CHAPTER IX

THE MELLIN-HANKEL TRANSFORM OF GENERALIZED FUNCTIONS.
9.1 Introduction:

In the theory of partial differential equations one of the most effective means of solving boundary value problems is by means of integral transforms. When the problems involve generalized functions, then the application of integral transforms requires finding out the integral transforms of generalized functions. It is quite well known that there are several problems which can be solved by the repeated application of the Mellin and Hankel transform. If we construct an integral transform for which the Kernel is the product of Mellin and Hankel Kernels, we may term this integral transform as Mellin-Hankel transform, then it is necessary to find the Mellin-Hankel transform of generalized functions.

Let \( \mathcal{R} = \{(x,y) \mid 0 < x < \infty, \ 0 < y < \infty\} \)

If the function \( \phi(x,y) \) is defined on \( \mathcal{R} \), then its Mellin-Hankel transform is defined by

\[
(9.1-1) \quad \overline{\Phi}(u, w) = \int_0^\infty \int_0^\infty x^{u-1} \cdot \sqrt{wx} \cdot J_\mu(\omega y) \phi(x,y) \, dx \, dy
\]

where \( J_\mu(\omega y) \) is the Bessel function of order \( \mu \) with \( \mu \) real.
The integral on the right hand side of
\[ (9.1-1) \] exists if and only if the integral \[ \int_0^\infty \int_0^\infty x^{u-1} \phi(x,y) dx dy \]
converges absolutely with \( \omega \) positive real and \( \mu > -\frac{1}{2} \).

If the Mellin-Hankel transform \( (9.1-1) \) exists with \( \omega \) positive real and \( \mu > -\frac{1}{2} \) then \( \phi(x,y) \) is repre- sented by

\[ (9.1-2) \phi(x,y)=\mathcal{H}_\mu^{-1}(\Phi)=\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} x^{-u} du \int_{\infty-\infty}^{\infty} \frac{\Phi(u\omega) du}{\sqrt{uy}} \cdot \mu(y) \]

for all \((x,y) \in \mathbb{R}^2\).

We shall require the following Parsevats' equation.

**Lemma 9.1-1:**

Let \( \phi(x,y) \) and \( \psi(x,y) \) are defined on \( \mathbb{R}^2 \). If the Mellin-Hankel transforms \( \Phi(u,\omega) \) and \( \Psi(1-u,\omega) \) of \( \phi(x,y) \) and \( \psi(x,y) \) exist with \( \mu > -\frac{1}{2} \) respectively and \( \omega \) positive real, then the Parsevats equation for Mellin-Hankel transformation is given by
(9.1-3) \[
\frac{1}{2\pi i} \int_0^\infty \int_{-i\infty}^{i\infty} \Phi(u,\omega) \cdot \Psi(1-u,\omega) \, du \, d\omega
\]
\[
= \int_0^\infty \int_0^\infty \Phi(x,y) \cdot \Psi(x,y) \, dx \, dy
\]

In this chapter we shall first of all define the conventional Mellin-Hankel transformation on complete locally convex, bornological Hausdorff topological \( \mathcal{M}_a,b,k \) and then we shall generalize the conventional Mellin-Hankel transformation to Mellin-Hankel transform of generalized functions.

9.2 The Conventional Mellin-Hankel Transformation:

\[ \text{on } \mathcal{M}_a,b,k \]

**Theorem 9.2-1:**

If \( \mu > -\frac{1}{2} \), \( 1-a < \Re u < 1-b \), then for every \( \Phi(x,y) \in \mathcal{M}_a,b,k \),

\[
(9.2-1) \quad \Phi(u,\omega) = \int_0^\infty \int_0^\infty x^{u-1} \cdot \omega y \cdot J_\mu(\omega y) \cdot \Phi(x,y) \, dx \, dy
\]

exists where \( 0 < \omega < \infty \) and \( a, b \in \mathbb{R}^1 \).
Proof:

Let \( \phi(x,y) \in \text{M}_{n \times k} \). Since \( \sqrt{uv} J_{\frac{\mu}{2}}(uv) = O(1) \) as \( u \to \infty \), we have

\[
| \Phi(u, v) | \leq \int_0^\infty \int_0^\infty |x^{u-1} \cdot \phi(x, y)| \, dx \, dy, \quad v > 0
\]

\[
= \int_0^1 \int_0^1 \mu \cdot P_{\alpha, \phi, o}(\phi) \cdot \frac{x^{u-1}}{x^{-a} \cdot y^{\mu-\frac{1}{2}}} \, dx \, dy
\]

\[
+ \int_0^\infty \int_0^\infty \mu \cdot P_{a, b, m, o}(\phi) \cdot \frac{x^{u-1}}{x^{-b} y^m \cdot y^{\mu-\frac{1}{2}}} \, dx \, dy
\]

\[
< \mu \cdot P_{\alpha, b, o, o}(\phi) \cdot \int_0^1 \int_0^1 x^{u-2+a} \cdot y^{\mu+\frac{1}{2}} \, dx \, dy
\]

\[
+ \mu \cdot P_{a, b, m, o}(\phi) \int_1^\infty \int_1^\infty x^{b+u-2} y^{\mu+\frac{1}{2} - m} \, dx \, dy
\]

the integral on the right hand side of above in equality exists if \( \mu > -\frac{1}{2} \), \( 1 - a < R \), \( u < 1 - b \) and \( m > \mu + \frac{3}{2} \) where \( m \) is a non-negative integer. (Q.E.D.)
Now we shall denote the Mellin-Hankel transform of $\phi(x,y) \in \text{MH}_{a,b,k}$ by

$$
\Phi(u,\omega) = \mathcal{M}_\mu \phi = \int_0^\infty \int_0^\infty x^{u-1} \sqrt{\omega y} \cdot J_\mu(\omega y) \phi(x,y) \, dx \, dy
$$

It is clear that if $\phi \in \text{MH}_{a,b,k}$, then its Mellin-Hankel transform $\mathcal{M}_\mu \phi = \Phi$ exists uniquely whenever $\mu > -\frac{1}{2}$, $1-a < Re u < 1-b$ and $0 < \omega < \infty$.

The set of all Mellin-Hankel transforms of the functions $\phi \in \text{MH}_{a,b,k}$ is denoted by $\tilde{\text{MH}}_{a,b,k} = \mathcal{M}_\mu [ \text{MH}_{a,b,k} ]$.

Evidently, $\tilde{\text{MH}}_{a,b,k}$ is a linear space.

The inverse of the transformation $\mathcal{M}_\mu$ is $\mathcal{M}_\mu^{-1}$ can be proved to be

$$
\phi(x,y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-u} \sqrt{\omega y} \cdot J_\mu(\omega y) \cdot \Phi(u,\omega) \, du \, d\omega
$$

for $\mu > -\frac{1}{2}$, $0 < \omega < \infty$, $c > Re u$.

9.3 Let us introduce the topology on $\tilde{\text{MH}}_{a,b,k}$ for which the Mellin-Hankel transformation $\mathcal{M}_\mu$ is an isomorphism from $\text{MH}_{a,b,k}$ onto $\tilde{\text{MH}}_{a,b,k}$.
particular, a sequence $\{ \phi_\mu \}_{\mu=1}^{\infty}$ converges in $\overset{\sim}{\mathcal{M}}_{a,b}^{\nu,k}$ to $\phi_0$ if and only if the sequence $\{ \phi_\mu \}_{\mu=1}^{\infty}$ where $\phi_\mu = \overline{M}_\mu \phi_\mu$ converges to $\phi = \overline{M_0} \phi$ in $\overset{\sim}{\mathcal{M}}_{a,b,k}$.

The space $\overset{\sim}{\mathcal{M}}_{a,b,k}$ turns out to be a testing function space.

Let $\overset{\sim}{\mathcal{M}}_{a,b}^{\nu,k}$ be the set of all continuous linear functionals over $\overset{\sim}{\mathcal{M}}_{a,b,k}$. $\overset{\sim}{\mathcal{M}}_{a,b,k}$ is a linear space. We assign to the space $\overset{\sim}{\mathcal{M}}_{a,b}^{\nu,k}$ the weak topology. Since $\overset{\sim}{\mathcal{M}}_{a,b,k}$ is a testing function space, $\overset{\sim}{\mathcal{M}}_{a,b,k}$, the dual of $\overset{\sim}{\mathcal{M}}_{a,b,k}$, is the space of all generalized functions defined on $\overset{\sim}{\mathcal{M}}_{a,b,k}$.

9.4 The Generalized Mellin-Hankel Transformation:

Let $\nu \gamma = \frac{1}{\tau_0}$. By 9.3 the mapping $R : \overline{M}_\mu (\phi) \rightarrow 2\pi i \phi$ is an isomorphism from $\overset{\sim}{\mathcal{M}}_{a,b}^{\nu,k}$ onto $\overset{\sim}{\mathcal{M}}_{a,b}^{\nu,k}$. We therefore define a generalized Mellin-Hankel transformation $\overline{M}_\mu$ as the adjoint of the mapping $R$ on $\overset{\sim}{\mathcal{M}}_{a,b}^{\nu,k}$. If $f \in \mathcal{M}_{a,b,k}$ and $\phi = \overline{M}_\mu (f) \in \overset{\sim}{\mathcal{M}}_{a,b,k}$, then the generalized Mellin-Hankel transform

$$F = R f = \overline{M}_\mu f$$
is defined by
(9.4-1) \[ \langle R' f, \overline{\phi} \rangle = 2\pi i \langle f, \phi \rangle \]

or \[ \langle \mathcal{M}_\mu f, \mathcal{M}_\mu \phi \rangle = 2\pi i \langle f, \phi \rangle \]

When the conventional function \( f \) is such that (i) the conventional Mellin-Hankel transform exists and (ii) \( f \) generates a regular distribution \( f \) in \( MH_{a,b,k} \), then \( M_\mu f \) generates a regular generalized functions \( \mathcal{M}_\mu f = F \) in \( MH_{a,b,k} \) through

\[ \langle \mathcal{M}_\mu f, \overline{\phi} \rangle = \int_{\sigma-i\infty}^{\sigma+i\infty} \int_{0}^{\infty} F(u, \omega) \cdot \Phi(u, \omega) du \ d\omega \]

where \( \Phi \in \widetilde{MH}_{a,b,k} \). Moreover \( M_\mu f = \mathcal{M}_\mu f \) in generalized sense. Indeed, by the result (9.1-3)

\[ \langle \mathcal{M}_\mu f, \overline{\phi} \rangle = 2\pi i \int_{0}^{\infty} \int_{0}^{\infty} f(x, y) \cdot \phi(x, y) dx \ dy \]

\[ = 2\pi i \langle f, \phi \rangle , \ \phi \in MH_{a,b,k} \]

\[ \therefore \langle \mathcal{M}_\mu f, \overline{\phi} \rangle = \langle \mathcal{M}_\mu f, \overline{\phi} \rangle . \]

In other words, we say that the generalized Mellin-Hankel transformation agrees with conventional Mellin-Hankel transformation.
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