CHAPTER – 6

SQUEEZE FILM CHARACTERISTICS OF RECTANGULAR PLATES WITH A COUPLE STRESS LUBRICANT – FINITE ELEMENT APPROACH


6.1 Introduction

6.2 Mathematical Analysis

6.3 Results and Discussions

6.1 Introduction:

Fluid film lubrication occurs when opposing bearing surfaces are completely separated by a lubricant film. The applied load is carried by
pressure generated within the fluid, and frictional resistance to motion arises entirely from the shearing of the viscous fluid. A finite time is required to squeeze the fluid out of the gap, and this action provides a useful cushioning effect in bearings. As the load is time dependent it is necessary to know the condition at which the lubricant film is likely to break down when time \( t \) becomes large enough.

In the present chapter an attempt has been made to study the influence of couple stress in squeeze film in rectangular parallel plates. An expression for the squeeze film pressure using finite element methods is obtained. Using this expression of film pressure the squeeze film characteristics are then studied. A numerical solution is also presented for the squeeze time and the results are analysed.

6.2 Mathematical Analysis:

Consider a squeeze film between two closely spaced parallel rectangular plates of dimensions \( 2a \) and \( 2b \). The lower plate is stationary and the upper plate moves slowly towards the lower plate. The fluid between the plates is incompressible and it is assumed that there are no body forces or body couples.

The equations of motion of fluid with couple stress satisfying the above assumptions are

\[
\frac{\partial p}{\partial t} + \nabla \cdot (p \vec{q}) = 0
\]  \hspace{1cm} (6.1)

\[
\rho \left( \frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla) \vec{q} \right) = -\nabla p + (\lambda + \mu + \eta \nabla^2) \nabla \cdot (\nabla \vec{q}) + (\mu - \eta \nabla^2) \nabla^2 \vec{q}
\]  \hspace{1cm} (6.2)
where \( \vec{v} = (u, v, w) \) is the velocity, \( \rho \) is the density, \( p \) is the pressure, \( \lambda \) and \( \mu \) are the classical viscosity coefficients and \( \eta \) is a material constant peculiar to a fluid with couple stress.

Solving the above equations (6.1) and (6.2) with the usual assumptions of hydrodynamic lubrication applicable to thin films we obtain

\[
\nabla \cdot \vec{q} = 0 \quad (6.3)
\]

\[
\mu \frac{\partial^2 u}{\partial z^2} - \eta \frac{\partial^4 u}{\partial z^4} = \frac{\partial p}{\partial x} \quad (6.4)
\]

\[
\mu \frac{\partial^2 v}{\partial z^2} - \eta \frac{\partial^4 v}{\partial z^4} = \frac{\partial p}{\partial y} \quad (6.5)
\]

\[
\frac{\partial p}{\partial z} = 0 \quad (6.6)
\]

The boundary conditions are

\[
u = v = w = 0 \quad \text{at } z = 0 \quad (6.7)
\]

\[
\frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 v}{\partial z^2} = 0 \quad \text{at } z = 0 \quad (6.8)
\]

\[
w = h \quad \text{at } z = h \quad (6.9)
\]

Integrating (6.4) with respect to \( z \) we obtain

\[
\mu \frac{\partial u}{\partial z} - \eta \frac{\partial^3 u}{\partial z^3} = \frac{\partial p}{\partial x} z + A \quad (6.10)
\]

\[
\mu u - \eta \frac{\partial^2 u}{\partial z^2} = \frac{\partial p}{\partial x} \frac{z^2}{2} + Az + B \quad (6.11)
\]

\[
\frac{\partial^2 u}{\partial z^2} \eta u = -\frac{1}{\eta} \left( \frac{\partial p}{\partial x} \frac{z^2}{2} + Az + B \right) \quad (6.12)
\]
Let $a_*^2 = \frac{\mu}{\eta}$ and $b_* = -\frac{1}{2\eta} p_*$

Using boundary conditions from (6.7), (6.8) and (6.9)

$$\frac{\partial^2 u}{\partial z^2} - a_*^2 u = -\frac{p_*}{2\eta} (z^2 - h z) \tag{6.13}$$

Solving (6.13) we obtain

$$u = Ce^{a_* z} + Ge^{-a_* z} - \frac{b_*}{a_*^2} (z^2 - h z) - \frac{2b_*}{a_*^4} \tag{6.14}$$

Using the boundary conditions

$$G = \frac{2b_*}{a_*^2} \left( \frac{e^{a_* h} - 1}{e^{a_* h} - e^{-a_* h}} \right) \text{ and}$$

$$C = \frac{2b}{a_*^4} \left( \frac{1 - e^{a_* h}}{e^{a_* h} - e^{-a_* h}} \right) \tag{6.15}$$

$$u = -\frac{b_*}{a_*^2} (z^2 - h z) - \frac{2b_*}{a_*^4} \left( (-1 + e^{a_* h}) - (e^{a_* h} - 1) \frac{e^{a_* h} - e^{-a_* h}}{(e^{a_* h} - e^{-a_* h})} \right) \tag{6.16}$$

Integrating (6.5) with respect to ‘z’ we obtain

$$\mu \frac{\partial v}{\partial z} - \eta \frac{\partial^3 v}{\partial z^3} = \frac{\partial p}{\partial y} z + Q \tag{6.17}$$

$$\mu v - \eta \frac{\partial^2 v}{\partial z^2} = \frac{\partial p}{\partial y} \frac{z^2}{2} + Q z + R \tag{6.18}$$

$$\frac{\partial^2 v}{\partial z^2} - \frac{\mu v}{\eta} = -\frac{1}{2} \left( \frac{\partial p}{\partial y} \frac{z^2}{2} + Q z + R \right) \tag{6.19}$$

Using boundary conditions from (6.7), (6.8) and (6.9)
\[
\frac{\partial^2 v}{\partial z^2} - a^2_* \frac{v}{2\eta} = -\frac{p_y}{2\eta} (z^2 - hz)
\]  \hspace{1cm} (6.20)

Solving (6.20) we obtain

\[
v = Se^{a_* z} + U e^{-a_* z} - \frac{c_*}{a_*^2} (z^2 - hz) - \frac{2c_*}{a_*^4}
\]  \hspace{1cm} (6.21)

Using the boundary conditions

\[
U = \frac{-2c_*}{a_*^4} \left( \frac{1 - e^{a_* h}}{e^{a_* h} - e^{-a_* h}} \right)
\text{ and }
\]

\[
S = \frac{2c_*}{a_*^4} \left( \frac{1 - e^{a_* h}}{e^{a_* h} - e^{-a_* h}} \right)
\]  \hspace{1cm} (6.22)

\[
v = \frac{-c_*}{a_*} (z^2 - hz) - \frac{2c_*}{a_*^4} \left( -1 + e^{a_* h} - (e^{a_* h} - 1) \frac{(e^{a_* z} - e^{-a_* z})}{(e^{a_* h} - e^{-a_* h})} \right)
\]  \hspace{1cm} (6.23)

Simplifying the above equations we obtain

\[
u = \frac{-b_*}{a_*^2} (z^2 - hz) - \frac{2b_*}{a_*^4} \left( \frac{1 + \sinh a_* (z - h) - \sinh (a_* z)}{\sinh (a_* h)} \right)
\]  \hspace{1cm} (6.24)

\[
v = \frac{-c_*}{a_*^2} (z^2 - hz) - \frac{2c_*}{a_*^4} \left( \frac{1 + \sinh a_* (z - h) - \sinh (a_* z)}{\sinh (a_* h)} \right)
\]  \hspace{1cm} (6.25)

where \( c_* = \frac{-1}{2\eta} p_y \)  \hspace{1cm} (6.26)
Let
\[ u = u_0, \quad v = v_0, \quad w = w_0 \quad \text{at} \quad z = 0 \quad \text{and} \]
\[ u = u_1, \quad v = v_1, \quad w = w_1 \quad \text{at} \quad z = h. \]

If the equation of continuity is integrated across the fluid film, it gives
\[ \int_0^h \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz = w_0 - w_1 \quad (6.27) \]

Using the leibnitz rule of differentiation under integral sign,
\[ \frac{\partial}{\partial x} \int_0^h udv + \frac{\partial}{\partial y} \int_0^h vdy = w_0 - w_1 + u_1 \frac{\partial h}{\partial x} + v_1 \frac{\partial h}{\partial y} \quad (6.28) \]

Hence
\[ \int_0^z \frac{b h^3}{6a^2} - \frac{2b}{a^4} \int_0^h \left( \frac{1 + \sinh (a(z-h)) - \sinh (a\cdot z)}{\sinh (a\cdot h)} \right) dz \quad (6.29) \]
\[ \int_0^z \frac{c h^3}{6a^2} - \frac{2c}{a^4} \int_0^h \left( \frac{1 + \sinh (a(z-h)) - \sinh (a\cdot z)}{\sinh (a\cdot h)} \right) dz \quad (6.30) \]

\[ \left[ h^3 - 12a^2 \left( h - 2a \cdot \tanh \left( \frac{h}{2a} \right) \right) \right] \left( p_{xx} + p_{yy} \right) = 12 \mu h \quad (6.31) \]

Solving (15) with the condition \( p = 0 \) on the boundary of the plates.

\[ p = -6h \sin (a, h) \hat{p} \quad (6.32) \]
where \( \hat{p} \) is the solution of \( p_{xx} + p_{yy} = -2 \) with \( \hat{p} = 0 \) on the boundary of the plates and 
\[
\hat{s}(a_*, h) = \left[ h^3 - 12 a_*^2 \left( h - 2 a_*, \tanh \left( \frac{h}{2 a_*} \right) \right) \right]^{-1}
\] (6.33)

As the solution is obviously symmetric about the lines \( x = 0 \) and \( y = 0 \), let
\[
\hat{p} = a_1 N_1 + a_2 N_2 + a_3 N_3
\] (6.34)

where
\[
N_1 = \cos \frac{\pi x}{2a} \cos \frac{\pi y}{2b},
\]
\[
N_2 = \cos \frac{3\pi x}{2a} \cos \frac{\pi y}{2b}
\]

and
\[
N_3 = \cos \frac{\pi x}{2a} \cos \frac{3\pi y}{2b}
\]

are even trial functions and \( a_1, a_2 \) and \( a_3 \) are constants.

By the application of weighted residual methods in solving
\[
\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = -2 \quad \text{with the boundary conditions and using the above trial functions it reduces to a system of linear equations of the form}
\]

\[
K a = f
\] (6.35)

\[
\begin{pmatrix}
  k_{11} & k_{12} & k_{13} \\
  k_{21} & k_{22} & k_{23} \\
  k_{31} & k_{32} & k_{33}
\end{pmatrix}
\begin{pmatrix}
  a_1 \\
  a_2 \\
  a_3
\end{pmatrix}
= 
\begin{pmatrix}
  f_1 \\
  f_2 \\
  f_3
\end{pmatrix}
\] (6.36)

where
\[
k_{lm} = \int_{-a}^{b} \int_{-b}^{b} \left( \frac{\partial^2 N_m}{\partial x^2} + \frac{\partial^2 N_m}{\partial y^2} \right) N_l \, dy \, dx, \quad 1 \leq l, m \leq 3
\] (6.37)

and
\[
f_i = \int_{-a}^{b} 2 N_i \, dy \, dx
\] (6.38)
Let \( k_1 = \frac{\pi}{2a} \), \( k_2 = \frac{3\pi}{2a} \), \( k_3 = \frac{\pi}{2b} \), \( k_4 = \frac{3\pi}{2b} \).

\[
k_{11} = \int_{-a}^{a} \int_{-b}^{b} \left( \frac{\partial^2 N_1}{\partial x^2} + \frac{\partial^2 N_1}{\partial y^2} \right) N_1 \, dy \, dx
\]

\[
= \int_{-a}^{a} \int_{-b}^{b} -\cos^2 k_1 x \cos^2 k_3 y \left( k_1^2 + k_3^2 \right) \, dy \, dx
\]

\[
= (k_1^2 + k_3^2) ab
\]

\[
k_{11} = \frac{\pi^2}{4ab} (a^2 + b^2)
\]

\[
k_{12} = -\int_{-a}^{a} \int_{-b}^{b} \left( \frac{\partial^2 N_2}{\partial x^2} + \frac{\partial^2 N_2}{\partial y^2} \right) N_1 \, dy \, dx
\]

\[
= \int_{-a}^{a} \int_{-b}^{b} (k_2^2 + k_3^2) \cos k_1 x \cos k_2 x \cos^2 k_3 y \, dy \, dx
\]

\[
k_{12} = 0
\]

\[
k_{13} = -\int_{-a}^{a} \int_{-b}^{b} \left( \frac{\partial^2 N_3}{\partial x^2} + \frac{\partial^2 N_3}{\partial y^2} \right) N_1 \, dy \, dx
\]

\[
= \int_{-a}^{a} \int_{-b}^{b} (k_1^2 + k_4^2) \cos^2 k_1 x \cos k_3 y \cos k_3 y \, dy \, dx
\]

\[
= (k_1^2 + k_4^2) \left[ \int_{-a}^{a} \cos^2 k_1 x \, dx \right] \left[ \int_{-b}^{b} \cos k_3 y \cos k_3 y \, dy \right]
\]

\[
k_{13} = 0
\]
\[ k_{22} = - \int_{-a}^{a} \int_{-b}^{b} \left( \frac{\partial^2 N_2}{\partial x^2} + \frac{\partial^2 N_2}{\partial y^2} \right) N_2 \, dy \, dx \]

\[ = - \int_{-a}^{a} \int_{-b}^{b} (k_2^2 + k_3^2) \cos^2 k_2 x \cos^2 k_3 y \, dy \, dx \]

\[ k_{22} = ab (k_2^2 + k_3^2) \]

\[ k_{22} = \frac{\pi^2}{4ab} (ab^2 + a^2) \quad (6.43) \]

\[ k_{21} = - \int_{-a}^{a} \int_{-b}^{b} \left( \frac{\partial^2 N_1}{\partial x^2} + \frac{\partial^2 N_1}{\partial y^2} \right) N_2 \, dy \, dx \]

\[ = (k_1^2 + k_4^2) \int_{-a}^{a} \int_{-b}^{b} \cos k_1 x \cos k_2 x \cos^2 k_3 y \, dy \, dx \]

\[ k_{21} = 0 \quad (6.44) \]

\[ k_{23} = - \int_{-a}^{a} \int_{-b}^{b} \left( \frac{\partial^2 N_3}{\partial x^2} + \frac{\partial^2 N_3}{\partial y^2} \right) N_2 \, dy \, dx \]

\[ = (k_1^2 + k_4^2) \int_{-a}^{a} \int_{-b}^{b} \cos k_1 x \cos k_2 x \cos k_3 y \, dy \, dx \]

\[ k_{23} = 0 \quad (6.45) \]

\[ k_{31} = - \int_{-a}^{a} \int_{-b}^{b} \left( \frac{\partial^2 N_3}{\partial x^2} + \frac{\partial^2 N_3}{\partial y^2} \right) N_1 \, dy \, dx \]

\[ = (k_1^2 + k_4^2) \int_{-a}^{a} \cos^2 k_1 x \, dx \int_{-b}^{b} \cos k_3 y \cos k_4 y \, dy \]

\[ k_{31} = 0 \quad (6.46) \]
\[ k_{32} = - \int_{-a}^{b} \int_{-b}^{a} \left( \frac{\partial^2 N_2}{\partial x^2} + \frac{\partial^2 N_2}{\partial y^2} \right) N_3 \, dx \, dy \]
\[ = (k_x^2 + k_y^2) \int_{-a}^{b} \int_{-b}^{a} \cos k_1 x \cos k_2 x \cos k_3 y \cos k_4 y \, dy \, dx \]
\[ k_{32} = 0 \quad (6.47) \]

\[ k_{33} = - \int_{-a}^{b} \int_{-b}^{a} \left( \frac{\partial^2 N_3}{\partial x^2} + \frac{\partial^2 N_3}{\partial y^2} \right) N_3 \, dx \, dy \]
\[ = - \int_{-a}^{b} \int_{-b}^{a} (k_x^2 + k_y^2) (\cos^2 k_1 x \cos^2 k_4 y) \, dy \, dx \]
\[ k_{33} = \frac{\pi^2}{4ab} (9a^2 + b^3) \quad (6.48) \]

\[ f_1 = \int_{-a}^{b} \int_{-b}^{a} 2N_1 \, dy \, dx = 2 \int_{-a}^{b} \int_{-b}^{a} \cos k_1 x \cos k_3 y \, dy \, dx \]
\[ f_1 = \frac{32ab}{\pi^2} \quad (6.49) \]

\[ f_2 = \int_{-a}^{b} \int_{-b}^{a} 2N_2 \, dy \, dx \]
\[ f_2 = 2 \int_{-a}^{b} \int_{-b}^{a} \cos k_2 x \cos k_3 y \, dy \, dx \]
\[ f_2 = \frac{32ab}{3\pi^2} \quad (6.50) \]

\[ f_3 = 2 \int_{-a}^{b} \int_{-b}^{a} N_3 \, dy \, dx \]
\[ f_3 = 2 \int_{-a}^{b} \int_{-b}^{a} \cos k_1 x \cos k_4 y \, dy \, dx \]
\[ f_3 = \frac{-32ab}{3\pi^2} \]  

(6.51)

Now the system reduces to

\[
\begin{pmatrix}
  k_{11} & 0 & 0 \\
  0 & k_{22} & 0 \\
  0 & 0 & k_{33}
\end{pmatrix}
\begin{pmatrix}
  a_1 \\
  a_2 \\
  a_3
\end{pmatrix}
= 
\begin{pmatrix}
  f_1 \\
  f_2 \\
  f_3
\end{pmatrix}
\]  

(6.52)

Solving the above system we obtain the values for \( a_1 \), \( a_2 \) and \( a_3 \).

\[ a_1 = \frac{c}{a^2 + b^2} \]  

(6.53)

\[ a_2 = \frac{-c}{3(a^2 + 9b^2)} \]  

(6.54)

\[ a_3 = \frac{-c}{3(9a^2 + b^2)} \]  

(6.55)

where \( c = \frac{128 a^2 b^2}{\pi^4} \)  

(6.56)

The load \( L \) is given by

\[ L = -\mu A^2 \int \int_A \hat{P}(x, y) \, dx \, dy \]  

(6.57)

where \( A \) is the area of the rectangular plate taken and

\[ D = \frac{6}{A^2} \int \int_A \hat{P}(x, y) \, dx \, dy \]  

(6.58)
\[ D = \frac{6}{4^2 a^2 b^2} \int \int (a_1 N_1 + a_2 N_2 + a_3 N_3) \, dx \, dy \quad (6.59) \]

Consider 
\[ \int \int a_1 N_1 \, dx \, dy = a_1 \int \int \cos \frac{\pi x}{2a} \cos \frac{\pi y}{2b} \, dx \, dy \]

\[ = \frac{16ab}{\pi^2} \left( \frac{e}{a^2 + b^2} \right) \]

\[ \int \int a_2 N_2 = a_2 \int \int \cos \frac{3\pi x}{2a} \cos \frac{3\pi y}{2b} \, dx \, dy \]

\[ = \frac{16ab}{3\pi^2} \left( \frac{-c}{3(a^2 + 9b^2)} \right) \]

\[ \int \int a_3 N_3 = a_3 \int \int \cos \frac{\pi x}{2a} \cos \frac{3\pi y}{2b} \, dx \, dy \]

\[ = \frac{16ab}{3\pi^2} \left( \frac{-c}{3(9a^2 + b^2)} \right) \]

\[ D = \frac{768ab}{\pi^6} \left( \frac{1}{(b^2 + a^2)} - \frac{1}{9(a^2 + 9b^2)} - \frac{1}{9(9a^2 + b^2)} \right) \quad (6.60) \]

The sinkage relation between the film thickness \( h \) and time \( t \) for a given load is

\[ T = D \int_{h_0}^{1} S_c (A_s, H) \, dH \quad (6.61) \]

where \(-T = \frac{Lh_0^2 t}{\mu A^2}, H = \frac{h}{h_0}, A_s = \frac{2a_s}{h_0}\) \quad (6.62)

so that
\[ S_1(A_*, H) = h_0^3 s_1(a_*, h) \]

\[ = \frac{1}{H^3 - 3A_*^3 \left( \frac{H}{A_*} \tanh \frac{H}{A_*} \right)} \]  

(6.63)

6.3 Results and Discussions:

The expression for squeeze film pressure obtained by finite element methods is used to study the squeeze film characteristics. Using this value of squeeze film pressure and for different values of the dimensionless parameter \( D \) the dimensionless time \( T \) is calculated by varying the ratio of the dimensions of the plates.

Fig.6.1 shows the values of dimensionless time \( T \) for the dimensionless film thickness \( H = 0.2 \) and for the values of the couple stress parameter \( A_* = 0.1, 0.2, 0.3, 0.4, \) and 0.5.

It is observed from Fig.6.1 the dimensionless time \( T \) increases as the couple stress parameter \( A_* \) increases from 0.1 to 0.5 for the dimensionless height \( H = 0.2 \).

Fig.6.2 shows the values of dimensionless time \( T \) for the dimensionless film thickness \( H = 0.3 \) and for the values of the couple stress parameter \( A_* = 0.1, 0.2, 0.3, 0.4, \) and 0.5.

It is observed from Fig.6.2 the dimensionless time \( T \) increases as the couple stress parameter \( A_* \) increases from 0.1 to 0.5 for the dimensionless height \( H = 0.3 \).
Fig. 6.3 shows the values of dimensionless time $T$ for the dimensionless film thickness $H = 0.4$ and for the values of the couple stress parameter $A_*= 0.5, 0.6$ and $0.7$.

It is observed from Fig. 6.3 the dimensionless time $T$ increases as the couple stress parameter $A_*$ increases from $0.1$ to $0.5$ for the dimensionless height $H = 0.4$.

Fig. 6.4 shows the values of dimensionless time $T$ for the dimensionless film thickness $H = 0.5$ and for the values of the couple stress parameter $A_*= 0.1, 0.2$ and $0.3$.

It is observed from Fig. 6.4 the dimensionless time $T$ increases as the couple stress parameter $A_*$ increases from $0.1$ to $0.5$ for the dimensionless height $H = 0.5$.

Fig. 6.5 shows the values of dimensionless time $T$ for the dimensionless film thickness $H = 0.2$, and $H = 0.4$ for a Newtonian fluid.
Fig. 6.1. Dimensionless Time T for H=0.2
Fig. 6.2. Dimensionless Time $T$ for $H=0.3$
Fig. 6.3. Dimensionless Time $T$ for $H=0.4$
6.4 Conclusion:

From the above results it is observed that the squeeze in couple stress fluid is slower than in Newtonian fluid films. The time of approach $T$ increases with the dimensionless parameter $D$ showing that couple stress fluids are better lubricants than Newtonian fluids. The presence of microstructures in the fluid film causes an enhancement of squeeze film characteristics.