CHAPTER VI

The Biharmonic Equation and Its Solution with Boundary Condition as a Distribution.
6.1 Introduction:

In this chapter we shall solve the Biharmonic equation with boundary condition as a distribution by the application of Hankel Transform of generalized functions and we use the definitions and results given in article 4.2.

6.2 Statement of the Problem:

The problem we have considered here is the problem of finding a conventional function \( u(\rho, z) \) which satisfies the biharmonic equation

\[
(6.2-1) \quad \Delta a \cdot \Delta u(\rho, z) = 0
\]

where

\[
\Delta a = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2}
\]

On the domain

\[
\mathcal{D} = \left\{ (\rho, z) \mid 0 < \rho < \infty, \ 0 < z < \infty \right\}
\]

with the following boundary conditions:
(a) \[ u(\rho, z) \to f(\rho) \in \mathcal{E}'(I) \text{ in some generalized sense as } z \to 0. \]

(b) \[ \frac{\partial u(\rho, z)}{\partial z} \to 0 \text{ uniformly as } z \to 0 \]

(c) \[ u(\rho, z) \to 0 \text{ uniformly as } z \to \infty \]

6.3 Solution of the problem:

Let \( \mathcal{U} (\rho, z) = \int f \cdot u(\rho, z) \) and \( g(\rho) = \int f \cdot f(\rho) \).

Here \( g(\rho) \in \mathcal{E}'(I) \). Then the equation (6.2-1) becomes

\[
(6.3-1) \quad \left( M_0 N_0 + \frac{\partial^2}{\partial z^2} \right)^2 \mathcal{U}(\rho, z) = 0
\]

where \[[2, p.154] \quad M_0 N_0 \mathcal{U} = \rho^{\frac{1}{2}} \frac{\partial}{\partial \rho} \frac{1}{\rho} \cdot \frac{\partial}{\partial \rho} \rho^{\frac{1}{2}} \mathcal{U} \].

By applying Hankel transformation of order-zero to (6.3-1) and formally interchanging \( h_0 \) with \( \frac{\partial^2}{\partial z^2} \) we get

\[
( - \xi^2 + \frac{\partial^2}{\partial z^2} )^2 \mathcal{U}(\xi, z) = 0 \text{ where }
\]

\[ \mathcal{U}(\xi, z) = h_0 \mathcal{U}(\rho, z) \]
(6.3-2) \[ (D^2 - \xi^2) \quad U(\xi, z) = 0 \quad \text{where} \quad D = \frac{\partial}{\partial z} \]

The solution of (6.3-2) is

(6.3-3) \[ U(\xi, z) = (C_1 + C_2 z) \cdot e^{\xi z} + (C_3 + C_4 z) \cdot e^{-\xi z} \]

In view of boundary conditions (a), (b) and (c) we have additional conditions:

\[ U(\xi, z) \to h_0 g(\rho) \quad \text{as} \quad z \to 0 \]

\[ \frac{\partial U(\xi, z)}{\partial z} \to 0 \quad \text{as} \quad z \to 0 \quad \text{and} \]

\[ U(\xi, z) \to 0 \quad \text{as} \quad z \to \infty. \]

Hence with respect to above boundary conditions we get \( C_1 = C_2 = 0, \quad C_3 = h_0 g(\rho) \) and \( C_4 = \xi h_0 \tilde{g}(\rho) \). Hence by theorem (5.6-3) \[2, 147\], we have

\[ C_3 = \langle g(\rho), \sqrt{\rho} \cdot J_0(\xi) \rangle \quad \text{and} \]

\[ C_4 = \langle g(\rho), \sqrt{\rho} \cdot J_0(\xi) \rangle. \quad \text{Therefore} \quad (6.3-3) \]

becomes
\[ U(\xi, z) = e^{-\xi z} \cdot (1 + \xi z) \cdot g(\rho) \cdot \int \rho \cdot J_0(\rho \xi) \, d\rho. \]

For each fixed \( z \), by theorems 5.6-1 and 5.6-2 \([2, p. 146]\), \( U(\xi, z) \) is a smooth function of \( \xi \) in \( L_1(0, \infty) \). Hence we may apply the conventional inverse Hankel transformation to get our formal solution

\[ (6.3-4) \quad \mathcal{H}(\rho, z) = \int_0^\infty U(\xi, z) \cdot \int \rho \xi \cdot J_0(\rho \xi) \, d\xi \]

\[ = \int_0^\infty e^{-\xi z} \cdot (1 + \xi z) \cdot g(\rho) \cdot \int \rho \xi \cdot J_0(\rho \xi) \cdot \int \xi^2 \cdot J_0(\xi^2) \, d\xi. \]

defined on \( -\infty \to \infty \).

Now we shall prove that (6.3-4) is the solution of (6.3-1). We know that \( J_0(\rho \xi) \) and \( J_1(\rho x) \) are bounded on \( 0 < \rho \xi < \infty \) and \( e^{-\xi z} \leq e^{-\xi z} \) for \( Z < z < \infty \), \( 0 < \xi < \infty \). These facts and by theorems 5.6-1 and 5.6-2 \([2, p. 146]\), we may interchange the differentiations in (6.3-1) with the integration in (6.3-4) since at every step the resulting integral converges uniformly on every compact subset of the domain \( -\infty \to \infty \). Since the integrand in (6.3-4) satisfies (6.3-1), we
conclude that \( \mathcal{U} (\rho, z) \) satisfies the equation (6.3-1).

For, from (6.3-4) we have

\[
\frac{\partial \mathcal{U}}{\partial z} = \int_0^\infty g(\rho, z) \mathcal{T}_o(\xi, z) \cdot \mathcal{T}_0(\xi, z) \cdot (-\frac{z}{\xi} \cdot e^{-\xi z}) d\xi
\]

(6.3-5) \( \frac{\partial \mathcal{U}}{\partial z} = \int_0^\infty g(\rho, z) \mathcal{T}_o(\xi, z) \cdot \mathcal{T}_0(\xi, z) \cdot \left( z \frac{z}{\xi} - \frac{z}{\xi} \right) e^{-\xi z} d\xi \)

and

(6.3-6) \( m_0 N_0 \mathcal{U} = \int_0^\infty e^{-\xi z} (1+\xi z) \cdot g(\rho, z) \mathcal{T}_o(\xi, z) \mathcal{T}_0(\xi, z) \mathcal{T}_o(\xi, z) \mathcal{T}_0(\xi, z) d\xi \)

By adding (6.3-5) and (6.3-6) we get

6.3-7 \( m_0 N_0 \mathcal{U} + \frac{\partial^2 \mathcal{U}}{\partial z^2} = W \) (say)

\[
= \int_0^\infty e^{-\xi z} \cdot g(\rho, z) \mathcal{T}_o(\xi, z) \cdot \mathcal{T}_0(\xi, z) \cdot (-\frac{z}{\xi} \cdot e^{-\xi z}) d\xi
\]

Now

\[
\frac{\partial W}{\partial z} = \int_0^\infty e^{-\xi z} \cdot g(\rho, z) \mathcal{T}_o(\xi, z) \cdot \mathcal{T}_0(\xi, z) \cdot (2\xi \frac{z}{\xi}) d\xi
\]
(6.3-8) \[ \frac{\partial^2 w}{\partial z^2} = \int_0^\infty e^{-\xi^2} \left< g(\xi), \int \xi p \cdot J_0(\xi \rho) \right> \left( \int \xi p \cdot J_0(\xi \rho) \right) (-2 \xi^2) \, d\xi \]

and

(6.3-9) \[ M_0 N_0 w = \int_0^\infty \langle \xi p \rangle, \left< \int \xi p \cdot J_0(\xi \rho) \right> \left( \int \xi p \cdot J_0(\xi \rho) \right) \langle \xi p \rangle (2 \xi^2) \, d\xi \]

By adding (6.3-8) and (6.3-9) we get

(6.3-10) \[ M_0 N_0 w + \frac{\partial^2 w}{\partial z^2} = (M_0 N_0 + \frac{\partial}{\partial z}) (M_0 N_0 + \frac{\partial}{\partial z}) U(\rho, z) = 0 \]

Hence the result. Therefore \( u(\rho, z) \) satisfies (6.3-1).

We now prove that the initial condition

(a) is satisfied. As a function of \( \xi \) for each fixed \( z \), \( U(\xi, z) \) is smooth function and for each fixed \( z > 0 \), it is absolutely integrable by virtue of theorems 5.6-1 and 5.6-2 \( [p. 146] \). Therefore \( U(\xi, z) \) satisfies the conditions under which the ordinary Hankel transformation \( h_0 \) is a special case of generalized Hankel transformation \( h_0^\prime \). According to (6.3-4), its Hankel transformation is \( U(\rho, z) \), so that, for any \( \phi \in H_0 \) and \( U = h_0 \phi \), by definition of the generalized Hankel transformation we have
\[ \langle \mathcal{U}(\rho, z), \phi(z) \rangle = \int_{0}^{\infty} \mathcal{U}(\xi, z) \cdot \prod_{\xi} \langle \xi \rangle d\xi \]

\[ = \int_{0}^{\infty} e^{-\xi^2 z} (1 + \xi^2 z) \cdot \langle g(\rho), \sqrt{\rho} \xi \cdot J_0(\xi \rho) \rangle \cdot \prod_{\xi} \langle \xi \rangle d\xi. \]

Since the term under integration is bounded, the integral on the right hand side converges uniformly on \( 0 < z < \infty \). Thus we may interchange the limiting process \( z \to 0 \) with the integration to get

\[ \text{It} \langle \mathcal{U}(\rho, z), \phi(z) \rangle = \int_{0}^{\infty} \langle g(\rho), \sqrt{\rho} \xi \cdot J_0(\xi \rho) \rangle \cdot \prod_{\xi} \langle \xi \rangle d\xi \]

\[ = \langle g(\rho), \phi(z) \rangle \]

by virtue of sec. 5.5 equation (4) [2, p.143].

\[ \therefore \mathcal{U}(\rho, z) \to g(\rho) \text{ in the sense of convergence in } H_0. \text{ Hence } u(\rho, z) \to f(\rho) \text{ in a generalized sense.} \]

To verify the condition (b) we know that

\[ (6.3-8) \langle g(\rho), \sqrt{\xi} \xi \cdot J_0(\xi \rho) \rangle \cdot \xi^2 \cdot z e^{-\xi^2 z} \]

is a smooth and for each fixed \( z > 0 \), it is absolutely integrable on \( 0 < \xi < \infty \), by virtue of theorem
5.6-1 and 5.6-2 we have

\[(6.3-9) \quad \frac{\partial u}{\partial z} = \int_0^\infty \big< g(\rho), \sqrt{\rho} \cdot J_0(\rho \xi) \big> (\xi z) e^{-\xi \cdot \xi} \int_0^\infty J_0(\xi \rho) d\xi.\]

According to (6.3-9) the Hankel transform of 6.3-8 is

\[\frac{\partial u}{\partial z}\]

and from (6.3-9) it is easy to see that \(\frac{\partial u}{\partial z} \rightarrow 0\)

uniformly as \(z \rightarrow 0\).

We shall now verify the condition (c). Let

\[|J_0(\xi \rho)| < B\] for \(0 < \xi \rho < \infty\) and we have

\[|u(\rho, z)| = \left| \frac{\mathcal{U}(\rho, z)}{1}\right|

\[< B \int_0^\infty \big< g(\rho), \sqrt{\rho} \cdot J_0(\xi \rho) \big> e^{-\xi \cdot \xi} (1 + \xi z) \cdot \sqrt{\xi} \big| d\xi\]

\[(6.3-10) < B \int_0^\infty \big< g(\rho), \sqrt{\rho} \cdot J_0(\xi \rho) \big> e^{-\xi \cdot \xi} \sqrt{\xi} \big| d\xi + B \int_0^\infty \big< g(\rho), \sqrt{\rho} \cdot J_0(\xi \rho) \big> \xi z e^{-\xi \cdot \xi} \sqrt{\xi} \big| d\xi.

It is easy to see that the 1st integral on the right
hand side of (6.3.10) tends to zero as $z \to \infty$. Since $z e^{-z} \to 0$ as $z \to \infty$, the second integral also to zero as $z \to \infty$. 
REFERENCES

