CHAPTER I

ON TOPOLOGY OF SPACES ASSOCIATED WITH FOURIER TRANSFORMS.

The main results of this chapter have already been published in the following journal.

   Marathwada University Journal of Science
1.0 Introduction:

As already mentioned (Chapter-0), after Doetsch (1943), the set of integral transformable functions is generally referred as the object space and the set of integral transformed functions as the image space. This has culminated in raising the question :- What is exactly the nature of the object space and image space i.e. the space of Fourier transformable and transformed functions ? . This chapter is devoted to answering this question by studying the object space and the image space. Of course, for this, the introduction of the suitable topology is necessary. Again, for application to analysis, some kind of algebraic structure should be introduced.

Recently Ganguly[1] has proved that the set of Laplace transformed functions is a topological space with a suitable metric defined on it and further he[2][3],[4] has also discussed the algebraic and topological structure in the sets of Laplace transformable and Laplace transformed functions, but he could not prove that the set of Laplace transformed functions and the set of Laplace transformable functions are linear topological spaces.
Doetsch developed the theory of Laplace transformation on using mainly notions of Riemann Integration. But modern workers generally use with Lebesgue integration. So here also Lebesgue integration is used. As in the theory of Lebesgue integration, functions which are equal almost everywhere, have been identified.

Let \( F_{\tau} \) be the set of all complex Fourier transformed functions \( F \) of \( f \in L^1 \) defined on the field \( R \) of real numbers, there exists \( F \in F_{\tau} \) such that

\[
(1.0.1) \quad F(p) = \int_{-\infty}^{\infty} e^{ipx} \cdot f(x) \, dx, \quad p \in (-\infty, \infty)
\]

exists in the Lebesgue sense.

In this chapter we employ the techniques used in [1] to the \( F_{\tau} \) and we shall prove—with a suitable metric defined on \( F_{\tau} \)—that the set \( F_{\tau} \) is a complete linear topological space. Further we shall prove that the set \( F_0 \) of all Fourier transformable functions belonging to \( L^1 \) is also a complete topological space. We shall investigate the problem of convergence and discuss how it is related in the two types of spaces.

In order to achieve this we shall first
introduce the algebraic structure in the sets $F_{\pi}$ and $F_{0}$.

1.1. Algebraic Structure of the set $F_{\pi}$.

Theorem 1.1-1.

The set $F_{\pi}$ is an abelian group with respect to the binary composition as usual addition.

Proof:

Let $F_{1}, F_{2}, \in F_{\pi}$, for the functions $f_{1}, f_{2} \in L^{1}$ defined on $\mathbb{R}$ and $P \in (-\infty, \infty)$ and $(F_{1} + F_{2}) (P)$ be the Fourier transform of $(f_{1} + f_{2})(x)$ then

$$(F_{1} + F_{2})(P) = \int_{-\infty}^{\infty} e^{ipx}. (f_{1} + f_{2})(x) \, dx$$

$$= \int_{-\infty}^{\infty} e^{ipx}. f_{1}(x) \, dx + \int_{-\infty}^{\infty} e^{ipx}. f_{2}(x) \, dx$$

exist. Therefore $(F_{1} + F_{2}) \in F_{\pi}$. Hence the set $F_{\pi}$ is closed with respect to the binary operation as
usual addition on $F_{\tau}$. As an immediate consequence we get the closure property in $F_{\tau}$.

The associative and the commutative properties, being true for all functions, do also hold for this set $F_{\tau}$ of functions. The identically zero complex Fourier transform acts as the additive identity in the set $F_{\tau}$.

If $f \in L^1$ is defined on $R$ and integrable in the sense of Lebesgue, then $-f \in L^1$ is also defined on $R$ and integrable in the Lebesgue sense. Hence if $F \in F_{\tau}$ then $-F \in F_{\tau}$ such that

$$
(F + (-F))(p) = \int_{-\infty}^{\infty} e^{ixp} (f + (-f))(x) \, dx
$$

$$
= \int_{-\infty}^{\infty} e^{ixp} (f-f)(x) \, dx
$$

$$
= 0
$$

$$
= (-F) + F(p)
$$

Therefore $-F \in F_{\tau}$ is the additive inverse of $F \in F_{\tau}$. 
Hence the theorem.

**Theorem 1.1-2.**

The set $F_\tau$ is a linear space.

**Proof:**

(i) By theorem (1.1-1) the set $F_\tau$ is an abelian group.

(ii) If $F \in F_\tau$ and $\alpha$ is any real or complex number then

$$\int_{-\infty}^{\infty} e^{ipx} \cdot (\alpha f)(x) \, dx = \alpha \int_{-\infty}^{\infty} e^{ipx} \cdot f(x) \, dx \text{ exists}$$

$$= \alpha F(p)$$

Therefore $\alpha \cdot F \in F_\tau$.

If $F_1, F_2 \in F_\tau$ and $\alpha, \beta$ are real or complex numbers then the following properties are easily verified.

(iii) $\alpha \cdot (\beta \cdot F_1)(p) = (\alpha \beta) \cdot F_1(p)$
(iv) \((\alpha + \beta) \cdot F_1(P) = \alpha \cdot F_1(P) + \beta \cdot F_1(P)\)

(v) \(\alpha(F_1 + F_2)(P) = \alpha \cdot F_1(P) + \alpha \cdot F_2(P)\)

(vi) \(1 \cdot F_1(P) = F_1(P)\)

where 1 is the multiplicative identity element in the field of real numbers.

Thus the set \(F_\tau\) is a linear space.

1.2 Metrization:

We now investigate the topology in the set \(F_\tau\).

We define a functional \(P\) on \(F_\tau\) as follows:

\[
P(F) = \int_{-\infty}^{\infty} e^{ipx} f(x) \, dx \quad \text{for every } F \in F_\tau
\]

\[
= \int_{-\infty}^{\infty} |f(x)| \, dx
\]

Then \(P\) is a norm on \(F_\tau\). Therefore \(F_\tau\) is a normed linear space.
If we define \( d(F_1, F_2) = P(F_1 - F_2) \) for \( F_1, F_2 \in F_\tau \), then the norm \( P \) induces a metric \( d \) on \( F_\tau \). Therefore \( F_\tau \) is a metric space with the metric \( d \) defined as above. Hence the metric \( d \) induces a topology on \( F_\tau \). Similarly it can be proved that \( F_\tau \) is also a metric space with respect to the metric

\[
\rho(F_1, F_2) = \frac{\int_\infty^\infty |f_1(x) - f_2(x)| \, dx}{1 + \int_{-\infty}^\infty |f_1(x) - f_2(x)| \, dx}
\]

where \( F_1, F_2 \in F_\tau \) and \( f_1, f_2 \in L^1 \).

Now in topologising the linear space \( F_\tau \) with an algebraic structure we now check up how far the algebraic operations are continuous with respect to the topology induced by the metric \( d \) on \( F_\tau \).

**Theorem (1.2-1):**

The addition operation is continuous in \( F_\tau \) with respect to the metric \( d \).
Proof:

Let $F_n^0$, $G_n^0 \in F^0$ for every $n$ (where $n$ is a natural number) such that $F_n \to F \in F^0$ and $G_n \to G \in F^0$, then

$$d(F_n + G_n, F + G) = P \left( (F_n + G_n) - (F + G) \right)$$

$$= P \left( (F_n - F) + (G_n - G) \right)$$

$$\leq P(F_n - F) + P(G_n - G)$$

$$= d(F_n, F) + d(G_n, G)$$

$$\to 0 \quad \text{as } n \to \infty$$

Therefore the additive operation is continuous on $F^0$.

Theorem 1.2-2:

The set $F^0$ is a linear metric space.

Proof:

In theorems 1.1-1, 1.2-1 and in article 1.2
we have proved that $F_\tau$ is a linear space, addition operation is continuous in $F_\tau$ and it is a metric space respectively.

Now let $F \in F_\tau$ and $\beta_n$ be a sequence of non-zero scalars such that $\beta_n \to \beta$ where $\beta$ is also a scalar, then

\[
d(\beta_n F, \beta F) = F(\beta_n F - \beta F)
\]

\[
= |\beta_n - \beta| F(F)
\]

\[
\longrightarrow 0 \text{ as } n \to \infty
\]

Hence $F_\tau$ is a linear metric space. Thus $F_\tau$ is a linear topological space.

Theorem 1.2-3:

$F_\tau$ is complete.

Proof:

Let $F_n$ be a Cauchy sequence in $F_\tau$. 
i.e. \( d(F_m, F_n) = \int_{-\infty}^{\infty} |f_m - f_n| \, dx \longrightarrow 0 \)

as \( n, m \rightarrow \infty \) independently. Therefore \( f_n \) is also a Cauchy sequence in \( L^1 \). Since \( L^1 \) is complete, there exists \( f \in L^1 \) such that \( f_n \rightarrow f \) as \( n \rightarrow \infty \). Now we claim that \( F \), the Fourier Transform of \( f \in L^1 \), is the limit of the sequence \( F_n \). For,

\[
d(F_n, F) = P(F_n - F) = \int_{-\infty}^{\infty} |f_n - f| \, dx
\]

\[\longrightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

Since there is one-one correspondence between the set \( L^1 \) and \( F_\tau \), the limit is unique. Hence \( F_\tau \) is a complete linear topological space.

1.3 Definition:

Let \( F_0 \) be the set of all Fourier transformable functions belonging to \( L^1 \). We say that \( f \in F_0 \) if \( \int_{-\infty}^{\infty} e^{ipx} \cdot f(x) \, dx \) exists in the sense of Lebesgue, \( P \in (-\infty, \infty) \).
1.4 Algebraic Structure in $F_0$

Theorem 1.4-1.

The set $F_0$ is an abelian group with respect to the binary operation as usual addition.

Proof:

Let $f_1, f_2 \in F_0$. Then $\int_{-\infty}^{\infty} e^{ipx} f_1(x) \, dx$ and $\int_{-\infty}^{\infty} e^{ipx} f_2(x) \, dx$ exist. Since

$$\int_{-\infty}^{\infty} e^{ipx} (f_1(x) + f_2(x)) \, dx = \int_{-\infty}^{\infty} e^{ipx} f_1(x) \, dx + \int_{-\infty}^{\infty} e^{ipx} f_2(x) \, dx$$

exists, $(f_1 + f_2) \in F_0$. Therefore the set $F_0$ is closed with respect to the binary composition as usual addition.

The commutative and associative properties are obvious. The identically zero function acts as identity in $F_0$. Since $f \in F_0$, $-f \in F_0$, then $-f$ is the additive identity of $F_0$. Thus the set $F_0$ is an abelian group.
Theorem 1.4-2:

The set \( F_0 \) is a linear space.

Proof:

The proof is obvious.

1.5. Metrization in \( F_0 \):

Let \( \gamma \) be a functional defined on \( F_0 \) as follows:

\[
\gamma(f) = \int_{-\infty}^{\infty} |f(x)| \, dx
\]

(1.5-1)

It is easy to see that \( \gamma(f) \) is a norm. Therefore \( F_0 \) is a normed linear space. If we define the distance function \( d_1 \) on \( F_0 \) as follows:

\[
d_1(f, g) = \gamma(f-g)
\]

(1.5-2)

then the norm defined in (1.5-1) induces a metric \( d_1 \) in \( F_0 \). Therefore the set \( F_0 \) is a metric space with the metric \( d_1 \). Hence this metric \( d_1 \) induces a topology on \( F_0 \).

Now we shall see that which of the operations are continuous in \( F_0 \).
Theorem (1.5-1):

The additive operation is continuous in \( F_0 \) with respect to the metric \( d_1 \).

Proof:

Let \( f_n, g_n \in F_0 \) for every \( n \) such that \( f_n \to f \in F_0 \) and \( g_n \to g \in F_0 \) then

\[
d_1(f_n + g_n, f + g) = \gamma \left[ (f_n + g_n) - (f + g) \right]
\]

\[
= \gamma \left[ (f_n - f) + (g_n - g) \right]
\]

\[
\leq \gamma (f_n - f) + \gamma (g_n - g)
\]

\[
= d_1(f_n, f) + d_1(g_n, g)
\]

\[
\to 0 \quad \text{as } n \to \infty.
\]

Therefore the additive operation is continuous on \( F_0 \).
Theorem 1.5-2:

The scalar multiplication operation is continuous on $F_0$.

Proof:

Let $f \in F_0$ and $\{\alpha_n\}$ be a sequence of non-zero scalars such that $\alpha_n \rightarrow \alpha$ (where $\alpha$ is also a scalar), then

$$d_1(\alpha_n f, \alpha f) = \gamma(\alpha_n f - \alpha f)$$

$$= | \alpha_n - \alpha | \gamma(f)$$

$$\rightarrow 0 \text{ as } n \rightarrow \infty.$$ 

Hence the scalar multiplication is continuous on $F_0$.

In view of theorems 1.4-2, 1.5-1, 1.5-2 and by step 1.5-2 the set $F_0$ is a linear topological space.

Theorem 1.5-3:

$F_0$ is complete.
Proof:

Let \( f_n \) be a Cauchy sequence in \( F_0 \), i.e. \( d_1(f_m, f_n) = \int_{-\infty}^{\infty} |f_m - f_n| \to 0 \) as \( m, n \to \infty \) independently. Since \( L^1 \) is complete and \( f_n \) is a Cauchy sequence in \( F_0 \), there exists \( f \in L^1 \) such that \( f_n \to f \) as \( n \to \infty \), then

\[
\int_{-\infty}^{\infty} |f_n - f| \to 0 \quad \text{as} \quad n \to \infty.
\]

Since \( f \in L^1 \), the limit is unique. Hence \( F_0 \) is complete. Thus we have proved that \( F_0 \) is a complete linear topological space.

1.6 Convergence:

In the previous part of this chapter the topological and algebraic structure of the space of Fourier transformable functions and the space of functions after transformation with metric defined suitably, has been discussed. We now like to investigate the problem, how convergence in the two types of spaces discussed previously are related.
Theorem 1.6-1:

Let \( F_n(P) = \int_{-\infty}^{\infty} e^{ipx} f_n(x) \, dx \) and

\( F_0(P) = \int_{-\infty}^{\infty} e^{ipx} f_0(x) \, dx \) where \( f_n, f_0 \in L^1 \)

then \( d(F_n, F_0) \to 0 \) as \( n \to \infty \) if and only if

\( d_1(f_n, f_0) \to 0 \) as \( n \to \infty \)

Proof:

We know that

\[
d(F_n, F_0) = \int_{-\infty}^{\infty} |f_n(x) - f_0(x)| \, dx
\]

and \( d_1(f_n, f_0) = \int_{-\infty}^{\infty} |f_n(x) - f_0(x)| \, dx \)

\[
\therefore \quad d(F_n, F_0) = \int_{-\infty}^{\infty} |f_n(x) - f_0(x)| \, dx
\]

\[
= d_1(f_n, f_0)
\]

\[
\therefore \quad d(F_n, F_0) = d_1(f_n, f_0)
\]

Hence the result.
Chap. 1

REFERENCES


   Institute Mathematique T.9(23), 1969.

