachi et al 1984).

Greenside et al [1982] tested Taken’s box counting algorithm on several dynamical systems, the capacity of which has been known by other methods. For low dimensional systems, \( D \leq 2 \), the method works and the number of points necessary to determine the capacity is within the limits.

They have also applied Taken’s algorithm in hydrodynamical models, three variable model by Lorenz (Lorenz 1963) and 14 variable model by Curry (Curry 1978). Taken’s algorithm converges for large \( \varepsilon \). However, for smaller \( \varepsilon \), the algorithm failed to converge for both models even when about a million points were used.

One of the fundamental reasons why enormously long time series may be needed to calculate the capacity by box-counting method is the exponential dependence of \( N(\varepsilon) \) on \( D_0 \). Two other reasons are, (i) the set \( S_D \) often fills out the attractor in a highly nonuniform way and (ii) dynamical system may rapidly contract the volumes in phase space, making it difficult to obtain the nonintegral part of the capacity, which arises from the fractal structure.

Hence the box-counting algorithm is not useful for high dimensional and rapidly contracting attractors, like those encountered in the Lorenz model. But this method gives successful results in the case of low dimensional systems like 2/3 Cantor set and Henon map which have slow rate of phase space contraction.
We have seen that fractal dimension $D_0$ is difficult to evaluate for higher dimensional systems. To override this difficulty a new dimension called correlation dimension $D_2$ is introduced. It has been suggested that strange attractors can be characterized by $D_2$ (Grassberger and Procaccia 1983a,b,c, Mizrachi et al. 1984). The Grassberger-Procaccia algorithm (GP algorithm) to evaluate $D_2$ are efficiently converging even with small number of experimental points and even at higher dimensions (Atmanspacher et al. 1986). For example, this algorithm yields accurate value of $D_2$ for higher dimensional system, with as small as 500 data points (Abraham et al. 1986).

Correlation dimension $D_2$ is a probabilistic type of dimension. It can be calculated in terms of correlation integral

$$
C_d(\varepsilon) = \lim_{N \to \infty} \frac{1}{N^2} \sum_{i,j=1}^{N} \theta(\varepsilon - |\tilde{x}_i - \tilde{x}_j|)
$$

(2.26)

$$
= \int \int \int \cdots \int_0^\varepsilon \cdots \int_0^\varepsilon \, \, \, d^d \varepsilon'
$$

where $\theta(x)$ is the Heaviside function and $C_d(\varepsilon)$ is the standard correlation function in $d$ dimensional space. $\theta(x) = 0$ for $x \leq 0$ and unity for $x > 0$. $N^{-2}$ is a normalization factor and $|X_i - X_j|$ represents the Euclidean norm of $(X_i - X_j)$. Equation (2.26) gives the number of vector difference which are less than $\varepsilon$. This can also be considered as the number of vector tips which lie in a hyperbox whose volume is $\varepsilon^d$ in the phase space and in this sense, one can interpret equation (2.26) as a probability measure. The $C_d(\varepsilon)$ behaves as a power of $\varepsilon$ for small $\varepsilon$. 

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In most of the cases $D_2$ is very close to $D_0$, but is never greater than $D_0$. Now let us find, how $D_2$, $D_1$ and $D_0$ are related.

Let us suppose that the attractor has dimension $D_0$ and cover this by hypercube of side length $\varepsilon$. Each cube will have a volume $\varepsilon^D$ and the number of cubes required to cover the attractor is

$$M(\varepsilon) \sim \varepsilon^{-D_0} \quad (2.28)$$

We have to get a measure of $M(\varepsilon)$ from the correlation function (2.26). Since the equation (2.26) gives the number of points on the attractor, we have to connect $M(\varepsilon)$ with the number of points. Following Grassberger and Procaccia [1983a], let $\mu_i$ be the number of points from the set $\{X_i\}$ which are in the $i^{th}$ nonempty cube. We then have

$$C_d(\varepsilon) \propto \frac{1}{N^2} \sum_{i=1}^{M(\varepsilon)} \mu_i^2 = \frac{M(\varepsilon)}{N^2} \langle \mu_i^2 \rangle \quad (2.29)$$

where we have used the box counting. Using Schwartz inequality, in the asymptotic state

$$\lim_{\varepsilon \to 0} \frac{C_d(\varepsilon)}{d} \geq \frac{M(\varepsilon)}{N^2} \langle \mu_i^2 \rangle = \frac{1}{N^2} \frac{M(\varepsilon)}{M(\varepsilon)} \left[ \sum \mu_i \right]^2 = \frac{1}{M(\varepsilon)} \sim \varepsilon^{-D_0} \quad (2.30)$$
as we take the limit of $d \to \infty$, and used the fact that $\sum \mu \equiv N$

$$D_2 \leq D_0 \quad (2.31)$$

Now, to show that $D_1 \leq D_0$, we consider the following equations

$$D_1 = \lim_{\epsilon \to 0} \frac{S(\epsilon)}{\ln(1/\epsilon)} \quad (2.32)$$

where

$$S(\epsilon) = -\sum_{i=1}^{M(\epsilon)} p_i \ln p_i$$

where $p_i$ is the probability for a point to fall in the $i^{th}$ cube, as $N \to \infty$. We write,

$$p_i = \frac{\mu_i}{N} \quad (2.33)$$

and for uniform coverage

$$p_i = \frac{1}{M(\epsilon)} \quad (2.34)$$

The entropy is defined as

$$S^*(\epsilon) = \ln M(\epsilon) = \text{constant} - D_0 \ln \epsilon \quad (2.35)$$

$S^*(\epsilon)$ is the maximum information needed.

In general,

$$S(\epsilon) < S^*(\epsilon) \quad (2.36)$$
therefore

\[ S(\varepsilon) = S_0 - D \ln \varepsilon \]

This means that

\[ D_1 \leq D_0 \] (2.37)

By combining equations (2.31) & (2.37), and following the arguments presented by Grassberger and Procaccia (1983a), we get

\[ D_2 \leq D_1 \leq D_0 \] (2.38)

Thus \( D_2 \) is a significant quantity to characterize the strangeness of the attractor, it is a lower bound on the Hausdorff dimension and it is easy to calculate \( D_2 \) from time series. It has also been shown that \( D_2 \) is a very prominent one among the set of \( D \)'s (Caputo and Atten 1987). It has also been established that \( D_2 \) is integer for regular system, noninteger for chaotic system and \( D_2 \sim d \) for completely stochastic system (see Table 2.1) (Atmanspacher and Scheingraber 1986).

Correlation integral becomes independent of \( d \) as \( d \to \infty \) and for small values of \( \varepsilon \),

\[ C_d(\varepsilon) \sim \varepsilon^{D_2} \]

therefore

<table>
<thead>
<tr>
<th>Table 2.1 Dimensions and entropies of different systems</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dimension</td>
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<tr>
<td>-----------</td>
</tr>
<tr>
<td>Regular</td>
</tr>
<tr>
<td>Chaotic</td>
</tr>
<tr>
<td>Stochastic</td>
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46
\[ D_2 = \lim_{{\varepsilon \to 0}} \lim_{{d \to \infty}} \frac{\ln C_d(\varepsilon)}{\ln \varepsilon} \]  

(2.39)

\( C_d(\varepsilon) \) can be calculated using the equation (2.26).

### 2.11 GENERALIZED ENTROPY

Unpredictability of chaotic systems due to the exponential rate of divergence of trajectories on strange attractors lead to the creation of information. But in predictable systems trajectories do not create new information. This is an essential difference between chaotic and regular systems. Kolmogorov entropy provides a quantitative measure to classify regular and chaotic system, and is defined to be the mean rate of creation of information (Farmer 1982)

Estimation of Kolmogorov entropy \( K \) directly from time signal enable us to quantify, how chaotic the system is, or it will help us to study the information flow in the system using isentropy curves.

To evaluate Kolmogorov entropy, consider a dynamical system with \( F \) degrees of freedom. Suppose that the \( F \) dimensional space is partitioned into boxes of size \( \varepsilon^F \) and that there is an attractor in the phase space. The trajectory \( \dot{x}(t) \) is assumed to be in the basin of attraction. The state of the system is now measured at intervals of time \( \tau \). Let \( p(i_1, i_2, i_3, \ldots, i_d) \) be the joint probability that \( \dot{x}(t=\tau) \) is in box \( i_1 \), \( \dot{x}(t=2\tau) \) is in box \( i_2 \), and \( \dot{x}(t=d\tau) \) is in \( i_d \). The Kolmogorov entropy \( K \) is then

\[ K = \lim_{{\varepsilon \to 0}} \lim_{{\tau \to 0}} \lim_{{d \to \infty}} \frac{1}{\tau d} \sum_{i_1 \ldots i_d} p(i_1 \ldots i_d) \ln p(i_1 \ldots i_d) \]  

(2.40)
As is well known (Schuster 1984, Grassberger and Procaccia 1983c, Cohen et al 1985) \( K=0 \) for an ordered system, and \( K=\infty \) for stochastic system and \( K \) is a nonzero constant for a chaotic system.

Calculation of Kolmogorov entropy \( K \) is not difficult for analytically defined models in terms of evolution of the distance between two initially close points, but it is very difficult to determine \( K \) directly from a measured time signal (Grassberger and Procaccia 1983c).

2.12 KOLMOGOROV SECOND ENTROPY (\( K_2 \))

Grassberger and Procaccia (1983c) defined a new quantity, viz., Kolmogorov Second entropy \( K_2 \), which can be extracted easily from an experimental signal. \( K_2 \) is an invariant measure of the system, and has the following properties (see table 2.1).

\( K \geq 0 \)

\( K \leq K \)

\( K_2 \) is infinite for random system (Completely stochastic system).

\( K_2 \neq 0 \) for chaotic system

\( K_2 =0 \) for ordered system

For typical cases, \( K_2 \) is close to \( K \). \( K_2 \) is a member of the set of generalized entropies (order-\( q \) Renyi entropies) which is defined as

\[
K_{q, \infty} = -Lt \lim_{\tau \to 0} \lim_{d \to \infty} \frac{1}{\tau d} \frac{1}{q-1} \ln \sum_{i_1 \ldots i_d} p^q (i_1 \ldots i_d)
\]

(2.41)

\( q \) can take any real values between \(-\infty \) and \(+\infty \) and \( p(i_1, i_2, \ldots, i_d) \) is the joint probability that the trajectory visit the boxes
First order entropy,

\[ K_1 = \lim_{q \to 1} K_q, \]

is the metric entropy which is a measure of the internal information production of the system during its temporal evolution. On putting

\[ p^q = p \exp(q-1) \ln p \]

in equation (2.41), we obtain

\[ K_1 = \lim_{q \to 1} \lim_{\varepsilon \to 0} \lim_{\tau \to 0} \lim_{d \to \infty} \frac{1}{\tau d} \sum_{i_1 \ldots i_d} p(i_1 \ldots i_d) \ln p(i_1 \ldots i_d) \quad (2.42) \]

Thus \( K_1 = K_q \), the Kolmogorov entropy. Here \( K_1 \) is approximately identical with the sum of positive LE's of the system. From equation (2.20) & (2.21), \( K_q \) can be defined in terms of correlation integral (Pawelzik and Schuster 1987) as

\[ K_q = \lim_{q \to 0} \lim_{d \to \infty} \left[ -\frac{1}{c} \ln C_d(q) \right] \quad (2.43) \]

\( K_2 \) gives the lower bound for the Kolmogorov entropy. It can be defined in terms of correlation integral as in the case of \( D_2 \). \( K_2 \) is singled out from \( K_q \) due to its ease of calculation from a time series. We shall establish this as follows, consider the equation (2.41) for \( q=2 \) and for any value of \( d \), and let \( \varepsilon \) be
The equation for $C_d^2(\varepsilon)$ (Grassberger et al. 1983c) is

$$C_d^2(\varepsilon) = \sum p_i^2$$  \hspace{1cm} (2.44)

where $p_i$ is the probability to visit the $i^{th}$ box and sum $i$ runs over all the boxes in phase space which contain a piece of attractor. This quantity $C_d^2(\varepsilon)$ can be easily calculated from a given time series.

We have already shown (Equation 2.27) that $C_d(\varepsilon)$ scales like

$$C(\varepsilon) \sim \varepsilon^2$$

Hence this equation and (2.41) would yield

$$C_d(\varepsilon) \sim \varepsilon^2 \exp(-dK_2)$$ \hspace{1cm} (2.45)

Then

$$C_{d+1}(\varepsilon) \sim \varepsilon^2 \exp(-d+1)K_2$$

and $K_{z,d}(\varepsilon)$ is

$$K_{z,d}(\varepsilon) = \frac{1}{\tau} \ln \left\{ \frac{C_d(\varepsilon)}{C_{d+1}(\varepsilon)} \right\}$$ \hspace{1cm} (2.46)

If we plot $\log C_d(\varepsilon)$ vs $\log(\varepsilon)$ we will get a series of lines with a linear part of slope $D_{2,d'}$ and which are separated from each other by a factor $\exp(-dK_{z,d})$. The second Kolmogorov entropy $K_2$ is
2.13 EVALUATION OF $D_2$ AND $K_2$ FROM TIME SERIES - GP ALGORITHM

GP algorithm is an efficient method to evaluate $D_2$ and $K_2$ from an experimental data obtained as a time series.

Consider the time series

$$\hat{X} = \left\{ X(t_1), X(t_2), \ldots, X(t_N) \right\}$$

(2.48)

where $X(t_i)$ is the voltage or temperature or density distribution or any fluctuations measured at the instant $t_i$. We usually take the time interval between two consecutive readings at constant $\tau$, and this series is rearranged in the following matrix form

$$
\begin{array}{cccc}
  x(t) & x(t+\tau) & \ldots & x(t+m-1\tau) \\
  x(t+\tau) & x(t+2\tau) & \ldots & x(t+m\tau) \\
  \vdots & \vdots & \ddots & \vdots \\
  x(t+m-1\tau) & x(t+m\tau) & \ldots & x(t+m+d-2\tau)
\end{array}
$$

(2.49)

This forms a matrix of $m$ columns and $d$ rows and is called a delayed matrix (Broomhead & King 1986). The matrix (2.49) can be considered as $m$ vectors (columns) defined in a $d$-dimensional phase space and $m \geq d$. The matrix (2.49) can be represented by the following,

$$
\hat{X}(t_i) = \left\{ X_i(t_i), X_i(t_i+\tau), \ldots, X_i(t_i+d-1\tau) \right\}
$$

(2.50)
where \( t_i = t + (i-1)\tau \), \( i \) running from 1 to \( m \). Equation (2.50) represents the various vectors (column) and using these vectors, one can evaluate the correlation integral (equation 2.26),

\[
C_d(\epsilon) = \sum_{i,j=1}^{N} \Theta (\epsilon - |\hat{x}_i - \hat{x}_j|),
\]

by counting the number of points whose distance is less than a pre assigned value \( \epsilon \), where \( \epsilon \) varies from small value ( \( \sim 0 \), say 0.0012), to large value ( \( \sim 1 \) ). \( C_d(\epsilon) \) is calculated using (2.26) for various \( \epsilon \) and for each particular dimension \( d \) of the constructed phase space. The plot of \( \log C_d(\epsilon) \) vs \( \log (\epsilon) \) for each \( d \) (Figure 2.10) will have a linear region with a slope \( \nu \).

![Log-Log plot of C_d(\epsilon)](image)

\[
\nu = \frac{\ln C_d(\epsilon)}{\ln (\epsilon)}
\]

2.14 CALCULATION OF \( D_2 \)

The slope \( \nu \) of the linear part of \( \log C_d(\epsilon) - \log (\epsilon) \) plot for each dimension \( d \) is evaluated. The plot of \( \nu \) vs dimension \( d \) (Figure 2.11), saturates to a finite value as \( d \) increases, and the saturated value of \( \nu \) is the second order dimension or correlation dimension \( D_2 \). If the data set consists of completely random noise, then the points would lie on the
straight line with 45° to the d-axis (Babloyantz et al 1986). On the other hand, if there exists a deterministic component in the system, the curve would saturate to $D_2$ and would become independent of $d$. The dimension $d$ at which the $v$ curve starts to saturate or the region at which the deterministic and random parts completely separate is the embedding dimension (In figure 2.11 saturation starts at $d=12$).

2.15 CALCULATION OF $K_2$

The second quantity of great interest is the Kolmogorov Second entropy $K_2$. This can be measured using the correlation integral by evaluating the ratio of spatial separation between the curves in Figure (2.10) for dimension $d$ and $d+1$. The mean value of $C_d(\varepsilon)/C_{d+1}(\varepsilon)$ over the linear range of $\varepsilon$ is calculated for each dimension and we write
\[
K_{2,d} = \lim_{\varepsilon \to 0} \frac{L_t}{\tau \to 0} \frac{1}{1} \ln \left[ \frac{C_d(\varepsilon)}{C_{d+1}(\varepsilon)} \right]
\]

\(K_{2,d}\) is plotted against dimension \(d\), and the curve will saturate as shown in Figure (2.12). The saturated value of \(K_{2,d}\) as \(d \to \infty\) is the Second Kolmogorov entropy (Equation 2.46).

\[
\lim_{d \to \infty} K_{2,d} \quad \longrightarrow \quad K_2
\]

We can classify the systems by comparing the values of \(D_2\) and \(K_2\) with values in the table (2.1).

The algorithm we shall be using in the thesis is the one developed by Atmanspacher and Scheingraber [1986] which has been modified for smaller data sets by Abraham et al [1986]. But, before any numerical scheme is used for the purpose of analysis of any unknown system, it should be subjected to certain known system, so that an evaluation of the efficiency and accuracy of the numerical code could be done.

2.16 PERIODIC SYSTEM

To test our algorithm, we used the sine series. The correlation integral was calculated for \(d\) varying from 1 to 30 from 512 data extracted from the digitized values of ten successive periods of the sine function (\(\sin x\)). Then \(\log C_d(\varepsilon)\) is plotted against \(\log \varepsilon\). All the curves are parallel to each other with equal slopes. Slopes of the curves (\(\nu\)) were plotted against dimension \(d\) and it is seen that the \(\nu\) curve is parallel to \(d\)-axis (Figure 2.13), and \(D_2=1\). The behaviour of entropy also shows that the system is ordered (\(K_2 \approx 0\)) (Figure 2.14).
function represents a periodic system with a single frequency, so

Fig 2.13 Slope of the linear part of log-log curves $G_d(x)$ of sine series, plotted against dimension.

that $D=1$ is an expected result. Thus the Second order dimension represent the number of frequencies present in the system, or it measures the number of independent parameters required to define the system. Therefore our algorithm is in good agreement with what we are expecting.

Fig 2.14 $K_{z,d}$ vs $d$ of sine series
2.17 CALCULATION OF $D_2$ AND $K_2$ FROM SMALL DATA SETS

Box-counting algorithm is not practical for dimension calculation since it needs large computer memory and time and also requires large amount of data to obtain satisfactory results.

The OP algorithm is suitable for $D_2$ and $K_2$ evaluation, but there is a usual assumption that a large number of data is required for accurate results. This has been first investigated by Abraham and his colleagues [1986]. They evaluated the dimension of logistic equation using 500, 1200, 5000 data, and in all these cases $D_2$ was found to be more or less same ($D_2$ is 0.92 for $N=500$, $D_2=0.94$ for $N=1200$, $D_2=0.93$ for $N=5000$). In the case of Henon attractor they got $D_2=1.28$ for $N=500$, $D_2=1.20$ for $N=1200$, $D_2=1.24$ for $N=4000$ and $D_2=1.24$ for $N=10,000$. These results show that the OP algorithm leads to successful results in the case of small data sets. Abraham et al [1986] calculated the slope in a slightly different way. They calculated slope of the curve $\log C_d(\varepsilon)$ vs $\log(\varepsilon)$ for each $\varepsilon$ and plotted it against $\log C_d(\varepsilon)$. For small values of $\varepsilon$, slope fluctuates due to noise, and for larger values of $\varepsilon$ it displays an increase in the slopes as the largest interpoint distances on the attractor are reached, and then saturate to a constant value of $\log C_d$.

We investigated the data dependence of OP algorithm in Neural system also. For this, we used a digitized Electroencephalogram data. We calculate $D_2$ for 1357 and 227 data points and obtained the values of $D_2$ as 3.56 and 3.50 respectively.
The correlation dimension $D_2$ as mentioned earlier, is one of an infinite set of dimensions that characterize the strange attractor. It is singled out by the ease of the actual calculation from time series (Caputo and Atten 1987). $D_2$ is defined in terms of correlation integral (equation 2.26) using the power law given in equation (2.27).

Now the question arises about the role of noise present in the time series. The basic idea is that when we have a deterministic motion on a strange attractor, the existence of noise will not ruin the structure of the attractor, but will cause fuzziness on the length scales that are much smaller or equal to the noise strength (Shaw 1981, Zardecki 1982). But the quantification of strangeness using GP algorithm gives a clear demarcation between deterministic and random part of the signal.

According to Mizrachi et al [1984] if we embed the attractor in a $d$-dimensional space, we expect that the noisy trajectory will be space filling on length scales smaller than the noise strength and scales like

$$C(\varepsilon) \sim \varepsilon^d$$

The plot of $\log C(\varepsilon)$ as a function $\log(\varepsilon)$ will have a slope of $D_2$ down to length scales characterized by the noise strength and then a slope of $d$.

To find $D_2$ according to GP algorithm, we construct a phase space with dimension $d$ and plot $\log C(\varepsilon)$ vs $\log(\varepsilon)$. For each value of $\varepsilon$, we get a curve with a linear part with slope equal to $D_2d$ above the length scales characterizing the noise strength. All curves will break at that value of $\varepsilon$, below which the slope is equal to $d$. Mizrachi et al [1984] explained it in terms of Mackey-Glass equation, and showed that for a given parameter the
strange attractor is characterized by $D = 1.95$. The noise strength is $10^{-3}$.

Thus GP algorithm has the following advantages:

1) Suitable for small data sets
2) It characterizes attractors
3) It gives information on the noise level of the system, i.e., the position of break in $\log_2 C(\varepsilon)$ vs $\log_2 (\varepsilon)$ plot
4) It separates deterministic and random components present in the system.

2.19 SPECTRA OF SCALING INDICES ($f(\alpha)$ SPECTRA) FOR FRACTAL GEOMETRIES

The subset of phase space to which a typical orbit of a chaotic noncommensurate system asymptotes with time is called a strange attractor and can have fractal geometry. Fractal measures can provide a phenomenological description of strange attractors. In order to obtain a more complete characterization we should consider the structure of scaling indices or singularities on a fractal measure (Halsey et al. 1986b). We consider a function $f(\alpha)$ where $\alpha$ is the scaling index of the measure about a point on the fractal and $f(\alpha)$ is the dimension of the set of points on the fractal with same value of $\alpha$.

Suppose that we have a time series of $N$ points on a strange attractor in the phase space of a dynamical system. Typically, trajectories in chaotic dynamics do not fill the $d$-dimensional phase space even when $N \to \infty$, because the trajectory lies on a strange attractor of dimension $D$, $D < d$. Defining

$$ p_i = \lim_{N \to \infty} \frac{N_i}{N} $$

where $N_i$ is the number of times the time series visits the
ith box, we generate the measure on the attractor. If the system is divided into pieces of size $l$, and defining a scaling exponent $a$, we can write

$$P_i \sim l^{-a}$$  \hspace{1cm} (2.52)

$a$ can take on a range of values, corresponding to different regions of the measure (Halsey et al 1986a). The number of times $a$ takes on a value between $a'$ and $a'+da'$ will be of the form

$$da' \rho(a') l^{-f(a')}$$  \hspace{1cm} (2.53)

where $f(a')$ is a continuous function (Halsey et al 1986c). The exponent $f(a')$ reflects the differing dimensions of the sets with singularities of strength $a'$. Thus we model fractal measures by interwoven sets of singularities of strength $a$, each characterized by its own dimension $f(a)$.

We can relate $f(a)$ to a set of dimensions which have been introduced by Hentschel and Procaccia (1983), the set of $D_q$ defined by (see section 2.7)

$$D_q = \lim_{l \to 0} \left\{ \frac{1}{q-1} \frac{\ln \chi(q)}{\ln l} \right\}$$  \hspace{1cm} (2.54)

where $\chi(q) = \sum_i p_i^q$

$D$ is the fractal dimension, $D_i$ is the information dimension and $D_2$ is the correlation dimension (Grassberger and Procaccia 1984, Grebogi et al 1988). Substituting for $p_i$
\[ \chi(q) = \int d\alpha' \rho(\alpha') e^{-f(\alpha')} i^{q\alpha'} \]  

(2.55)

since \( l \) is small, the integral will be dominated by value of \( \alpha' \) which makes \( qa' - f(\alpha') \) smallest (Sato et al 1987)

Thus,

\[ \frac{d}{d\alpha'} [qa' - f(\alpha')] = 0 \]  

(2.56)

also

\[ \frac{d^2}{d(\alpha')^2} [qa' - f(\alpha')] > 0 \]

so that

\[ f'(\alpha(q)) = q \]  

(2.57a)

\[ f''(\alpha(q)) < 0 \]  

(2.57b)

It follows that (Halsey et al 1986b)

\[ D_q = \frac{1}{(q-1)} [qa(q) - f(\alpha(q))] \]  

(2.58)

Thus if we know the \( f(\alpha) \) spectrum, we can find \( D_q \). Alternatively, knowing \( D_q \) we can find \( \alpha(q) \) since
and hence $f(\alpha)$ can be evaluated from equation (2.58).

If a continuous scaling spectrum $f(\alpha)$ exists, then the above relationships implies that it must be convex. Furthermore, $f_{\text{max}}(\alpha)$ will be equal to the dimension $D_o$ of the attractor. The minimum scaling exponent $\alpha_{\text{min}}$ will correspond to the most concentrated region of measure on the attractor and $\alpha_{\text{max}}$ will correspond to the most rarefied region of the measure (Halsey et al 1986c).

$f(\alpha)$ spectrum can be measured experimentally and will result in new tests of scaling theories of nonlinear systems. We have evaluated $f(\alpha)$ spectra from EEG recording, results of which are included in later chapters.