Chapter 3

Inventory Models for Price-Sensitive Trapezoidal Demand with Net Credit
3.0 Introduction

In this chapter, integrated supplier-buyer inventory models are analyzed when market demand is price-sensitive trapezoidal and the supplier offers a choice between discount in unit price and delay in settlement of account against the purchases made. This type of trade credit is termed as ‘net credit’. ‘Net credit’ means if the buyer pays within offered time period $M_1$, then the buyer receives a cash discount; otherwise the full account must be made by the time $M_2$; where $M_2 > M_1 \geq 0$. The objective is to maximize the joint profit per unit time.

In this chapter, following four models are formulated.

*Model 3.1* Optimal shipments, ordering and payment policies for an integrated supplier-buyer inventory model with price-sensitive trapezoidal demand and net credit

*Model 3.2* Optimal transfers, ordering and payment policies for joint supplier-buyer inventory model with variable price-sensitive trapezoidal demand and net credit

*Model 3.3* Optimal shipments, ordering and payment policies for integrated supplier-buyer deteriorating inventory system with price-sensitive trapezoidal demand and net credit

*Model 3.4* Co-ordinated shipments, ordering and payment policies for deteriorating inventory of two players in supply chain with variable price-sensitive trapezoidal demand and net credit scenario
3.1 Optimal Shipments, Ordering and Payment Policies for an Integrated Supplier – Buyer Inventory Model with Trapezoidal Demand and Net Credit

The basic assumption of the classical EOQ model that the buyer settles the account immediately on the arrival of units in the warehouse does not hold in the business transactions. In this section, we discuss an integrated inventory policy for price-sensitive trapezoidal demand. It is assumed that the supplier offers a choice of a cash discount in purchase price if payment is made before a specified period; otherwise, the buyer has to settle the full payment before the due date of the trade credit. The optimal payment time, retail price, ordering policy and number of shipments from the supplier to the buyer are the decision variables to maximize the joint total profit.

3.1.1 Assumptions and Notations

The proposed problem is formulated using following assumptions and notations along with those given in A.1 and N.1.

3.1.1.1 Assumptions

The following assumptions are used in this chapter:

1. The supply chain consists of single supplier – single buyer.
2. Shortages are not allowed.
3. The market demand rate \( R(= R(P,t)) \) for the item is price-sensitive trapezoidal. So, as per assumption A.2, functional form of the demand
\[ R(P,t) = \begin{cases} f(t)P^{-\eta}, & 0 \leq t \leq u_1 \\ R_0P^{-\eta}, & u_1 \leq t \leq u_2 \\ g(t)P^{-\eta}, & u_2 \leq t \leq T \end{cases} \]

and hence, demand function can be written as

\[ R(P,t) = \begin{cases} R_1(P,t), & 0 \leq t \leq u_1 \\ R_2(P,t), & u_1 \leq t \leq u_2 \\ R_3(P,t), & u_2 \leq t \leq T \end{cases} \]

where

\[ R_1(P,t) = a(1 + b_1t)P^{-\eta} \]
\[ R_2(P,t) = a(1 + b_1u_1)P^{-\eta} \]
\[ R_3(P,t) = a(1 + b_1u_1)e^{-b_2(t-u_2)}P^{-\eta} \]

where \( \eta > 1 \) is a price-elasticity mark-up.

4. To increase cash inflow and reduce the risk of cash flow shortage, the supplier offers a discount \( \beta \ (0 < \beta < 1) \) off the purchase price, if the buyer settles the account within time \( M_1 \), otherwise, full payment against the purchases made is to be settled within offered credit period \( M_2 \), where \( M_2 > M_1 \geq 0 \). By offering a delay period to the buyer, the seller compromises with late cash inflow. As a result, the supplier incurs an opportunity cost during the delivery and payment of the product.

5. During the available credit period, the buyer generates the revenue by selling the product which is deposited in an interest earning account. At the end of this period, the supplier charges to the buyer on the unsold stock.

6. During time period \([M_1, M_2]\), a cash flexibility rate \( f_{sc} \) is used to quantize the advantage of early cash income for the supplier.
3.1.1.2 Notations

$TBP$ : Total buyer’s profit ($ per unit time)

$TSP$ : Total supplier’s profit ($ per unit time)

$\pi$ : $(= TSP + TBP)$ Joint profit of the supply chain ($ per unit time$)

3.1.2 Mathematical Model

The inventory changes due to trapezoidal demand for both the players of the supply chain. The rate of change of inventory at any instant of time $t$ is given by the differential equation

$$\frac{dI(t)}{dt} = -R(P,t), 0 \leq t \leq T$$

with the boundary condition $I(T) = 0$.

The solution of the differential equation is

$$I(t) = \begin{cases} I_1(t) = aP^{-\eta} \left(u_1 - t + \frac{1}{2}b_1 \left(u_1^2 - t^2\right)\right) + aP^{-\eta} \left(1 + b_1u_1\right)(u_2 - u_1) \\
+ \frac{aP^{-\eta}}{b_2} \left(1 + b_1u_1\right) \left(1 - e^{-b_2(T-u_2)}\right), & 0 \leq t \leq u_1 \\
I_2(t) = aP^{-\eta} \left(1 + b_1u_1\right)(u_2 - t) + \frac{aP^{-\eta}}{b_2} \left(1 + b_1u_1\right) \left(1 - e^{-b_2(T-u_2)}\right), & u_1 \leq t \leq u_2 \\
I_3(t) = \frac{aP^{-\eta}}{b_2} \left(1 + b_1u_1\right) \left(e^{-b_2(t-u_2)} - e^{-b_2(T-u_2)}\right), & u_2 \leq t \leq T \end{cases}$$

Initially, $Q$ units are procured. Equivalently,

$$Q = I(0) = aP^{-\eta} \left(u_1 + \frac{1}{2} b_1 u_1^2\right) + aP^{-\eta} \left(1 + b_1u_1\right)(u_2 - u_1) + \frac{aP^{-\eta}}{b_2} \left(1 + b_1u_1\right) \left(1 - e^{-b_2(T-u_2)}\right)$$

(3.1.1)

The buyer’s order quantity is $Q$ units in every transfer. So the supplier manufactures in the batches of size $nQ$ and saves set-up cost.
The supplier ships first \( Q \) units produced and thereafter, during every \( T \) time units, \( Q \) units are shipped until the supplier’s inventory depletes to zero.

**Supplier’s total profit per unit time**

During each production run, the supplier produces \( nQ \) units in batches and bears a batch set-up cost \( A_s \). The supplier’s set-up cost per unit time is \( A_s / (nT) \). Using Joglekar (1988), the supplier’s inventory holding cost per unit time is given by

\[
\frac{1}{T} \left[ C_s (I_s + I_{sp}) \left[ (n-1)(1-\gamma) + \gamma \right] \right] \int_0^T I(t) \, dt.
\]

For each unit of an item, the supplier charges \((1 - K_j \beta)\nu\), when buyer pays the payment at time \( M_j \); where \( j = 1, 2; K_1 = 1, K_2 = 0 \). Hence, for the offered trade credit, the opportunity cost per unit time is

\[
\frac{1}{T} \left(1 - K_j \beta\right)\nu I_{sp} M_j Q; \quad j = 1, 2; \quad K_1 = 1, K_2 = 0.
\]

When the buyer pays at time \( M_1 \), the supplier can use the revenue \((1 - \beta)\nu\) to reduce a cash flow crisis during time \( M_2 - M_1 \). This early payment incurs gain at the cash flexibility rate per unit time as

\[
\frac{1}{T} (1 - \beta)\nu f_{sc}(M_2 - M_1)Q.
\]

Hence, the supplier’s total profit per unit time is sales revenue minus total cost which comprises of the manufacturing cost, set-up cost, inventory holding cost and opportunity cost for permissible delay period and plus the interest earned on the early payment is given by

\[
TSP_j(n) = \frac{(1 - K_j \beta)\nu Q}{T} - \frac{C_s Q}{nT} - \frac{A_s}{T} \left[ C_s (I_s + I_{sp}) \left[ (n-1)(1-\gamma) + \gamma \right] \right] \int_0^T I(t) \, dt
\]

\[
- \frac{(1 - K_j \beta)\nu I_{sp} M_j Q}{T} + \frac{(1 - \beta)\nu f_{sc}(M_2 - M_1)Q}{T}; \quad j = 1, 2; \quad K_1 = 1, K_2 = 0.
\]
**Buyer’s total profit per unit time**

The ordering cost per unit time is \( \frac{A_b}{T} \) for each shipment of size \( Q \) units. The buyer’s purchase cost per unit time is \( \frac{(1-K_j \beta) v Q}{T} \) and inventory holding cost per unit time is \( \frac{1}{T} \left(1-K_j \beta \right) v I_b \int_0^T I(t) dt \); where \( j = 1,2; \ K_1 = 1, \ K_2 = 0 \).

The following two cases may arise depending on the choice of payment time of the buyer.

1. \( T < M_j \)
2. \( T \geq M_j; j = 1,2 \).

**Case 1: \( T < M_j \) (Figure 3.1.1)**

In this case, buyer sells of all units before the allowable delay period. So, the opportunity cost for the buyer is zero. The interest earned on the generated revenue at the rate \( I_{be} \) per unit time is given by

\[
\frac{1}{T} PI_{be} \left[ \int_0^T t \cdot R(P,t) dt + Q(M_j - T) \right]; j = 1,2.
\]

Hence, buyer’s total profit per unit time is

\[
TBP_{j1}(P,T) = \frac{PQ}{T} - \frac{(1-K_j \beta) v Q}{T} - \frac{A_b}{T} - \frac{1}{T} \left(1-K_j \beta \right) v I_b \int_0^T I(t) dt
+ \frac{1}{T} PI_{be} \left[ \int_0^T t \cdot R(P,t) dt + Q(M_j - T) \right]; j = 1,2; \ K_1 = 1, \ K_2 = 0
\]
Figure 3.1.1 Interest earned when 

\[ T < M_j; \ j = 1, 2 \]

**Case 2:** \( T \geq M_j; \ j = 1, 2 \) \( \text{(Figure 3.1.2)} \)

In this case, the buyer’s offered payment time ends on or before the cycle time. During \([0, M_j]; \ j = 1, 2\), the buyer earns interest at the rate \( I_{be} \) on the generated revenue. The interest earned per unit time is

\[
\frac{1}{T} PI_{be} \int_0^{M_j} t \cdot R(P, t) dt
\]

\[
= \frac{1}{T} PI_{be} \left\{ \int_0^{u_1} t \cdot R_1(P, t) dt + \int_{u_1}^{M_j} t \cdot R_2(P, t) dt \right\}, \ 0 < M_j < u_1
\]

\[
= \frac{1}{T} PI_{be} \left\{ \int_0^{u_1} t \cdot R_1(P, t) dt + \int_{u_1}^{u_2} t \cdot R_2(P, t) dt + \int_{u_2}^{M_j} t \cdot R_3(P, t) dt \right\}, \ u_1 < M_j < u_2; \ j = 1, 2
\]

\[
= \frac{1}{T} PI_{be} \left\{ \int_0^{u_1} t \cdot R_1(P, t) dt + \int_{u_1}^{u_2} t \cdot R_2(P, t) dt + \int_{u_2}^{M_j} t \cdot R_3(P, t) dt \right\}, \ u_2 < M_j < T
\]

During \([M_j, T]; \ j = 1, 2\), the supplier will charge the interest rate \( I_{bc} \) for unsold stock. Hence, the interest charged per unit time to the buyer is
\[
\frac{1}{T}(1-K_j\beta)vI_{bc} T M_j \int I(t)dt \\
= \left\{ \frac{1}{T}(1-K_j\beta)vI_{bc} \left\{ \begin{array}{l}
\frac{u_1}{M_j} \int I_1(t)dt + \frac{u_2}{u_1} \int I_2(t)dt + \frac{T}{u_2} \int I_3(t)dt \\
\end{array} \right\} , M_j < u_1 < T \\
\frac{1}{T}(1-K_j\beta)vI_{bc} \left\{ \begin{array}{l}
\frac{u_2}{M_j} \int I_2(t)dt + \frac{T}{u_2} \int I_3(t)dt \\
\end{array} \right\} , M_j < u_2 < T \\
\frac{1}{T}(1-K_j\beta)vI_{bc} \int I_3(t)dt , u_2 < M_j < T
\right\}
\]

; \ j=1,2; K_1 = 1, K_2 = 0

Therefore, total profit of buyer per unit time is

\[
TBP_{j2}(P,T) = \left\{ \begin{array}{l}
TBP_{j2}\left((P,T): 0 < M_j < u_1 \right) \\
TBP_{j2}\left((P,T): u_1 < M_j < u_2 \right) \\
TBP_{j2}\left((P,T): u_2 < M_j < T \right) ; \ j=1,2
\end{array} \right\}
\]

where

\[
TBP_{j2}\left((P,T): 0 < M_j < u_1 \right)
= \frac{PQ}{T} - \frac{(1-K_j\beta)vQ}{T} - \frac{A_b}{T} \left\{ \begin{array}{l}
\frac{1}{T}(1-K_j\beta)vI_{bc} \left\{ \begin{array}{l}
\frac{u_1}{M_j} \int I_1(t)dt + \frac{u_2}{u_1} \int I_2(t)dt + \frac{T}{u_2} \int I_3(t)dt \\
\end{array} \right\}
\end{array} \right\}
+ \frac{1}{T} \int_{0}^{M_j} p_{be} t \cdot R_1(P,t)dt - \frac{1}{T} \left(1-K_j\beta\right)vI_{b} \int_{0}^{T} I(t)dt , 0 < M_j < u_1
\]
The total profit per unit time for the buyer is

\[ TBP_{j2} \left( (P, T): u_1 < M_j < u_2 \right) = \frac{PQ}{T} - \frac{(1-K_j\beta)vQ}{T} - \frac{A_b}{T} \left( \frac{1}{T} - (1-K_j\beta)vl_{bc} \right) \left( \int_{u_1}^{u_2} I_2(t) \, dt + \frac{T}{u_2} \left( I_3(t) \, dt \right) \right) \]

\[ + \frac{1}{T} PI_{be} \left\{ \int_{0}^{u_1} t \cdot R_1(P,t) \, dt + \int_{u_1}^{u_2} t \cdot R_2(P,t) \, dt \right\} - \frac{1}{T} \left( 1 - K_j\beta \right)vl_{b} \int_{0}^{T} I(t) \, dt \]

\( , u_1 < M_j < u_2 \)

The supplier-buyer joint profit per unit time is given by

\[ TBP_j(P,T) = \begin{cases} TBP_{j1}(P,T) , & T < M_j \\ TBP_{j2}(P,T) , & T \geq M_j \end{cases} \]

**Joint total profit per unit time**

The supplier-buyer joint profit per unit time is given by

\[ \pi_j(n,P,T) = \begin{cases} \pi_{j1}(n,P,T) , & T < M_j \\ \pi_{j2}(n,P,T) , & T \geq M_j ; \ j = 1,2 \end{cases} \]

where

\[ \pi_{j1}(n,P,T) = TSP_j(n) + TBP_{j1}(P,T) \]

\[ \pi_{j2}(n,P,T) = TSP_j(n) + TBP_{j2}(P,T) ; \ j = 1,2 \]
The goal is to determine optimal values of discrete variable $n$ and continuous variables $P$ and $T$, which maximize $\pi_j(n,P,T)$, $j = 1, 2$. We adopt following procedure to optimize the objective function.

### 3.1.3 Computational Procedure

To maximize joint profit following steps are performed.

**Step 1:** Assign parametric values in proper units to all model parameters.

**Step 2:** Set $n = 1$.

**Step 3:** Solve $\frac{\partial \pi_j}{\partial P} = 0$ and $\frac{\partial \pi_j}{\partial T} = 0$; $j = 1, 2$ simultaneously for $P$ and $T$.

**Step 4:** Increase $n$ by 1.

**Step 5:** Continue steps 3 and 4 until, we get

$$\pi_j(n-1,P(n-1),T(n-1)) \leq \pi_j(n,P,T) \geq \pi_j(n+1,P(n+1),T(n+1)); j = 1, 2$$

**Step 6:** Stop.

Knowing the optimal solution $(n,P,T)$, the optimal procurement quantity $Q$ per shipment for the buyer can be computed from equation 3.1.1.

### 3.1.4 Numerical example

Consider the following numerical values for the model parameters.

$$\alpha = 90,000 \text{ units}, \ b_1 = 7\%, \ b_2 = 5\%, \ \eta = 1.25, \ u_1 = 15 \text{ days}, \ u_2 = 45 \text{ days},$$

$$\gamma = 0.9, \ C_s = \$ \ 2/\text{unit}, \ v = \$ \ 4.5/\text{unit}, \ A_s = \$ \ 1,000/\text{set-up}, \ A_b = \$ \ 300/\text{order},$$

$$I_s = 5\% /\text{unit/year}, \ I_b = 8\% /\text{unit/year}, \ I_{sp} = 9\% /\$ /\text{year}, \ I_{bc} = 16\% /\$ /\text{year},$$
\[ I_{be} = 12\% \quad dollar/year \quad and \quad f_{sc} = 17\% \quad dollar/year. \] The credit term considered is ‘3/10 net 30’ means if buyer pays by 10 days then he will be offered 3% discount in the purchase price otherwise buyer has to settle full account by 30 days.

From Table 3.1.1, it is observed that for 14-shipments, the buyer’s selling price is $10.87/unit and cycle time 103 days maximizing joint total profit of $38,411, supplier’s profit as $10,179 and that of buyer as $28,232. The optimum purchase quantity per order is 1,287 units. Optimal payment time is 10 days in 3/10 net 30 policy.

**Table 3.1.1 Optimal solution for various credit terms**

<table>
<thead>
<tr>
<th>M₁ (days)</th>
<th>M₂ (days)</th>
<th>n</th>
<th>P ($)</th>
<th>T(days)</th>
<th>Q (units)</th>
<th>Profit ($)</th>
<th>Profit (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>16</td>
<td>10.79</td>
<td>89</td>
<td>26820</td>
<td>10885</td>
</tr>
<tr>
<td>0</td>
<td>30</td>
<td>30</td>
<td>16</td>
<td>10.53</td>
<td>89</td>
<td>1153</td>
<td>26961</td>
</tr>
<tr>
<td>10</td>
<td>30</td>
<td>10</td>
<td>14</td>
<td>10.87</td>
<td>103</td>
<td>1287</td>
<td>28232</td>
</tr>
<tr>
<td>20</td>
<td>30</td>
<td>20</td>
<td>14</td>
<td>10.97</td>
<td>103</td>
<td>1278</td>
<td>28360</td>
</tr>
<tr>
<td>0</td>
<td>60</td>
<td>60</td>
<td>15</td>
<td>10.40</td>
<td>95</td>
<td>1262</td>
<td>27265</td>
</tr>
<tr>
<td>10</td>
<td>60</td>
<td>60</td>
<td>15</td>
<td>10.47</td>
<td>96</td>
<td>1258</td>
<td>28080</td>
</tr>
<tr>
<td>20</td>
<td>60</td>
<td>60</td>
<td>15</td>
<td>10.57</td>
<td>96</td>
<td>1248</td>
<td>28219</td>
</tr>
</tbody>
</table>

The concavity of total joint profit with respect to number of shipments \( n \) and retail sale price \( P \) are exhibited in figures 3.1.3 and 3.1.4. 3-D plot (figure 3.1.5) of total joint profit for \( n = 14 \) establishes the concavity of the total joint profit.
The performance of variations in allowable credit period; $M_1$ and $M_2$ on decision variables and total joint profit is concerted in Table 3.1.1. The profit gain is compared with benchmark of no credit period. The percentage profit gain is calculated as $\left[ \frac{\text{Profit with trade credit}}{\text{Profit without trade credit}} - 1 \right] \times 100\%$. 

Figure 3.1.3 Concavity of joint profit w. r. t. number of shipments $n$

Figure 3.1.4 Concavity of joint profit w. r. t. retail price $P$

Figure 3.1.5 Concavity of joint profit w. r. t. cycle time $T$ and retail price $P$
It is seen that profit gain is positive in joint decision establishing that players of the supply chain is benefited under two-level trade credits. It is also seen that buyer gets to be encouraged to pay earlier in net credit scenario of ‘3/10 net 30’ with maximum profit.

In Table 3.1.2, independent and joint decisions are compared. The optimal solutions under both the decisions suggests that the offer of trade credit decreases retail price of the buyer and attracts for purchase of larger order. The retail price of buyer is almost double in independent decision compared to joint strategy. Reverse is the scenario for purchase quantity. This lowers the joint profit in independent decision. It is also seen that the buyer’s profit decreases and that of the supplier increases, which encourages the buyer to be sole decision maker. Goyal (1976) gave concept of readjustment of profit for encouraging the players of the supply chain to opt for joint decision.

Readjust profit of buyer and supplier as follows:

Buyer’s profit = \[ \pi(n,P,T) \times \frac{TBP(P,T)}{[TBP(P,T)+TSP(n)]} \]

= \[ 38,411 \times \frac{32,242}{(32,242+3,916)} = 34,251 \]

Supplier’s profit = \[ \pi(n,P,T) \times \frac{TSP(n)}{[TBP(P,T)+TSP(n)]} \]

= \[ 38,411 \times \frac{3,916}{(32,242+3,916)} = 4,160 \]

The adjusted profit is written in the last row of Table 3.1.2.
Table 3.1.2 Optimal solutions for different strategies

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Credit Term</th>
<th>Optimal Payment Time (days)</th>
<th>n</th>
<th>P ($)</th>
<th>T (days)</th>
<th>R(P,T) (units)</th>
<th>Q (units)</th>
<th>Profit ($)</th>
</tr>
</thead>
<tbody>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Buyer</td>
</tr>
<tr>
<td>Independent</td>
<td>Cash on delivery</td>
<td>0</td>
<td>22</td>
<td>23.17</td>
<td>108</td>
<td>77</td>
<td>523</td>
<td>31222</td>
</tr>
<tr>
<td></td>
<td>Trade Credit 3/10 net 30</td>
<td>10</td>
<td>15</td>
<td>22.62</td>
<td>151</td>
<td>155</td>
<td>752</td>
<td>32242</td>
</tr>
<tr>
<td>Joint</td>
<td>Cash on delivery</td>
<td>0</td>
<td>16</td>
<td>10.79</td>
<td>89</td>
<td>138</td>
<td>1127</td>
<td>26820</td>
</tr>
<tr>
<td></td>
<td>Trade Credit 3/10 net 30</td>
<td>10</td>
<td>14</td>
<td>10.87</td>
<td>103</td>
<td>181</td>
<td>1287</td>
<td>28232</td>
</tr>
<tr>
<td>Adjusted</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>34251</td>
</tr>
</tbody>
</table>

The sensitivity analysis for modal parameters is carried out by changing parameter as -20%, -10%, 10% and 20% as shown in figure 3.1.6.

Figure 3.1.6 Sensitivity analysis for model parameters on joint profit

It is observed from figure 3.1.6 that the joint profit has positive impact due to the utilization factor and the constant scale demand. This suggests that the manufacturer should maintain the production and demand ratio near to 1. The
joint profit is very sensitive to the buyer’s ordering cost indicating that the buyer should save in the transportation by placing larger order. The joint profit is negatively related to the markup, the supplier’s production cost, the interest charged to the buyer and the supplier’s opportunity cost. The first and last parameters are not under control but supplier can reduce production cost by deploying latest machinery. The joint profit has positive impact when demand increases linearly with time. So both the players are advantageous when the demand is in increasing phase.

3.2 Optimal Transfers, Ordering and Payment Policies for Joint Supplier-Buyer Inventory Model with Price-Sensitive Trapezoidal Demand and Net Credit

In market, the retail price of the product hardly remains fixed. It keeps changing with time depending on demand pattern. So, in this section, the retail price of the inventory products is taken different in each phase of demand.

3.2.1 Assumptions and Notations

3.2.1.1 Assumptions

The section uses all assumptions of section 3.1 with the demand rate as follows.

1. The demand rate \( R(P_1, P_2, P_3, t) \) is variable price-sensitive trapezoidal and is defined as
Analysis of Inventory Policies in Supply Chain under Trapezoidal Demand

\[
R\left(P_1, P_2, P_3, t\right) = \begin{cases} 
  f(t)P_1^{-\eta}, & 0 \leq t < u_1 \\
  R_0P_2^{-\eta}, & u_1 \leq t < u_2 \\
  g(t)P_3^{-\eta}, & u_2 \leq t \leq T 
\end{cases}
\]

So the demand function is

\[
R\left(P_1, P_2, P_3, t\right) = \begin{cases} 
  R_1\left(P_1, t\right) = a(1 + b_1t)P_1^{-\eta}, & 0 \leq t < u_1 \\
  R_2\left(P_2, t\right) = a(1 + b_1u_1)P_2^{-\eta}, & u_1 \leq t < u_2 \\
  R_3\left(P_3, t\right) = a(1 + b_1u_1)P_3^{-\eta}e^{-b_2(t-u_2)}, & u_2 \leq t \leq T 
\end{cases}
\]

### 3.2.1.2 Notations

The section uses all notations of previous section with the buyer’s unit sale price as follows.

- **\(P_1\)**: Buyer’s unit sale price ($/unit) when demand is linearly increasing with time (a decision variable)
- **\(P_2\)**: Buyer’s unit sale price ($/unit) when demand is constant with time (a decision variable)
- **\(P_3\)**: Buyer’s unit sale price ($/unit) when demand is decreasing exponentially with time (a decision variable)

Note: \(P_1 < P_2 < P_3\) and \(P_i > (1 - \beta)v > C_s\) for \(i = 1, 2, 3\).

### 3.2.2 Mathematical Model

The rate of change of inventory at any instant of time \(t\) is given by the differential equation

\[
\frac{dI(t)}{dt} = -R\left(P_1, P_2, P_3, t\right); 0 \leq t \leq T \text{ with the boundary condition } I(T) = 0.
\]
The solution of the differential equation is

\[
I(t) = \begin{cases}
I_1(t) = aP_1^{-\eta} \left( u_1 - t + \frac{1}{2} b_1 \left( u_1^2 - t^2 \right) \right) + aP_2^{-\eta} (1 + b_1 u_1) (u_2 - u_1) \\
+ \frac{aP_3^{-\eta}}{b_2} (1 + b_1 u_1) \left( 1 - e^{-b_2(T-u_2)} \right) & , 0 \leq t \leq u_1 \\
I_2(t) = aP_2^{-\eta} (1 + b_1 u_1) (u_2 - t) + \frac{aP_3^{-\eta}}{b_2} (1 + b_1 u_1) \left( 1 - e^{-b_2(T-u_2)} \right) & , u_1 \leq t \leq u_2 \\
I_3(t) = \frac{aP_3^{-\eta}}{b_2} (1 + b_1 u_1) \left( e^{-b_2(t-u_2)} - e^{-b_2(T-u_2)} \right) & , u_2 \leq t \leq T
\end{cases}
\]

Initial procurement quantity is

\[Q = I(0) = aP_1^{-\eta} \left( u_1 + \frac{1}{2} b_1 u_1^2 \right) + aP_2^{-\eta} (1 + b_1 u_1) (u_2 - u_1) + \frac{aP_3^{-\eta}}{b_2} (1 + b_1 u_1) \left( 1 - e^{-b_2(T-u_2)} \right)\]

(3.2.1)

Here, while calculating the joint total profit, only the buyer’s interest earned for both cases (i) when \( T < M_j; \ j = 1,2 \) and (ii) when \( T \geq M_j; \ j = 1,2 \) are different than section 3.1 and can be calculated as follows.

When \( T < M_j; \ j = 1,2 \), the interest earned on the generated revenue at the rate \( I_{be} \) per unit time is given by

\[
\frac{P_1 I_{be}}{T} \int_0^{u_1} t \cdot R_1 \left( P_1, t \right) dt + \frac{P_2 I_{be}}{T} \int_{u_1}^{u_2} t \cdot R_2 \left( P_2, t \right) dt + \frac{P_3 I_{be}}{T} \int_{u_2}^{T} t \cdot R_3 \left( P_3, t \right) dt .
\]
Hence, the buyer’s total profit per unit time is

\[
TBP_{j1}(P_1, P_2, P_3, T) = \frac{1}{T} \left[ \int_0^{u_1} P_1 \int_0^{M_j} t \cdot R_1(P_1, t) \, dt + \int_0^{u_2} P_2 \int_0^{M_j} t \cdot R_2(P_2, t) \, dt + \int_0^{u_2} P_3 \int_0^{M_j} t \cdot R_3(P_3, t) \, dt \right]
- \frac{(1 - K_j \beta) vQ}{T} - \frac{A_b}{T} \int_0^{T} I(t) \, dt
+ \frac{I_{be}}{T} \left[ \int_0^{u_1} P_1 \int_0^{M_j} t \cdot R_1(P_1, t) \, dt + \int_0^{u_2} P_2 \int_0^{M_j} t \cdot R_2(P_2, t) \, dt + \int_0^{T} P_3 \int_0^{M_j} t \cdot R_3(P_3, t) \, dt \right]
+ \frac{I_{be}}{T} \left[ \int_0^{u_1} P_1 \int_0^{T} t \cdot R_1(P_1, t) \, dt + \int_0^{u_2} P_2 \int_0^{T} t \cdot R_2(P_2, t) \, dt + \int_0^{T} P_3 \int_0^{T} t \cdot R_3(P_3, t) \, dt \right]

; j = 1, 2; K_1 = 1, K_2 = 0

When \( T \geq M_j \); \( j = 1, 2 \), the interest earned per unit time is

\[
\left\{ \begin{array}{ll}
\frac{1}{T} P_{1,be} \int_0^{M_j} t \cdot R_1(P_1, t) \, dt & , 0 \leq M_j < u_1 \\
\frac{1}{T} P_{1,be} \left[ \int_0^{u_1} t \cdot R_1(P_1, t) \, dt + \int_0^{M_j} t \cdot R_2(P_2, t) \, dt \right] & , u_1 \leq M_j < u_2 \\
\frac{1}{T} P_{1,be} \left[ \int_0^{u_1} t \cdot R_1(P_1, t) \, dt + \int_0^{u_2} t \cdot R_2(P_2, t) \, dt + \int_0^{M_j} t \cdot R_3(P_3, t) \, dt \right] & , u_2 \leq M_j \leq T \\
\end{array} \right.
; j = 1, 2

Therefore, the total profit of the buyer per unit time is

\[
TBP_{j2}(P_1, P_2, P_3, T) = \left\{ \begin{array}{ll}
TBP_{j2}\left((P_1, P_2, P_3, T) : 0 < M_j < u_1 \right) \\
TBP_{j2}\left((P_1, P_2, P_3, T) : u_1 \leq M_j < u_2 \right) \\
TBP_{j2}\left((P_1, P_2, P_3, T) : u_2 \leq M_j < T \right) & ; j = 1, 2
\end{array} \right.
\]
where; $TBP_{j2} \left((P_1, P_2, P_3, T): \ 0 < M_j < u_1 \right)$

$$= SR - \frac{(1-K_j\beta)vQ}{T} - \frac{A_b}{T} - \frac{(1-K_j\beta)vl_{b}}{T} \left[ I(t)dt \right]_{0}^{T}$$

$$- \frac{1}{T} (1-K_j\beta)vl_{bc} \left\{ \frac{u_1}{M_j} \int I_1(t)dt + \frac{u_2}{u_1} \int I_2(t)dt + \frac{T}{u_2} \int I_3(t)dt \right\}$$

$$+ \frac{1}{T} P_1 |_{0}^{M_j} t \cdot R_1(P_1, t) dt$$

$\quad , \ 0 < M_j < u_1$

$TBP_{j2} \left((P_1, P_2, P_3, T): \ u_1 < M_j < u_2 \right)$

$$= SR - \frac{(1-K_j\beta)vQ}{T} - \frac{A_b}{T} - \frac{(1-K_j\beta)vl_{b}}{T} \left[ I(t)dt \right]_{0}^{T}$$

$$- \frac{1}{T} (1-K_j\beta)vl_{bc} \left\{ \frac{u_2}{M_j} \int I_2(t)dt + \frac{T}{u_2} \int I_3(t)dt \right\}$$

$$+ \frac{1}{T} |_{0}^{M_j} P_1 t \cdot R_1(P_1, t) dt + \frac{u_1}{u_2} \int t \cdot R_2(P_2, t) dt$$

$\quad , \ u_1 \leq M_j < u_2$

$TBP_{j2} \left((P_1, P_2, P_3, T): \ u_2 < M_j < T \right)$

$$= SR - \frac{(1-K_j\beta)vQ}{T} - \frac{A_b}{T} - \frac{(1-K_j\beta)vl_{b}}{T} \left[ I(t)dt \right]_{0}^{T}$$

$$- \frac{1}{T} (1-K_j\beta)vl_{bc} \left\{ \frac{T}{M_j} \int I_3(t)dt \right\}$$

$$+ \frac{1}{T} |_{0}^{M_j} P_1 t \cdot R_1(P_1, t) dt + \frac{u_1}{u_2} \int t \cdot R_2(P_2, t) dt + \frac{u_2}{u_3} \int t \cdot R_2(P_3, t) dt$$

$\quad , \ u_2 \leq M_j < T$

where

sales revenue $SR = \frac{1}{T} \left[ P_1 |_{0}^{u_1} R_1(P_1, t) dt + \frac{u_2}{u_1} \int R_2(P_2, t) dt + \frac{T}{u_2} \int R_3(P_3, t) dt \right]$. 

The total profit of the buyer per unit time is

$$TBP_j(P_1, P_2, P_3, T) = \begin{cases} TBP_{j1}(P_1, P_2, P_3, T), & T < M_j \\ TBP_{j2}(P_1, P_2, P_3, T), & T \geq M_j \\ \end{cases}$$
Joint total profit per unit time

The supplier-buyer joint profit per unit time is given by

\[
\pi_j(n, P_1, P_2, P_3, T) = \begin{cases} 
\pi_{j1}(n, P_1, P_2, P_3, T), & T < M_j \\
\pi_{j2}(n, P_1, P_2, P_3, T), & T \geq M_j, j = 1, 2
\end{cases}
\]

where

\[
\pi_{j1}(n, P_1, P_2, P_3, T) = TSP_j(n) + TBP_{j1}(P_1, P_2, P_3, T) \\
\pi_{j2}(n, P_1, P_2, P_3, T) = TSP_j(n) + TBP_{j2}(P_1, P_2, P_3, T); j = 1, 2
\]

The total joint profit per unit time is function of discrete variable \( n \) and continuous variables \( P_1, P_2, P_3 \) and \( T \). We use following procedure to maximize the total joint profit per unit time.

3.2.3 Computational Procedure

Step 1: Assign parametric values in proper units to all the model parameters.

Step 2: Set \( n = 1 \).

Step 3: Solve \( \frac{\partial \pi_j}{\partial P_i} = 0 \) and \( \frac{\partial \pi_j}{\partial T} = 0 \), \( i = 1, 2, 3; j = 1, 2 \) simultaneously for \( T \) and \( P_i, i = 1, 2, 3 \).

Step 4: Increase \( n \) by 1.

Step 5: Continue steps 3 and 4 until, we get

\[
\pi_j(n-1, P_1(n-1), P_2(n-1), P_3(n-1), T(n-1)) \leq \pi_j(n, P_1, P_2, P_3, T) \\
\geq \pi_j(n+1, P_1(n+1), P_2(n+1), P_3(n+1), T(n+1)); j = 1, 2
\]

Step 6: Stop.

Knowing the optimal solution \( (n, P_1, P_2, P_3, T) \), the optimal procurement quantity \( Q \) per shipment for the buyer can be calculated from equation 3.2.1.
3.2.4 Numerical Example

With the data of section 3.1.4; 14-shipments, the retail prices for the buyer $P_1 = $ 10.03/unit, $P_2 = $ 10.35/unit and $P_3 = $ 11.37/unit in different demand structures and the cycle time for the buyer of 104 days maximize the joint total profit of $38,428. The corresponding order quantity is 1,304 units. For the obtained optimal solution, supplier’s profit is $10,215 and buyer’s profit is $28,213. The concavity of total joint profit with respect to number of shipments $n$ and retail sale prices $P_1$, $P_2$ and $P_3$ are shown in figures 3.2.1 - 3.2.4. 3-D plots (figures 3.2.5 – 3.2.7) of total joint profit for 14-shipments establish the concavity of the total joint profit with respect to $(P_1,T)$, $(P_2,T)$ and $(P_3,T)$. The variations in allowable credit periods and its effect on decision variables and total joint profit are tabulated in Table 3.2.1.

Table 3.2.1 Optimal solution for various credit terms

<table>
<thead>
<tr>
<th>$M_1$ (days)</th>
<th>$M_2$ (days)</th>
<th>Optimal Payment Time (days)</th>
<th>$n$</th>
<th>$P_1$ ($)</th>
<th>$P_2$ ($)</th>
<th>$P_3$ ($)</th>
<th>$T$ (days)</th>
<th>$Q$ (units)</th>
<th>Buyer Profit ($)</th>
<th>Supplier Profit ($)</th>
<th>Joint Profit ($)</th>
<th>Buyer Profit (%)</th>
<th>Supplier Profit (%)</th>
<th>Joint Profit (%)</th>
<th>$R(P,T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>16</td>
<td>10.39</td>
<td>10.63</td>
<td>11.03</td>
<td>89</td>
<td>1130</td>
<td>26809</td>
<td>10900</td>
<td>37709</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>136</td>
</tr>
<tr>
<td>0</td>
<td>30</td>
<td>30</td>
<td>16</td>
<td>10.13</td>
<td>10.37</td>
<td>10.77</td>
<td>89</td>
<td>1156</td>
<td>26949</td>
<td>11391</td>
<td>38340</td>
<td>0.52</td>
<td>4.31</td>
<td>1.65</td>
<td>138</td>
</tr>
<tr>
<td>10</td>
<td>30</td>
<td>10</td>
<td>14</td>
<td>10.03</td>
<td>10.35</td>
<td>11.37</td>
<td>104</td>
<td>1304</td>
<td>28213</td>
<td>10215</td>
<td>38428</td>
<td>4.98</td>
<td>-6.71</td>
<td>1.87</td>
<td>179</td>
</tr>
<tr>
<td>20</td>
<td>30</td>
<td>20</td>
<td>14</td>
<td>10.13</td>
<td>10.45</td>
<td>11.48</td>
<td>105</td>
<td>1294</td>
<td>28341</td>
<td>9994</td>
<td>38335</td>
<td>5.41</td>
<td>-9.07</td>
<td>1.63</td>
<td>179</td>
</tr>
<tr>
<td>0</td>
<td>60</td>
<td>60</td>
<td>15</td>
<td>9.78</td>
<td>10.02</td>
<td>10.74</td>
<td>96</td>
<td>1275</td>
<td>27254</td>
<td>11768</td>
<td>39022</td>
<td>1.63</td>
<td>7.38</td>
<td>3.36</td>
<td>163</td>
</tr>
<tr>
<td>10</td>
<td>60</td>
<td>60</td>
<td>15</td>
<td>9.89</td>
<td>10.12</td>
<td>10.85</td>
<td>97</td>
<td>1271</td>
<td>28071</td>
<td>10853</td>
<td>38924</td>
<td>4.50</td>
<td>-0.43</td>
<td>3.12</td>
<td>164</td>
</tr>
<tr>
<td>20</td>
<td>60</td>
<td>60</td>
<td>15</td>
<td>9.99</td>
<td>10.22</td>
<td>10.95</td>
<td>97</td>
<td>1261</td>
<td>28209</td>
<td>10617</td>
<td>38827</td>
<td>4.96</td>
<td>-2.67</td>
<td>2.88</td>
<td>163</td>
</tr>
</tbody>
</table>

The positive profit gain suggests that the joint decision is beneficial.
Figure 3.2.1 Concavity of joint profit w. r. t. number of shipments $n$

Figure 3.2.2 Concavity of joint profit w. r. t. retail price $P_1$

Figure 3.2.3 Concavity of joint profit w. r. t. retail price $P_2$

Figure 3.2.4 Concavity of joint profit w. r. t. retail price $P_3$
In Table 3.2.2, independent and joint decisions are compared. Reallocation of the profit is as follows:
Buyer’s profit = $\pi(n,P_1,P_2,P_3,T) \times \frac{TBP(P_1,P_2,P_3,T)}{TBP(P_1,P_2,P_3,T) + TSP(n)}$

= $38,428 \times \frac{32,245}{(32,245 + 3,921)} = 34,262$

Supplier’s profit = $\pi(n,P_1,P_2,P_3,T) \times \frac{TSP(n)}{TBP(P_1,P_2,P_3,T) + TSP(n)}$

= $38,428 \times \frac{3,921}{(32,245 + 3,921)} = 4,166$

Table 3.2.2 Optimal solutions for different strategies

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Credit Term</th>
<th>Optimal Payment Time (days)</th>
<th>n</th>
<th>$P_1$ ($)</th>
<th>$P_2$ ($)</th>
<th>$P_3$ ($)</th>
<th>T (days)</th>
<th>$R(P,T)$ (units)</th>
<th>Q (units)</th>
<th>Buyer ($$$)</th>
<th>Supplier ($$$)</th>
<th>Joint ($$$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Independent</td>
<td>Cash on delivery</td>
<td>0</td>
<td>21</td>
<td>23.31</td>
<td>23.24</td>
<td>23.11</td>
<td>107</td>
<td>77</td>
<td>523</td>
<td>31222</td>
<td>4051</td>
<td>35273</td>
</tr>
<tr>
<td></td>
<td>Trade Credit 3/10 net 30</td>
<td>10</td>
<td>15</td>
<td>21.81</td>
<td>21.92</td>
<td>22.94</td>
<td>151</td>
<td>154</td>
<td>754</td>
<td>32245</td>
<td>3921</td>
<td>36166</td>
</tr>
<tr>
<td>Joint</td>
<td>Cash on delivery</td>
<td>0</td>
<td>16</td>
<td>10.39</td>
<td>10.63</td>
<td>11.03</td>
<td>89</td>
<td>136</td>
<td>1130</td>
<td>26809</td>
<td>10900</td>
<td>37709</td>
</tr>
<tr>
<td></td>
<td>Trade Credit 3/10 net 30</td>
<td>10</td>
<td>14</td>
<td>10.03</td>
<td>10.35</td>
<td>11.37</td>
<td>89</td>
<td>179</td>
<td>1304</td>
<td>28213</td>
<td>10215</td>
<td>38428</td>
</tr>
</tbody>
</table>

The reallocated profits are given in the bottom line of Table 3.2.2.

The effect of changes in inventory parameters on joint total profit of the supply chain is given in figure 3.2.8. The parameters are varied as -20%, -10%, 10%, and 20%. 

Figure 3.2.8 Sensitivity analysis of inventory parameters on joint total profit

The impact of variations in inventory parameters on joint profit is same as observed in section 3.1.4.

3.3 Optimal Shipments, Ordering and Payment Policies for Integrated Supplier-Buyer Deteriorating Inventory System with Price-Sensitive Trapezoidal Demand and Net Credit

The present inventory model is an extension of model discussed in section 3.1 with the assumption that inventory products get deteriorated at constant rate.

3.3.1 Assumptions

The present section uses following additional assumption with those in section 3.1.

1. The units in the inventory system of both the player deteriorate at a constant rate $\theta$ ($0 < \theta < 1$). The deteriorated units can neither be repaired nor replaced during the period under review.
3.3.2 Mathematical Model

The rate of change of inventory at any instant of time \( t \) is governed by the differential equation

\[
\frac{dI(t)}{dt} = -R(P, t) - \theta I(t); 0 \leq t \leq T \quad \text{with the boundary condition } I(T) = 0.
\]

The solution of the differential equation is

\[
I(t) = \begin{cases} 
I_1(t), & 0 \leq t \leq u_1 \\
I_2(t), & u_1 \leq t \leq u_2 \\
I_3(t), & u_2 \leq t \leq T
\end{cases}
\]

where

\[
I_1(t) = \frac{aP^{-\eta} \left[ -\frac{1+b_1 t}{\theta} + \frac{b_1}{\theta^2} + \frac{1+b_1 u_1}{\theta} e^{\theta u_1 - \theta t} - \frac{b_1}{\theta^2} e^{\theta u_1 - \theta t} \right]}{1+e^{\theta u_2 - \theta t}}
\]

\[
I_2(t) = \frac{aP^{-\eta}(1+b_1 u_1)}{\theta} \left[ -1 + e^{\theta u_2 - \theta t} \right]
+ \frac{aP^{-\eta} \left[ \frac{a(1+b_1 u_1) e^{b_2 u_2}}{\theta - b_2} \right]}{1+e^{\theta u_2 - \theta t}} \left[ e^{-b_2 T + \theta T - \theta t} - e^{-b_2 u_2 + \theta u_2 - \theta t} \right]
\]

\[
I_3(t) = P^{-\eta} \left[ \frac{a(1+b_1 u_1) e^{b_2 u_2}}{\theta - b_2} \right] \left[ e^{-b_2 T + \theta T - \theta t} - e^{-b_2 t} \right]
\]
Using $I(0) = Q$, we get,

$$Q = \frac{e^{\theta u_2} - e^{\theta u_1}}{\theta - b_2}$$

Following the same procedure as discussed in section 3.1, the optimal solution $(n,P,T)$ and hence maximum value of $\pi_j(n,P,T)$, $j=1,2$ can be computed.

### 3.3.3 Numerical example

Consider the following numerical values for the model parameters.

$a = 1,00,000$ units, $b_1 = 7\%$, $b_2 = 5\%$, $\eta = 1.25$, $u_1 = 15$ days, $u_2 = 45$ days, $\gamma = 0.9$, $C_s = $ 2/unit, $v = $ 4.5/unit, $A_s = $ 1,000/set-up, $A_b = $ 300/order, $I_s = 5\%$/unit/year, $I_b = 8\%$/unit/year, $I_{sp} = 9\%$/$/year, $I_{bc} = 16\%$/$/year$, $I_{be} = 12\%$/$/year$ and $f_{sc} = 17\%$/$/year$ and $\theta = 12\%$. The credit term considered is ‘3/10 net 30’.

From Table 3.3.1, we see that for 10-shipments, the buyer’s selling price is $6.59/unit and cycle time is 122 days maximizing joint total profit of $25,319 of the integrated system. The corresponding profit of the supplier is $13,507 and that of buyer is $11,812. Each transfer is of 2,018 units. Optimal payment time is 10 days in ‘3/10 net 30’ credit terms. The concavity of joint
total profit with respect to number of shipments $n$ and retail sale price $P$ are shown in figures 3.3.1 and 3.3.2 respectively.

Figure 3.3.1 Concavity of joint profit w. r. t. number of shipments $n$

Figure 3.3.2 Concavity of joint profit w. r. t. retail price $P$

Figure 3.3.3 Concavity of joint profit w. r. t. cycle time $T$ and retail price $P$

3-D plot given in figure 3.3.3 for $n = 10$ establishes the convexity of the total joint profit. The variations in permissible delay periods; $M_1$ and $M_2$ are
worked out to study the changes in decision variable and total joint profit in Table 3.3.1.

Table 3.3.1 Optimal solution for various credit terms

<table>
<thead>
<tr>
<th>$M_1$ (days)</th>
<th>$M_2$ (days)</th>
<th>Optimal Payment Time (days)</th>
<th>n</th>
<th>P ($)</th>
<th>T (days)</th>
<th>Q (units)</th>
<th>Buyer ($)</th>
<th>Supplier ($)</th>
<th>Joint ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>11</td>
<td>6.53</td>
<td>112</td>
<td>1878</td>
<td>10336</td>
<td>14463</td>
<td>24800</td>
</tr>
<tr>
<td>0</td>
<td>30</td>
<td>30</td>
<td>11</td>
<td>6.37</td>
<td>111</td>
<td>1925</td>
<td>10130</td>
<td>15149</td>
<td>25279</td>
</tr>
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<td>30</td>
<td>10</td>
<td>10</td>
<td>6.59</td>
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<td>11812</td>
<td>13507</td>
<td>25319</td>
</tr>
<tr>
<td>20</td>
<td>30</td>
<td>20</td>
<td>10</td>
<td>6.66</td>
<td>123</td>
<td>2005</td>
<td>12021</td>
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<td>11</td>
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<td>10152</td>
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<td>25790</td>
</tr>
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<td>60</td>
<td>11</td>
<td>6.41</td>
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<td>1997</td>
<td>11597</td>
<td>14266</td>
<td>25863</td>
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<td>20</td>
<td>60</td>
<td>60</td>
<td>11</td>
<td>6.41</td>
<td>116</td>
<td>1995</td>
<td>11595</td>
<td>14140</td>
<td>25735</td>
</tr>
</tbody>
</table>

The positive profit gain proves that the players of the supply chain are advantageous under the two-level trade credit policy.

In Table 3.3.2, independent and joint decisions are compared under different credit terms. Reallocated profits of the buyer and the supplier are shown in last row of Table 3.3.2.

Table 3.3.2 Optimal solutions for different strategies

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Credit Term</th>
<th>Optimal Payment Time (days)</th>
<th>n</th>
<th>P ($)</th>
<th>T (days)</th>
<th>R(P,T) (units)</th>
<th>Q (units)</th>
<th>Buyer ($)</th>
<th>Supplier ($)</th>
<th>Joint ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Independent</td>
<td>Cash on delivery</td>
<td>0</td>
<td>13</td>
<td>14.15</td>
<td>168</td>
<td>198</td>
<td>886</td>
<td>16991</td>
<td>4388</td>
<td>21379</td>
</tr>
<tr>
<td></td>
<td>Trade Credit 3/10 net 30</td>
<td>10</td>
<td>13</td>
<td>13.72</td>
<td>167</td>
<td>206</td>
<td>925</td>
<td>17534</td>
<td>4348</td>
<td>21882</td>
</tr>
<tr>
<td>Joint</td>
<td>Cash on delivery</td>
<td>0</td>
<td>11</td>
<td>6.53</td>
<td>112</td>
<td>283</td>
<td>1878</td>
<td>10336</td>
<td>14463</td>
<td>24800</td>
</tr>
<tr>
<td></td>
<td>Trade Credit 3/10 net 30</td>
<td>10</td>
<td>10</td>
<td>6.59</td>
<td>122</td>
<td>330</td>
<td>2018</td>
<td>11812</td>
<td>13507</td>
<td>25319</td>
</tr>
<tr>
<td>Adjusted</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>20288</td>
<td>5031</td>
<td>25319</td>
</tr>
</tbody>
</table>
The sensitivity analysis for the model parameters is carried out by changing parameter as -20%, -10%, 10% and 20% as shown in figure 3.3.4.

Figure 3.3.4 Sensitivity analysis for model parameters on joint profit

It is observed from figure 3.3.4 that increase in the deterioration rate of inventory units decreases the joint profit. However, the supplier can reduce deterioration rate by using latest storage facilities.

3.4 Co-ordinated Shipments, Ordering and Payment Policies for Deteriorating Inventory of Two Players in Supply Chain with Variable Price-Sensitive Trapezoidal Demand and Net Credit Scenario

The present inventory model is an extension of model discussed in section 3.2 with the assumption that inventory products get deteriorated at constant rate.
3.4.1 Assumptions

The present section uses following additional assumptions with those in section 3.2.

1. The units in the inventory system of both the player deteriorate at a constant rate $\theta \ (0 < \theta < 1)$. The deteriorated units can neither be repaired nor replaced during the period under review.

3.4.2 Mathematical Model

The rate of change of inventory is due to variable price-sensitive trapezoidal demand and deterioration of units. It is governed by the differential equation

$$\frac{dI(t)}{dt} = -R(P_1, P_2, P_3, t) - \theta I(t); 0 \leq t \leq T \text{ with the boundary condition } I(T) = 0.$$ 

The solution of the differential equation is

$$I(t) = \begin{cases} I_1(t), & 0 \leq t \leq u_1 \\ I_2(t), & u_1 \leq t \leq u_2 \\ I_3(t), & u_2 \leq t \leq T \end{cases}$$

where

$$I_1(t) = aP_1^{-\eta} \left[ \frac{1 + b_1 t}{\theta} + \frac{b_1}{\theta^2} + \frac{1 + b_1 u_1}{\theta} e^{\theta u_1 - \theta t} - \frac{b_1}{\theta^2} e^{\theta u_1 - \theta t} \right]$$

$$+ P_2^{-\eta} \frac{a(1+b_1 u_1)}{\theta} \left[ e^{\theta u_2 - \theta t} - e^{\theta u_1 - \theta t} \right]$$

$$+ P_3^{-\eta} \frac{a(1+b_1 u_1)e^{b_2 u_2}}{\theta - b_2} \left[ e^{-b_2 T + \theta T - \theta T} - e^{-b_2 u_2 + \theta u_2 - \theta T} \right]$$

$$I_2(t) = aP_2^{-\eta}(1+b_1 u_1) \left[ -1 + e^{\theta u_2 - \theta t} \right]$$

$$+ P_3^{-\eta} \frac{a(1+b_1 u_1)e^{b_2 u_2}}{\theta - b_2} \left[ e^{-b_2 T + \theta T - \theta T} - e^{-b_2 u_2 + \theta u_2 - \theta T} \right]$$
\[ I_3(t) = P_3^{-\eta} \left[ \frac{a(1 + b_1 u_1) e^{b_2 u_2}}{\theta - b_2} \right] \left[ e^{-b_2 T + \theta t - \theta t} - e^{-b_2 t} \right] \]

Initial purchase quantity is given by

\[ Q = I(0) = aP_1^{-\eta} \left[ -\frac{1}{\theta} + \frac{1}{\theta^2} + \frac{1 + b_1 u_1}{\theta} e^{\theta u_1} - \frac{b_1}{\theta^2} e^{\theta u_1} \right] \]
\[ + P_2^{-\eta} \frac{a(1 + b_1 u_1)}{\theta} \left[ e^{\theta u_2} - e^{\theta u_1} \right] \]
\[ + P_3^{-\eta} \left[ \frac{a(1 + b_1 u_1) e^{b_2 u_2}}{\theta - b_2} \right] \left[ e^{-b_2 T + \theta t} - e^{-b_2 u_2 + \theta u_2} \right] \]

Following the same procedure as discussed in section 3.2, optimal solution \((n, P_1, P_2, P_3, T)\) and hence maximum value of \(\pi_j(n, P_1, P_2, P_3, T)\); \(j = 1, 2\) can be computed.

### 3.4.3 Numerical example

Consider the same numerical example discussed in section 3.2 with deterioration rate \(\theta = 12\%\).

From Table 3.4.1, 14-shipments maximize the joint total profit of $24,684 for a supply chain. The optimal selling prices set by the buyer in corresponding demand structures are \(P_1 = $ 6.03/\text{unit},\ P_2 = $ 6.27/\text{unit}\) and \(P_3 = $ 6.87/\text{unit}\). The buyer orders at every 90 days and order quantity is 1,513 units. For the computed optimal solution, the supplier’s profit is $13,783 and the buyer’s profit is $10,901. The concavity of the total joint profit with respect to the number of shipments \(n\) and the retail sale prices, \(P_1, P_2, P_3\) are shown in figures 3.4.1-3.4.4. Figures 3.4.5-3.4.7 exhibit
concavity of the total joint profit for 14-shipments with respect to $(P_1, T)$, $(P_2, T)$ and $(P_3, T)$. The variations in permissible payment periods and its effect on decision variables and total joint profit are tabulated in Table 3.4.1.

Table 3.4.1 Optimal solutions for various credit terms

<table>
<thead>
<tr>
<th>$M_1$ (days)</th>
<th>$M_2$ (days)</th>
<th>Optimal Payment Time (days)</th>
<th>$n$</th>
<th>$P_1$ ($)</th>
<th>$P_2$ ($)</th>
<th>$P_3$ ($)</th>
<th>$T$ (days)</th>
<th>$Q$ (units)</th>
<th>Buyer Profit ($)</th>
<th>Supplier Profit ($)</th>
<th>Joint Profit ($)</th>
<th>Buyer Profit (%)</th>
<th>Supplier Profit (%)</th>
<th>Joint Profit (%)</th>
<th>$R(P, T)$</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>16</td>
<td>6.24</td>
<td>6.43</td>
<td>6.73</td>
<td>86</td>
<td>1424</td>
<td>9729</td>
<td>14462</td>
<td>24191</td>
<td>-</td>
<td>-</td>
<td>-</td>
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<td>30</td>
<td>14</td>
<td>5.97</td>
<td>6.21</td>
<td>6.81</td>
<td>89</td>
<td>1516</td>
<td>9710</td>
<td>14790</td>
<td>24500</td>
<td>0.03</td>
<td>3.39</td>
<td>1.26</td>
<td>175</td>
</tr>
<tr>
<td>10</td>
<td>30</td>
<td>10</td>
<td>14</td>
<td>6.03</td>
<td>6.27</td>
<td>6.87</td>
<td>90</td>
<td>1513</td>
<td>10901</td>
<td>13783</td>
<td>24684</td>
<td>10.75</td>
<td>-4.93</td>
<td>2.00</td>
<td>176</td>
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<td>20</td>
<td>14</td>
<td>6.10</td>
<td>6.33</td>
<td>6.93</td>
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<td>1499</td>
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<td>3.85</td>
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<td>6.13</td>
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<td>6.01</td>
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<td>3.61</td>
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</table>

The positive profit gain establishes that the joint decision with the two-level trade credit intensive is advantageous.
Figure 3.4.1 Concavity of joint profit w. r. t. number of shipments $n$

Figure 3.4.2 Concavity of joint profit w. r. t. retail price $P_1$

Figure 3.4.3 Concavity of joint profit w. r. t. retail price $P_2$

Figure 3.4.4 Concavity of joint profit w. r. t. retail price $P_3$
In Table 3.4.2, the independent and joint decisions are analyzed. Reallocated profits of the supply chain are exhibited in last row of the Table.
Table 3.4.2 Optimal solutions for different strategies

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Credit Term</th>
<th>Optimal Payment Time (days)</th>
<th>n</th>
<th>$P_1$ ($)</th>
<th>$P_2$ ($)</th>
<th>$P_3$ ($)</th>
<th>$T$ (days)</th>
<th>$R(P,T)$ (units)</th>
<th>$Q$ (units)</th>
<th>Profit ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Independent</td>
<td>Cash on delivery</td>
<td>0</td>
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<td>14.07</td>
<td>14.14</td>
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<td>99</td>
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<tr>
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<td>Trade Credit 3/10 net 30</td>
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<td>16</td>
<td>13.12</td>
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<td>14.15</td>
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<td>144</td>
<td>775</td>
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</tr>
<tr>
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<td>6.24</td>
<td>6.43</td>
<td>6.73</td>
<td>86</td>
<td>161</td>
<td>1424</td>
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</tr>
<tr>
<td></td>
<td>Trade Credit 3/10 net 30</td>
<td>10</td>
<td>14</td>
<td>6.03</td>
<td>6.27</td>
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<td>176</td>
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<td>10901</td>
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<td>19736</td>
</tr>
</tbody>
</table>

The inventory parameters are varied as -20%, -10%, 10% and 20% to study the changes in the joint total profit of the supply chain (figure 3.4.8).

Figure 3.4.8 Sensitivity analysis of inventory parameters on joint profit
The impact of variations in inventory parameters on joint profit is same as observed in section 3.3.4.

Conclusions

In this chapter, an integrated supplier-buyer inventory policy is discussed when the demand is price-sensitive trapezoidal. The analysis is focused on two payment scenarios namely ‘net credit’. The joint total profit is maximized with respect to the number of shipments from the supplier to the buyer, the suitable payment time, the retail price and the cycle time. In next section, different retail prices in different demand phases are decided to get the optimal profit. Also, constant deterioration rate for the inventory products is considered. It is observed that the joint decision is advantageous for both the players. The comparison between sole and joint decision between the players of the supply chain reveals that by reallocating profit, the buyer can be enticed to opt for the joint decision. This study helps the buyer to make a decision between two promotional tools, namely, price discount and trade credit.