CHAPTER 3

A THEORETICAL STUDY ON THE MICROPOLAR FLUID BETWEEN TWO PARALLEL PLATES

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3.1 Introduction

3.2 Mathematical Analysis

3.3 Results and Discussions

3.1 Introduction:

Several authors have considered the effects of porous boundaries on steady, laminar and incompressible flow of micropolar fluids. The laminar flow through parallel and uniformly porous walls of different permeability of a micropolar fluid has been analysed by Terrill and Shrestha (1968) and obtained the series solution using perturbation method and compared it
with the numerical solution by calculating the skin friction at the lower and the upper wall. The developments in microcontinuum fluid mechanics has been reviewed by Ariman in (1973). Ahmadi (1976) obtained a similarity solution for micropolar boundary layer flow over a semi-infinite plate. The radial flow of a viscous fluid between porous discs was analysed by Elert (1976). The flow of a micropolar fluid between porous walls of different permeability has been discussed by Agarwal et al (1987).

The fluid flow of a micropolar fluid between a rotating and a porous stationary disc has been investigated by Agarwal et al (1987). Cox (1991) had analysed the steady flow in a channel with one porous wall, or with accelerating walls. A finite element solution of micropolar fluid flow and heat transfer between two discs has been obtained by Takhar (2000). A detailed description on the unsteady stokes flow of micropolar fluid between two parallel porous plates has been studied by Srinivasacharya et al (2001). A numerical solution of steady viscous flow of a micropolar fluid driven by injection between two porous discs has been investigated by Anwar Kamal et al (2006).

The purpose of this chapter is to analyse the flow characteristics of a micropolar fluid between parallel porous plates with a suction of velocity at the lower plate and injection of velocity at the upper plate. The velocities are expressed in terms of a stream function and a similarity transformation is introduced to solve the resulting system of differential equations.

3.2. Mathematical Analysis:

Consider a steady flow of an incompressible micropolar fluid between two porous plates coinciding with the plates \( y = 0 \) and \( y = h \) along the direction of the X axis. Since the flow is along x direction all the variables are
independent of z. Assume that there is a suction of velocity \( v_1 \) at the lower plate and injection of velocity \( v_2 \) at the upper plate. Hence we choose the velocity vector \( \vec{q} \), microrotation vector \( \vec{q}_1 \) in the form

\[
\vec{q} = u(x, y) \hat{i} + v(x, y) \hat{j} \quad \text{and} \quad \vec{q}_1 = c(x, y) \hat{k}.
\]

The equations governing the flow as given by Eringen are

\[
\nabla \cdot \vec{q} = 0 \quad (3.1)
\]

\[
0 = -\nabla p + k \left[ \nabla \times \vec{q}_1 \right] - (\mu + k) \left( \nabla \times \nabla \times \vec{q} \right) \quad (3.2)
\]

\[
0 = -2kc + k \left[ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right] + \gamma \left( \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} \right) \quad (3.3)
\]

where \( p = p(x, y) \) is the fluid pressure and \( \mu, k \) are the viscosity coefficients and \( \gamma \) is the gyroviscosity coefficient.

Then equations (3.1) and (3.2) reduces to

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (3.4)
\]

\[
0 = -\frac{\partial p}{\partial x} + k \frac{\partial c}{\partial y} + (\mu + k) \nabla^2 u \quad (3.5)
\]

\[
0 = -\frac{\partial p}{\partial y} - k \frac{\partial c}{\partial x} + (\mu + k) \nabla^2 v \quad (3.6)
\]

Introducing the stream function \( \psi(x, y) \) we have

\[
u(x, y) = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v(x, y) = -\frac{\partial \psi}{\partial x} \quad (3.7)
\]

Then the above equations reduce to
\[
0 = -\frac{\partial p}{\partial x} + k \frac{\partial c}{\partial y} + (\mu + k) \frac{\partial}{\partial y} \nabla^2 \psi \tag{3.8}
\]
\[
0 = \frac{\partial p}{\partial y} + k \frac{\partial c}{\partial x} + (\mu + k) \frac{\partial}{\partial x} \nabla^2 \psi \tag{3.9}
\]
\[
0 = -2kc + k (-\psi_{xx} - \psi_{yy}) + \gamma \nabla^2 c
\]
\[
0 = -2kc - k \nabla^2 \psi + \gamma \nabla^2 c \tag{3.10}
\]

The boundary conditions are

At \(y = 0\)

\[u = c = 0 \text{ and } v = v_1\] \tag{3.11}

At \(y = h\)

\[u = c = 0 \text{ and } v = v_2\] \tag{3.12}

Introducing similarity transformations solves the equations governing the flow.

Assume \(\psi = xd(\zeta)\) and \(c = xb(\zeta)\) where \(\zeta = \frac{y}{h}\) \tag{3.13}

From equation (3.8) and (3.9)

\[
0 = -\frac{\partial^2 p}{\partial y \partial x} + k \frac{\partial^2 c}{\partial y^2} + (\mu + k) \frac{\partial^2}{\partial y^2} \nabla^2 \psi \tag{3.14}
\]
\[
0 = \frac{\partial^2 p}{\partial y \partial x} + k \frac{\partial^2 c}{\partial x^2} + (\mu + k) \frac{\partial^2}{\partial x^2} \nabla^2 \psi \tag{3.15}
\]

Solving the above equations

\[0 = k \nabla^2 c + (\mu + k) \nabla^4 \psi \tag{3.16}\]
From Equation (3.10)

$$\nabla^2 \psi = \frac{1}{k} (\gamma \nabla^2 c - 2kc) \quad (3.17)$$

Using Equations (3.13) in (3.17) we obtain

$$kd''(\zeta) = [\gamma b''(\zeta) - 2kh^2 b(\zeta)] \quad (3.18)$$

Hence from Equations (3.7) and (3.13)

$$\frac{\partial \psi}{\partial y} = \frac{xd'(\zeta)}{h} = u(x, y) \quad (3.19)$$

$$\frac{\partial \psi}{\partial x} = d(\zeta) = -v(x, y) \quad (3.20)$$

Now the boundary conditions become

At \( \zeta = 0, \ u(x, y) = 0 \)

(i.e) \( u(x, 0) = 0 \) \quad (3.21)

At \( \zeta = 1, \ u(x, y) = 0 \)

(i.e) \( u(x, h) = 0 \) \quad (3.22)

At \( \zeta = 0, \ v(x, y) = v_1 \)

(i.e) \( v(x, 0) = 0 \) \quad (3.23)

At \( \zeta = 1, \ v(x, y) = v_2 \)

(i.e) \( v(x, h) = v_2 \) \quad (3.24)

from Equations (3.16) and (3.17)
\[ 0 = k \nabla^2 c + \left( \frac{\mu + k}{k} \right) \left[ \gamma \nabla^4 c - 2k \nabla^2 c \right] \]  

(3.25)

\[ \nabla^4 c - \ell^2 \nabla^2 c = 0 \]  

(3.26)

Here \[ \ell = \left( \frac{2\mu k + k^2}{\gamma (\mu + k)} \right) \]  

(3.27)

Using (3.13) in the above equations

\[ b(\zeta) = (A\zeta + B) + Ge^{i\zeta} + He^{-i\zeta} \]  

(3.28)

From equation (3.17)

\[ kd''(\zeta) = (\gamma D^2 - 2k \ell^2) b(\zeta) \]  

(3.29)

\[ kd''(\zeta) = (\gamma D^2 - 2k \ell^2) (A\zeta + B + Ge^{i\zeta} + He^{-i\zeta}) \]  

From equations (3.28) and (3.29)

\[ d''(\zeta) = Ac_1 \zeta + Bc_2 + e^{i\zeta} c_3 G + c_4 H e^{-i\zeta} \]  

(3.30)

where \[ c_1 = c_2 = -2k \ell^2 \]  

(3.31)

\[ c_3 = c_4 = \frac{\gamma \ell^2}{k} - 2k \ell^2 \]  

(3.32)

Hence \[ d''(\zeta) = \frac{Ac_1 \zeta^2}{2} + Bc_1 \zeta + \frac{G c_3 e^{i\zeta}}{\ell} - \frac{c_4 H e^{-i\zeta}}{\ell} \]  

(3.33)

\[ d(\zeta) = \frac{Ac_1 \zeta^3}{6} + \frac{Bc_1 \zeta^2}{2} + \frac{Gc_3 e^{i\zeta}}{\ell^2} + \frac{Hc_4 e^{-i\zeta}}{\ell^2} + E\zeta + F \]  

(3.34)

The values of the unknowns in the above equations are found using the boundary conditions in (3.28) and (3.34)

Since \[ b(0) = 0 \]
\[ B + G + H = 0 \]  
Also \( b(1) = 0 \)  
\[ A + B + Ge^\ell + He^{-\ell} = 0 \]  
Also \( d'(0) = 0 \)  
gives \( c_3 G - c_4 H + Ep = 0 \)  
\[ d'(1) = 0 \]  
gives  
\[ a_1 A + a_2 B + a_3 G + a_4 H + a_5 E = 0 \]  
where \( a_1 = c_1 \ell, \quad a_2 = 2c_2 \ell, \quad a_3 = 2c_3 e^\ell, \quad a_4 = -2c_4 e^{-\ell} \)  
and \( a_5 = 2p \)  
\[ d(0) = -v_1 \text{ gives } F = -v_1 \]  
Solving  
\[ d'(1) = -v_2 \text{ we obtain} \]  
\[ a_6 A + a_7 B + a_8 G + a_9 H + a_{10} E = a_{11} \]  
where  
\[ a_6 = c_1 p^2, \]  
\[ a_7 = 3c_1 p^2, \]  
\[ a_8 = 6c_3 e^\ell, \]  
\[ a_9 = 6c_4 e^{-\ell} \]
\[ a_{10} = 6l^2 \text{ and} \]

\[ a_{11} = 6l^2 v_1 - v_2 \]

Eliminating $E$ from the above equations we obtain

\[ a_{12}A + a_{13}B + a_{14}G + a_{15}H = -a_{11}a_5 \tag{3.41} \]

where

\[ a_{12} = a_1 a_{10} - a_5 a_6 \]

\[ a_{13} = a_2 a_{10} - a_5 a_7 \]

\[ a_{14} = a_3 a_{10} - a_5 a_8 \]

\[ a_{15} = a_4 a_{10} - a_5 a_9 \]

Also from equation (3.36)

\[ a_{12}A + a_{13}B + a_{12}e^\ell G + a_{12}e^{-\ell} H = 0 \tag{3.42} \]

Solving equations (3.41) and (3.42) we obtain

\[ a_{16}B + a_{17}G + a_{18}H = -a_{11}a_5 \tag{3.43} \]

where

\[ a_{16} = a_{13} - a_{12} \]

\[ a_{17} = a_{14} - a_{12}e^\ell \]

\[ a_{18} = a_{15} - a_{12}3^{-\ell} \]

From equation (3.35)

\[ a_{16}B + a_{16}G + a_{16}H = 0 \tag{3.44} \]
Solving Equations (3.43) and (3.44) we obtain

\[
G \left( a_{17} - a_{16} \right) + H \left( a_{18} - a_{16} \right) = -a_{11}a_5
\]  

(3.45)

Also

\[
A + G \left( e^\ell - 1 \right) + H \left( e^{-\ell} - 1 \right) = 0
\]  

(3.46)

From the above Equations

\[
a_{19} G + a_{20} H = a_{11}a_5
\]  

(3.47)

Where

\[
a_{19} = a_{12} \left( e^\ell - 1 \right) + a_{13} - a_{14}
\]

\[
a_{20} = a_{12} \left( e^{-\ell} - 1 \right) + a_{13} - a_{15}
\]

Eliminating \( H \) from Equations (3.45) and (3.47) we obtain

\[
a_{21} G = a_{22}
\]

\[
G = \frac{a_{22}}{a_{21}}
\]  

(3.48)

where

\[
a_{22} = -a_{11} a_5 \left( a_{20} + a_{18} - a_{16} \right)
\]

\[
a_{21} = \left[ a_{17} a_{20} - a_{16} a_{20} - a_{18} a_{17} + a_{16} a_{19} \right]
\]

Hence

\[
H = \left[ \frac{a_{11} a_5 a_{21} - a_{22} a_{19}}{a_{20} a_{21}} \right]
\]  

(3.49)

Again

\[
B = \left[ \frac{-a_{22} a_{20} - a_{11} a_5 a_{21} + a_{22} a_{19}}{a_{20} a_{21}} \right]
\]  

(3.50)

\[
A = \frac{-a_{22}}{a_{21}} \left( e^\ell - 1 \right) + \left( e^{-\ell} - 1 \right) \left[ \frac{a_{22} a_{17} - a_{11} a_5 a_{21}}{a_{20}} \right]
\]  

(3.51)
\[ E = \frac{1}{\ell} \left[ \frac{(c_4 a_{11} a_{15} a_{31} - c_4 a_{22} a_{17}) a_{21} - c_3 a_{22} a_{20}}{a_{20} a_{21}} \right] \]  \hspace{1cm} (3.52)

Now \( u = \frac{\partial \psi}{\partial y} \)

\[ = x d' (\zeta) \frac{d}{dy} (\zeta) \]

\[ u = \frac{x}{h} \left[ A c_1 \frac{\zeta^2}{2} + B c_1 \zeta + \frac{G c_3 e^{i\xi}}{\ell} - c_4 \frac{H e^{-i\xi}}{\ell} + E \right] \]  \hspace{1cm} (3.53)

Again

\[ v = -\frac{\partial \psi}{\partial x} \]

\[ = -\frac{\partial}{\partial x} \left[ x d (\zeta) \right] \]

\[ = -d (\zeta) \]

\[ v = -\left[ \frac{A c_1 \zeta^3}{6} + c_1 \frac{B \zeta^3}{\alpha} + G c_3 \frac{e^{i\xi}}{\ell^2} + c_4 \frac{H e^{-i\xi}}{\ell^2} + E \zeta + F \right] \]  \hspace{1cm} (3.54)

Also \( q = c(x, y) \)

\[ = x b (\zeta) \]

\[ q = x (A \zeta + B) + G e^{i\xi} + H e^{-i\xi} \]  \hspace{1cm} (3.55)

\[ c_1 = -2 \frac{h^2}{\ell} = c_2 \]

\[ c_3 = c_4 = \frac{\gamma p^2}{k} - 2 \frac{h^2}{\ell} \]

\[ a_1 = -2 \frac{h^2}{\ell} \]

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\[ a_2 = -4 \, k^2 \, \ell \]

\[ a_3 = 2 \, c_3 \, e^{i \left[ \frac{\gamma \ell^2}{k} - 2 \, h^2 \right]} \]

\[ a_4 = 2 \, e^{-i \left( 2 \, h^2 - \frac{\gamma \ell^2}{k} \right)} \]

\[ a_5 = 2 \, \ell \]

\[ a_6 = -2 \, h^2 \, \ell \]

\[ a_7 = -6 \, h^2 \, \ell \]

\[ a_8 = 6 \, e^{i \left( \frac{\gamma \ell^2}{k} - 2 \, h^2 \right)} \]

\[ a_9 = 6 \, e^{i \ell} \]

\[ a_{10} = 6 \, \ell^2 \]

\[ a_{11} = 6 \, \ell^2 \, \nu_1 - \nu_2 \]

\[ a_{12} = -8 \, h^2 \, \ell^3 \]

\[ a_{13} = -12 \, \ell^3 \, h^2 \]

\[ a_{14} = 12 \, \ell \, e^{i \ell} (\ell - 1) \left( \frac{\gamma \ell^2}{k} - 2 \, h^2 \right) \]

\[ a_{15} = 12 \, \ell \, e^{-i \ell} (\ell + 1) \left( 2 \, h^2 - \frac{\gamma \ell^2}{k} \right) \]

\[ a_{16} = -4 \, \ell^2 \, h^2 \]

\[ a_{17} = 12 \, \ell \left( \frac{\gamma \ell^2}{k} - 2 \, h^2 \right) e^{i \ell (\ell - 1)} + 8 \, h^2 \, \ell^3 \, e^{i \ell} \]
\[ a_{18} = e^{-\ell} \left( 8h^2 \ell^3 + 12 \ell (\ell + 1) \left( 2h^2 - \frac{\gamma \ell^2}{k} \right) \right) \]

\[ a_{19} = -8h^2 \ell^3 (e^\ell - 1) - 12 \ell^3 \frac{\gamma \ell^2}{k} + 12 \ell \left( \frac{\gamma \ell^2}{k} - 2h^2 \right) e^\ell (\ell - 1) \]

\[ a_{20} = h^2 \ell^3 (-8e^\ell - 4) - 12 \ell (\ell + 1) e^{-\ell} \left( \frac{2h^2 \gamma \ell^2}{k} \right) \]

The shear stress is given by

\[ \tau = (\mu + k) \left( \frac{x}{h^2} \frac{d^2}{d\zeta^2} (\zeta) \right) + k x b(\zeta) \quad (3.56) \]

\[ \tau = (\mu + k) \left( \frac{x}{h^2} (A c_1 c_2 + B c_2 + e^{i\zeta} c_3 G + c_4 H e^{-i\zeta}) \right) + k x b(\zeta) \quad (3.57) \]

At the lower plate \( y = 0 \)

\[ \tau = x (Bc_2 + c_3 G + c_4 H) \left( \frac{\mu + k}{h^2} \right) \quad (3.58) \]

At the upper plate \( y = h \)

Hence \( \tau = (\mu + k) \left( \frac{x}{h^2} \right) (Ac_1 + Bc_2 + c_3 e^{i\ell} + c_4 e^{-i\ell}) \quad (3.59) \)

Co-efficient of skin friction is

\[ C_f = \frac{2 \tau}{\rho U^2} \text{ where } U \text{ is the entrance velocity.} \]
Fig. 3.1. Variation of Velocity Component $u$
Fig. 3.3. Velocity Distribution
Fig. 3.4. Microrotation Distribution
Fig. 3.5. Values of Skin Friction on Lower Plate
Fig. 3.6. Values of Skin Friction on Upper Plate.
3.3. Results and Discussions:

1. The variation of the components of velocity and microrotation distribution with $\zeta$ are analysed by varying $x$. From Figure 3.1. it is found that an increase in the $\zeta$ increases the velocity $u$ for any given value of $x$. Also an increase in the value of $x$ decreases the velocity $u$ for any fixed value of $\zeta$.

2. From Figure 3.2. it is seen that the velocity component $v$ at first gradually increases as $\zeta$ attains a maximum value and then slowly decreases as the parameter $\zeta$ approaches one.

3. The velocity distribution for any given value of $\zeta$ gradually increases with $x$. As $\zeta$ approaches one the velocity decreases for any given value of $x$. Figure 3.4. indicates this.

4. The microrotation velocity for any given value of $\zeta$ gradually increases with $x$. Figure 3.3. indicates this.

5. The skin friction at lower plate for a given value of $x$ decreases as the parameter $\zeta$ approaches one. For any given value of $\zeta$, the skin friction decreases as $x$ increases. Figures 3.5 and 3.6. indicates the same results. Similar results are also seen for the upper plate also.

6. Figures 3.5 and 3.6 also reveals that the skin friction with $\zeta$ is more pronounced for the upper plate than the lower plate.