CHAPTER 6

WAITING TIME PROBLEMS - II
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6.1 Introduction:

We have discussed in the previous chapter the waiting time problems with balls of two colours. In these problems the number of urns and the number of balls of one colour were fixed. Here, we discuss the waiting time problem with fixed number of urns, however the balls of both the colours are the variables.

This waiting time problem is an extension of the waiting time problem studied by Arnold (1972), EL-Desouky and Hussen (1990) and Mecuven (1988). Their experiment can be explained as follows.

One urn contains a number of balls which are labelled. Balls are drawn one by one from this urn with replacement and with equal probability until the first repetition of a ball occurred.

This problem seems to be similar to the one in which the balls are thrown one by one into a number of urns until any of the urn receives two balls. This problem is extended to the balls of two colours and is discussed in this chapter. The proposed waiting time problem is explained as follows.
We consider $m$ identical urns. We throw white balls one by one into these urns until any of the urn receives two white balls. After throwing of white balls is stopped, the same exercise is repeated with balls of red colour. Here both types of balls are thrown under usual assumptions mentioned in Section 3.3. At the end of above procedure, urns get divided into four types

(i) urns containing only white balls

(ii) urns containing only red balls

(iii) urns containing at least one white and at least one red ball.

(iv) empty urns.

Number of white balls and number of red balls required are denoted by $N_w$ and $N_r$ respectively. It is easy to note that the random variables $N_w$ and $N_r$ have the same distribution and they are independent.

Since $N_w$ white balls are required, it means that $(N_w-1)$ urns out of $m$ urns are occupied, of which each of $(N_w-2)$ urns contain one white ball and one urn contains two white balls.

The probability function of $N_w$ is given by
\[ P(N_w = n_w) = \left( \frac{1}{m} \right) \frac{(m-1)}{m} \ldots \frac{(m-n_w+2)}{m} \cdot \frac{n_w}{m} \]

; \quad n_w = 2, 3, \ldots, (m+1)

= 0, \quad \text{Otherwise.}

This can be written as

\[ P(N_w = n_w) = \binom{m}{n_w-1} \frac{(n_w-1)!}{n_w} \frac{(n_w-1)}{m} \quad ; \quad n_w = 2, 3, \ldots, (m+1) \]

= 0; \quad \text{Otherwise} \quad (6.1.1)

Since \( N_w \) and \( N_r \) have same distribution, distribution of \( N_r \) is given by

\[ P(N_r = n_r) = \binom{m}{n_r-1} \frac{(n_r-1)!}{n_r} \frac{(n_r-1)}{m} \quad ; \quad n_r = 2, 3, \ldots, (m+1) \]

= 0; \quad \text{Otherwise} \quad (6.1.2)

Number of urns containing at least one white and at least one red ball are denoted by \( S \). In the next section, we study distribution of \( S \) and expected number of \( S \). An interesting relation between factorial moments of \( S \) and factorial moments of \( N_w \) has been established in the next section. Chapter is concluded with remarks in the section 6.3.
6.2 - Distribution and factorial moments of $S$.

At first, we throw the white balls one by one into the urns until any of the urn receives two white balls. Since the white balls required are denoted by $N_w$, $(N_w-1)$ urns are occupied with these white balls. Afterwards, the red balls are thrown one by one into the urns until any of the urn receives two red balls. Number of red balls required are denoted by $N_r$. Out of these $N_r$ balls, some balls may go in some of the $(N_w-1)$ urns containing white balls. These urns contain at least one white and at least one red ball. The number of such urns is denoted by $S$.

For convenience, we use the following notations for conditional and unconditional distributions.

$$P(S = s | N_w = n_w, N_r = n_r) = p(s | n_w, n_r)$$

$$P(S = s) = p(s).$$

It is easily noted that the conditional distribution of $S = s$ for a given $N_w = n_w$ and $N_r = n_r$ is hypergeometric and is given by
\[
p(s|n_w, n_\tau) = \frac{\binom{n_w}{s} \binom{m-n_w+1}{n_\tau-1-s}}{\binom{m}{n_\tau-1}}
\]

for \(s = 0, 1, \ldots, \min(n_w-1, n_\tau-1)\)

\[= 0; \quad \text{Otherwise.}\]

Using equations (6.1.1) and (6.1.2), unconditional probability function of \(S\) is given by

\[
p(s) = \sum_{n_\tau=2}^{m+1} \sum_{n_w=2}^{m+1} \left[ \frac{\binom{n_w-1}{s} \binom{m-n_w+1}{n_\tau-1-s}}{\binom{m}{n_\tau-1}} \right]
\]

for \(s = 0, 1, 2, \ldots, m\)

\[= 0; \quad \text{Otherwise} \quad \text{(6.2.1)}\]

**Remark:** Here we do not get compact form for \(p(s)\).

Now we evaluate mean of \(S\).

From equation (3.3.14) we get conditional expectation of \(S\) for given values of \(N_\tau = n_\tau, N_w = n_w\) as
\[
E[S|n_r,n_w] = \frac{(n_r-1)(n_w-1)}{m}.
\]

Hence unconditional expectation of \( S \) is

\[
E(S) = \frac{E(N_r-1)E(N_w-1)}{m}
\]

Since \( N_r \) and \( N_w \) have the same distribution.

\[
E(S) = \frac{[E(N_r) - 1]^2}{m}
\]

According to Mecuven (1988)

\[
E(N_r) = e^m m^{-m} (\Gamma[m+1], m)
\]

Where \( \Gamma[m+1], m \) is an incomplete gamma function which is given by

\[
(\Gamma[m+1], m) = \int_m^\infty e^{-t} t^m \, dt
\]

Therefore

\[
E(S) = \frac{1}{m} [e^m m^{-m} (\Gamma[m+1], m) - 1]^2
\]
Using equation (3.3.15) we get the factorial moments of $S$ for given values of $N_r = n_r$ and $N_w = n_w$ as

$$E[S^{(k)}|N_r = n_r, N_w = n_w] = \frac{(n_r-1)^{(k)}(n_w-1)^{(k)}}{m(k)}$$

Where $S^{(k)} = S(S-1) \ldots (S-k+1)$

Hence unconditional factorial moments of $S$ are given by

$$E[S^{(k)}] = \frac{[E(N_r-1)^{(k)}]^2}{m(k)}$$

The above expression is somewhat interesting. However, evaluation of $[E(N_r-1)^{(k)}]$ involves complicated algebra and hence the explicit expression for $E(S^{(k)})$ could not be obtained.

Number of urns containing only white balls and containing only red balls are given by

$$W = N_w - 1 - S,$$

$$R = N_r - 1 - S$$ respectively.

Using the above technique expected values of $W$ and $R$ can be also evaluated. It is noted that
\[ E(W) = E(R) = E(N_r) - E(S) - 1 \]

Substituting \( E(N_r) \) and \( E(S) \) we get

\[ E(W) = E(R) = e^m m^{-m} \left( \frac{m}{m+1}, m \right) - \frac{1}{m} \left[ e^m m^{-m} \left( \frac{m+1}{m+1}, m \right) - 1 \right]^2 - 1 \]

Section 6.3: Concluding remarks:

We have succeeded in evaluating probability distribution of \( S \). But the explicit distribution could not be derived. Also factorial moments of \( S \) could not be derived explicitly.