Chapter 7

Lot Size Model for Reverse Logistics

with Quadratic Demand
7.0 Introduction

Taking care of nature is the prime responsibility of the companies now-a-days. This chapter proposes the inventory model with reverse logistics for the environmental concerns. By establishing the reverse supply chain, resources as well as energy can be conserved. Customers return used products and after remanufacturing such products, they become as good as new products. Now, demand is satisfied by newly produced as well as remanufactured products (figure 7.1). Quadratic demand is discussed in the present chapter. Due to unavoidable situations, we may not be in a position to fulfill entire demand always and therefore, shortages are also allowed to take place. Optimal cost is worked out for the present system with respect to the cycle time. Numerical example and its graphical analyses are provided. Sensitivity analysis and managerial insights are discussed.

![Flow chart for inventory products for the inventory system](image)

Figure 7.1 Flow chart for inventory products for the inventory system
7.1 Assumptions and Notations

The following additional assumptions and notations other than those given in A.1 and N.1 are used to formulate the proposed model.

7.1.1 Assumptions

1. Fresh products are produced at a rate of $P_m(t)$.

2. Products once used are collected at a rate of $R_R(t)$. These products are remanufactured at a rate of $P_{rm}(t)$.

3. Demand is satisfied from fresh and remanufactured products at rate of $R(t)$.

4. $R(t) = a_i \left(1 + b_i t - c_i t^2\right)$, $P_m(t) = a_2 \left(1 + b_2 t - c_2 t^2\right)$, $P_{rm}(t) = a_3 \left(1 + b_3 t - c_3 t^2\right)$ and $R_R(t) = a_4 \left(1 + b_4 t - c_4 t^2\right)$ where $a_i > 0$ is scale demand, manufacturing, remanufacturing and return respectively, $0 \leq b_i < 1$ denotes linear rate of change of demand, manufacturing, remanufacturing and return respectively with respect to time and $0 \leq c_i < 1$ denotes quadratic rate of change of demand, manufacturing, remanufacturing and return respectively; where; $i = 1, 2, 3, 4$.

5. Shortages are allowed. $C_{sh}$ is the shortage cost per unit.
7.1.2 Notations

\( P_m(t) \) : Rate of production (units/year)

\( P_{rm}(t) \) : Rate of remanufacturing (units/year)

\( R_R(t) \) : Returned rate (units/year)

\( I_{rm}(t) \) : Inventory level at any time during remanufacturing

\( h_{rm} \) : Holding cost per unit per unit time during remanufacturing (in $)

(excluding interest charges)

\( A_{rm} \) : Set up cost per cycle during remanufacturing (in $)

\( C_{rm} \) : Remanufacturing cost per unit per unit time (in $)

\( q_1 \) : Maximum inventory level during remanufacturing (units)

\( I_m(t) \) : Inventory level at any time during manufacturing

\( h_m \) : Holding cost per unit per unit time during manufacturing (in $)

(excluding interest charges)

\( A_m \) : Set up cost per cycle during manufacturing (in $)

\( C_{pm} \) : Manufacturing cost per unit per unit time (in $)

\( C_m \) : Procurement cost per unit per unit time (in $)

\( q_2 \) : Maximum inventory level during manufacturing (units)

\( C_R \) : Return cost per unit per unit time (in $)

\( I_R(t) \) : Inventory level at any time for returned items
$h_R$ : Holding cost per unit per unit time for returned items (in $)
(excluding interest charges)

$A_R$ : Set up cost per returned cycle for returned items (in $)

$A_{rm}$ : Acquisition cost related to returned items (in $)

$C_{sh}$ : Shortage cost per unit per unit time (in $)

### 7.2 Mathematical Model

As shown in figure 7.2, the cycle starts at time $t = t_0$ with shortages. At this, the remanufacturing starts at the rate of $P_{rm}(t)$ and at $t = t_1$, all the shortages are satisfied. Inventory level $I_{rm}(t)$ increases by the remanufacturing at the rate of $P_{rm}(t) - R(t)$. At time $t = t_2$, the inventory reaches to the maximum.
Remanufacturing stops at time $t = t_2$ and the inventory level depletes due to demand $R(t)$. At time $t = t_3$, the inventory level reaches to zero and the shortages starts to occur till time $t = t_4$. At time $t = t_4$, the manufacturing of fresh products starts at the rate $P_m(t)$. At time $t = t_5$, all the shortages are satisfied. Inventory level $I_m(t)$ increases by the manufacturing at the rate $P_m(t) - R(t)$. At time $t = t_6$, the inventory level reaches to its maximum. Production stops at time $t = t_6$ and the inventory decreases due to demand $R(t)$. At time $t = t_7$, the inventory level becomes zero and shortages again occur up to time $t = t_8$.

In one cycle, all the returned products (during time $t = t_0$ to $t = t_8$) are remanufactured (during time $t = t_0$ to $t = t_2$). No product is thrown as a scrap.

Following are the differential equations with boundary conditions which governs the inventory levels.

\[
\frac{dI_{rm}(t)}{dt} = P_{rm}(t) - R(t), \quad 0 \leq t \leq t_2, \quad I_{rm}(t_1) = 0
\]

\[
\frac{dI_{rm}(t)}{dt} = -R(t), \quad t_2 \leq t \leq t_4, \quad I_{rm}(t_3) = 0
\]

\[
\frac{dI_m(t)}{dt} = P_m(t) - R(t), \quad t_4 \leq t \leq t_6, \quad I_m(t_5) = 0
\]

\[
\frac{dI_m(t)}{dt} = -R(t), \quad t_6 \leq t \leq t_8, \quad I_m(t_7) = 0
\]

\[
\frac{dI_R(t)}{dt} = R_R(t) - P_{rm}(t), \quad 0 \leq t \leq t_2, \quad I_R(t_2) = 0
\]

\[
\frac{dI_R(t)}{dt} = R_R(t), \quad t_2 \leq t \leq t_8, \quad I_R(t_2) = 0
\]
Solutions of the above differential equations are

\[ I_{nm1} (t) = -\left[ \frac{1}{3} (c_3 a_3 - a_1 c_1) (t^3 - t_1^3) + \frac{1}{2} (b_3 a_3 + a_1 b_1) (t^2 - t_1^2) + (a_3 - a_1) (t_1 - t) \right], \quad 0 \leq t \leq t_1 \]

\[ I_{nm2} (t) = \left[ \frac{1}{3} (c_3 a_3 - a_1 c_1) (t^3 - t_1^3) + \frac{1}{2} (b_3 a_3 + a_1 b_1) (t^2 - t_1^2) + (a_3 - a_1) (t_1 - t) \right], \quad t_1 \leq t \leq t_2 \]

\[ I_{nm3} (t) = \left[ \frac{1}{3} a_1 c_1 (t^3 - t_3^3) + \frac{1}{2} a_1 b_1 (t^2 - t_3^2) + a_1 (t_3 - t) \right], \quad t_2 \leq t \leq t_3 \]

\[ I_{nm4} (t) = -\left[ \frac{1}{3} a_1 c_1 (t^3 - t_3^3) + \frac{1}{2} a_1 b_1 (t^2 - t_3^2) + a_1 (t_3 - t) \right], \quad t_3 \leq t \leq t_4 \]

\[ I_{n1} (t) = -\left[ \frac{1}{3} (c_2 a_2 - a_1 c_1) (t^3 - t_5^3) + \frac{1}{2} (b_2 a_2 + a_1 b_1) (t^2 - t_5^2) + (a_2 - a_1) (t_5 - t) \right], \quad t_4 \leq t \leq t_5 \]

\[ I_{n2} (t) = \left[ \frac{1}{3} (c_2 a_2 - a_1 c_1) (t^3 - t_5^3) + \frac{1}{2} (b_2 a_2 + a_1 b_1) (t^2 - t_5^2) + (a_2 - a_1) (t_5 - t) \right], \quad t_5 \leq t \leq t_6 \]

\[ I_{n3} (t) = \left[ \frac{1}{3} a_1 c_1 (t^3 - t_7^3) + \frac{1}{2} a_1 b_1 (t^2 - t_7^2) + a_1 (t_7 - t) \right], \quad t_6 \leq t \leq t_7 \]

\[ I_{n4} (t) = -\left[ \frac{1}{3} a_1 c_1 (t^3 - t_7^3) + \frac{1}{2} a_1 b_1 (t^2 - t_7^2) + a_1 (t_7 - t) \right], \quad t_7 \leq t \leq t_8 \]

\[ I_{R1} (t) = \left[ \frac{1}{3} (c_4 a_4 - c_3 a_3) (t_2^3 - t^3) + \frac{1}{2} (b_3 a_3 - b_4 a_4) (t_2^2 - t^2) + (a_4 - a_3) (t - t_2) \right], \quad 0 \leq t \leq t_2 \]

\[ I_{R2} (t) = \left[ \frac{1}{3} a_4 c_4 (t_2^3 - t^3) + \frac{1}{2} a_4 b_4 (t_2^2 - t^2) + a_4 (t - t_2) \right], \quad t_2 \leq t \leq t_8 \]

Now, we calculate all the related costs as follows.

Purchase cost, \( PC = \int_{t_4}^{t_6} P_m (t) \, dt \)
Return cost, \( RC = C_R \int_0^{t_8} R_R (t) \, dt \)

Production cost, \( PRC = C_{pm} \int_{t_4}^{t_6} P_m (t) \, dt \)

Remanufacturing cost, \( RMC = C_{rm} \int_0^{t_2} P_m (t) \, dt \)

Holding cost, \( HC = h_{rm} \left( \int_{t_1}^{t_2} I_{m2} (t) \, dt + \int_{t_2}^{t_3} I_{m3} (t) \, dt \right) \)

\[ + h_{m} \left( \int_{t_4}^{t_5} I_{m2} (t) \, dt + \int_{t_5}^{t_6} I_{m3} (t) \, dt \right) + h_R \left( \int_{t_3}^{t_4} I_{R1} (t) \, dt + \int_{t_4}^{t_8} I_{R2} (t) \, dt \right) \]

Shortage cost, \( SC = C_{sh} \left( \int_{t_1}^{t_2} I_{m1} (t) \, dt + \int_{t_2}^{t_3} I_{m4} (t) \, dt + \int_{t_3}^{t_4} I_{m4} (t) \, dt + \int_{t_4}^{t_5} I_{m1} (t) \, dt + \int_{t_5}^{t_7} I_{m4} (t) \, dt \right) \)

Ordering cost, \( OC = A_{rm} + A_m + A_R \)

Now, we take relation \( t_i = \lambda_i t_8; \, i = 1 \text{to} 7 (0 < \lambda_i < 1) \) and hence total cost \( TC \) can be defined as

\[ TC (t_8) = \frac{1}{t_8} \left[ PC + RC + PRC + RMC + HC + SC + OC \right] \]

The necessary condition for the existence of the solution is \( \frac{d}{dt_8} TC (t_8) = 0 \) provided that \( \frac{d^2}{dt_8^2} TC (t_8) > 0 \).

### 7.3 Numerical Example and Sensitivity Analysis

**Example:** Take \( a_1 = 2,000 \text{ units}, \ b_1 = 0.75\%, \ c_1 = 1\%, \ a_2 = 2,500 \text{ units}, \ b_2 = 1\%, \)
\( c_2 = 1\%, \ a_3 = 3,000 \text{ units}, \ b_3 = 1\%, \ c_3 = 0.33\%, \ a_4 = 1,500 \text{ units}, \ b_4 = 1.87\%, \)
\( c_4 = 0.93\%, \ C_m = $50 / \text{unit}, \ C_R = $10 / \text{unit}, \ C_{pm} = $20 / \text{unit}, \ C_{rm} = $10 / \text{unit}, \)
Analysis of Inventory Policies in Supply Chain under Trapezoidal Demand

\[ h_{rm} = \$ 1 / \text{unit/year}, \quad h_m = \$ 5 / \text{unit/year}, \quad h_R = \$ 2 / \text{unit/year}, \quad C_{sh} = \$ 5 / \text{unit}, \]

\[ A_{rm} = \$ 80 / \text{order}, \quad A_m = \$ 150 / \text{order}, \quad A_R = \$ 100 / \text{order}, \quad \lambda_1 = 0.05, \quad \lambda_2 = 0.09, \]

\[ \lambda_3 = 0.14, \quad \lambda_4 = 0.20, \quad \lambda_5 = 0.24, \quad \lambda_6 = 0.30, \quad \lambda_7 = 0.42. \]

Then optimum cycle time is 0.72 years and corresponding optimum total cost is $36,154.

The convexity of the total cost for the obtained solution is shown in figure 7.3.

![Figure 7.3 Convexity of total cost w. r. t. cycle time](image)

At last, we examine the effects in total cost function and the decision variable by altering the inventory parameters as -40\%, -20\%, 20\% and 40\% to deduce managerial decisions. Figures 7.4 and 7.5 show the sensitivity analysis for the cycle time and total cost respectively.
Figure 7.4 Sensitivity analysis for cycle time

Figure 7.5 Sensitivity analysis for total cost
Table 7.1 Summary of sensitivity analyses

<table>
<thead>
<tr>
<th>Inventory Parameters</th>
<th>Cycle time</th>
<th>Total Cost</th>
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<tbody>
<tr>
<td>$a_1$</td>
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<tr>
<td>$\lambda_7$</td>
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</tbody>
</table>

$\uparrow$ shows increasing pattern.  
$\downarrow$ shows decreasing pattern.
Figures 7.4, 7.5 and Table 7.1 show that the player incurs increase in the total cost of the inventory system when ordering cost of returned items and remanufacturing is considered. This can be controlled by using inspection at the manufacturing department. The remanufacturing cost is directly proportional to the total cost. It is observed that overall remanufacturing is preferable but the player will have to deploy strategies to control the total cost of the inventory system. It also suggests that when remanufacturing is available, shortages should be minimized as far as possible.

Also, exhausting remanufacturing of inventory \( (t_3) \) at a faster rate, fresh manufacturing after clearing shortages \( (t_4) \) and satisfying demand of the manufactured items \( (t_7) \) at a faster rate reduces the total cost of the inventory system. Remaining time points have negative impact on the total cost. So the player should control these time points.

**Conclusions**

Environmental consciousness can be seen at each and every industry now-a-days and in particular, electronic, chemical and plastic industries. The present chapter focuses on the same and brings into light the concept of reverse logistics. The chapter discusses the inventory model in which manufacturing of new products and remanufacturing of used ones is considered. Both the newly produced items and remanufactured items are similar in each and every quality aspects and hence demand can be fulfilled by both items. Quadratic demand is proposed here which is suitable for the products for which demand increases
initially and after some point of time it decreases gradually. As shortages are unavoidable part of mostly every system, we have incorporated them as well. Finally, we have calculated the optimal cost for the cycle time. It is established that the rework is advantageous which results in the good quality items as the fresh one. So it is concluded that the customers should be encouraged to return used products for remanufacturing.