Chapter 3

The mixing angle as a function of neutrino mass ratio

In the quark sector, we experience a correlation between the mixing angles and the mass ratios. A partial realization of the similar tie-up in the neutrino sector helps to constrain the parametrization of masses and mixing, and hints for a predictive framework. We derive five hierarchy dependent textures of neutrino mass matrix with minimum number of parameters (≤ 4), following a model-independent strategy.

3.1 Introduction

The neutrino mass matrix plays the central role in the study of neutrino physics, as it contains the information of both the masses and mixing. In that sense, it is more fundamental than the PMNS matrix. It is always desirable to derive a texture of the mass matrix which leads to significant prediction. If the neutrino mass matrix, \( \mathcal{M}_\nu \) follows \( \mu-\tau \) symmetry \([112, 142, 169–175]\), we obtain two constraints on the matrix elements; they are: \((\mathcal{M}_\nu)_{12} = (\mathcal{M}_\nu)_{13}\) and \((\mathcal{M}_\nu)_{22} = (\mathcal{M}_\nu)_{33}\). These two constraints generate: \(\theta_{13} = 0\) and \(\theta_{23} = 45^0\). But the \(\mu-\tau\) symmetric texture does not tell anything about the neutrino mass hierarchy and solar angle. The texture becomes predictive only when it is associated with certain flavor symmetries\([59–61, 63–66, 69, 176]\). On the contrary, the present experimental data strongly rule out any possibility of a vanishing reactor angle\([128–130]\) and the central value...
of $\theta_{23}$ is more than $45^0$ (and is close to $49^0$) [1]. These deviations undoubtedly questions the credibility of $\mu$-$\tau$ symmetry.

Visualization of a more realistic neutrino mixing pattern and mass matrix, demands perturbation to the $\mu$-$\tau$ symmetry [177–179]. In the present article we emphasize more on the possibility to perceive an exact texture of $M_\nu$ which is model independent, with minimum number of efficient elements, than following perturbation techniques. Our approach is bottom-up and inspired by the phenomenology of quark sector.

### 3.2 The angle and the mass ratio

The Cabibbo angle ($\theta_c$) [149, 180] is a parameter which plays a significant role in describing the quark masses and mixing. It is anticipated that this angle might be a function of the ratio of down and strange quark masses [181],

$$\sin \theta_c \simeq \sqrt{\frac{m_d}{m_s}}. \quad (3.1)$$

It is an esteemed endeavor of particle physicists to unify the quark and lepton sectors, or to realize similar kind of happenings in both the sectors otherwise. Based on this, is it possible to extend a similar idea in the form of an ansatz in the neutrino sector also, as in the following,

$$\sqrt{\frac{m_i}{m_j}} = \sin \theta_{ij}. \quad (i, j = 1, 2, 3)? \quad (3.2)$$

Undoubtedly there are several hurdles which will arise both from the theoretical and phenomenological perspectives. The reason lies in the difference between the mixing mechanism in both the sectors. The CKM matrix is very close to unit matrix and the spectra of “up” and “down” quarks are strongly hierarchical. But, for neutrinos we are ignorant of the exact hierarchy of the masses. Unlike the quarks, the mixing is quite large in lepton sector and the PMNS matrix is far from being an unit matrix.

So long the reactor angle was predicted to be vanishing, such development in Eq.(3.2) seems obsolete. But, in the light of present data, when, $\theta_{13} \sim \theta_c$, one
cannot deny the possible existence of the following relation,

\[
\sqrt{\frac{m_1}{m_3}} \simeq \sin \theta_{13} = \epsilon, \quad \text{(say)},
\] (3.3)

in the non-degenerate spectrum of neutrino masses, obeying normal ordering (see Fig.(3.1)). We emphasize that the realization of the ansatz in Eq. (3.2) is not “full”, but “partial”. Because, it seems impossible to realize all the three possibilities (See in Eq. (3.2)) simultaneously. For example, if we realize Eq. (3.3), then another possibility,

\[
\sqrt{\frac{m_2}{m_3}} \simeq \sin \theta_{23} = \eta;
\] (3.4)

is ruled out and vice-versa (See Fig.(3.1)). Again, a similar realization in 1-2 sector is forbidden because of the smallness of the solar mass squared difference, which insists the mass ratio, \(m_1 : m_2\) to be constant, and this ratio tends towards unity.

Because of similar reasons, for inverted ordering of the neutrino masses, only the 2-3 realization,

\[
\sqrt{\frac{m_3}{m_2}} \simeq \sin \theta_{23} = \eta;
\] (3.5)

is possible.

Now even though there are several clampdowns, a partial realization of the ansatz (Eq. (3.2)) clutches certain positive aspects like it puts some constraints on the
parametrization of neutrino masses and mixing. Let us discuss some implications of the ansatz in Eq. (3.2).

- It categorizes the parametrization in both ways: “$\epsilon$-based” or “$\eta$-based”.
- The “$\epsilon$-based” parametrization only encompasses the Normal ordering of the masses with non-degenerate spectrum (NH-ND), with absolute mass scale, $0.047 \, eV \leq m_3 \leq 0.05 \, eV$. This parametrization rules out any possibility of vanishing $m_1$ (Strict-NH case). Since the reactor angle is not zero as depicted in Eq. (3.3).
- “$\eta$-based” parametrization encompasses both the Normal ordering and inverted ordering of neutrino masses with degenerate spectrum (NH-QD and IH-QD) only, with absolute mass scale, $0.05 \, eV \leq m_3 (m_2) \leq 0.067 \, eV$.
- The present ansatz (Eq. (3.2)), rules out the possibility of non-degenerate inverted spectrum of neutrino masses.
- In the degenerate limit, the ansatz sets the upper limit on the sum of the neutrino masses, $\Sigma m_i \leq 0.17 \, eV$. This prediction is relevant in the light of present Cosmological observation.

### 3.3 The Parametrization

We can see that although the ansatz, (Eq. (3.2)) cannot solve the hierarchy issue, yet it can put some constraints on the mass spectrum. This are subjected to the sensitivities of the future experiments. The $\epsilon$ and $\eta$ based mass spectrum can be represented in the following way,

\[
M_{\nu}^d(\epsilon, s) = \begin{bmatrix} s\epsilon^2 & 0 & 0 \\ 0 & s\epsilon & 0 \\ 0 & 0 & 1 \end{bmatrix} m_3, \quad \text{(NH-ND)}
\]  

\[
M_{\nu}^d(\eta, c) = \begin{bmatrix} c\eta^2 & 0 & 0 \\ 0 & \eta^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} m_3, \quad \text{(NH-QD)}
\]
\[
M^d_\nu(\eta, c) = \begin{bmatrix}
c & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \eta^2 \\
\end{bmatrix} m_2, \quad \text{(IH-QD)} \tag{3.8}
\]

where, \(s\) and \(c\) are \(O(1)\) coefficients: \(\epsilon < s < \epsilon^{-1}\), \(\eta < c < \eta^{-1}\). The positivity of \(\Delta m^2_{21}\) enforces, \(c < 1\).

We are working in a basis, where the charged lepton mass matrix is diagonal. We represent the PMNS matrix, \(U(\epsilon)\) which is “\(\epsilon\)” motivated as in the following,

\[
U(\epsilon, d, f) \approx \begin{bmatrix}
1 - \frac{1}{2} f^2 \epsilon^2 - \frac{c^2}{2} & \epsilon & f \\
-f \epsilon - de^2 & 1 - \frac{1}{2} f^2 \epsilon^2 - \frac{d^2 c^2}{2} & de \\
-\epsilon + cde^2 & -de - f \epsilon^2 & 1 - \frac{1}{2} d^2 \epsilon^2 - \frac{c^2}{2} \\
\end{bmatrix}, \quad \text{(PMNS-I)} \tag{3.9}
\]

Where, \(d\) and \(f\) are \(O(1)\) coefficients. We put forward another possible form of PMNS matrix, which is motivated by \(\eta\)-based parametrization, \(U(\eta)\),

\[
U(\eta, b, c) \approx \begin{bmatrix}
\sqrt{\frac{2}{3}} \epsilon' & \frac{c}{\sqrt{3}} & b \gamma \\
-\frac{cs}{\sqrt{3}} - \sqrt{\frac{2}{3}} b \gamma \eta c' & \sqrt{\frac{2}{3}} \kappa c' - \frac{bc \gamma \eta}{\sqrt{3}} & \eta \\
\frac{cn}{\sqrt{3}} - \sqrt{\frac{2}{3}} b \gamma \kappa c' & -\frac{bc \gamma c}{\sqrt{3}} - \sqrt{\frac{2}{3}} \eta c' & \kappa \\
\end{bmatrix}, \quad \text{(PMNS-II)} \tag{3.10}
\]

where, \(\gamma(\eta) = \eta^8\), \(\kappa(\eta) = \cos \sin^{-1}(\eta)\) and \(c' = (3 - c^2)^{1/2}/2\). Let us summarize some important features of the above two non-familiar parametrization of PMNS matrix.

- This is to be highlighted that in either of the two possibilities \textbf{PMNS-I} or \textbf{PMNS II}, a vanishing reactor angle is not possible. If this is so, the mass eigenvalues will also disappear. Hence, the present parametrization cannot hold the Tri-Bimaximal (TB) and Bi-maximal (BM) framework in exact form, which assumes predominantly the reactor angle to be zero.

- The \textbf{PMNS-II} allows only the possibilities, \(\theta_{23} > 45^0\) and \(\theta_{12} \lesssim \sin^{-1}(1/\sqrt{3})\).

- The free parameter, \(\epsilon \sim O(\lambda)\) and \(\eta \sim 4 \lambda\), where \(\lambda\) is the Wolfenstein parameter.
$A \approx -f^2 se^4 + f^2 se^3 - se^4 + se^2 + \epsilon^2$

$B \approx -\frac{1}{2} d^2 f se^4 - dse^3 - \frac{d^2 \epsilon}{2} + d \epsilon - \frac{1}{2} f^2 se^4 - \frac{1}{2} fse^4 - fse^3 + fse^2$

$C \approx -\frac{d^2 \epsilon^3}{2} + df se^4 - df se^3 - f^2 se^4 - se^3 - \frac{\epsilon^3}{2} + \epsilon$

$D \approx -d^2 se^3 - d^2 \epsilon^3 + d^2 \epsilon^2 - 2df se^4 + f^2 se^4 - f^2 se^3 + se$

$E \approx \frac{1}{2} d^3 se^4 - \frac{d^2 \epsilon^3}{2} + df^2 se^4 - dse^2 - d \epsilon^3 + d \epsilon + fse^4 - fse^3$

$F \approx d^2 se^3 + d^2 \epsilon^4 - d^2 \epsilon^2 - 2df se^4 + se^4 + \frac{\epsilon^4}{4} - \epsilon^2 + 1$

| Table 3.1: The elements of the general neutrino mass matrix (Normal Hierarchy-non-degenerate (NH-ND) case) |

3.4 The Texture of the Neutrino mass matrix

Next, we try to understand how the ansatz in Eq. (3.2) will help to understand the texture of neutrino mass matrix? The ansatz reduces the number of free parameters, and the number of working parameters are less than that of physical ones. Hence we expect that the mass matrix is little predictive. To construct the same, we concentrate on the finding out some exact sum rules to relate different matrix elements.

Here, we construct the neutrino mass matrix, $M_\nu = U M_\nu^d U^T$. For a lucid flow of the present discussion we keep aside the numerical description of the internal parameters like $\epsilon, \eta$ etc. The details of the same can be found in the Table.(3.4). We have the general texture of left-handed Majorana neutrino mass matrix, as shown below.

$$M_\nu \sim \begin{bmatrix} A & B & C \\ B & D & E \\ C & E & F \end{bmatrix}.$$  \hspace{1cm} (3.11)

If, the parametrization is $\epsilon$-based, we have $M_\nu = M_\nu(\epsilon, d, f, s)$ and $M_\nu = M_\nu(\eta, b, c)$, if it is $\eta$-based. We start with the NH-ND case, parametrization of which is $\epsilon$ based. The matrix elements are tabulated in Table. (3.1).
Choosing the working parameters, \((\epsilon, d, f, s)\) properly, we derive the following exact relations connecting the matrix elements.

**Sum rule 1:** \(2 (\mathcal{A} + \mathcal{C}) - (\mathcal{E} - \mathcal{B}) = 0\), \(\text{(3.12)}\)

**Sum rule 2:** \(\mathcal{D} - (\mathcal{A} + \mathcal{F}) = 0\), \(\text{(3.13)}\)

**Sum rule 3:** \(2\mathcal{A} - (\mathcal{B} + \mathcal{C}) = 0\), \(\text{(3.14)}\)

**Sum rule 4:** \(2\mathcal{A} - (\mathcal{D} - \mathcal{E}) = 0\), \(\text{(3.15)}\)

These sum-rules promote a framework with,

\[\theta_{13} \approx 8.73^0, \quad \theta_{23} \approx 48.96^0, \quad \theta_{12} \approx 32.51^0.\] \(\text{(3.16)}\)

Except the solar angle which is consistent with \(2\sigma\) range (by sacrificing one of the sum rules, the solar angle can be made precise.), the rest lies within \(1\sigma\) range.

The above sum rules lead to the following texture,

\[
M_\nu = \begin{bmatrix}
\mathcal{A} & 2\mathcal{A} - \mathcal{C} & \mathcal{C} \\
2\mathcal{A} - \mathcal{C} & 6\mathcal{A} + \mathcal{C} & 4\mathcal{A} + \mathcal{C} \\
\mathcal{C} & 4\mathcal{A} + \mathcal{C} & 5\mathcal{A} + \mathcal{C}
\end{bmatrix} m_3, \quad \text{(Texture-I)} \quad \text{(3.17)}
\]

where, \(\mathcal{A} > \mathcal{C}\). It is interesting to note that the **Texture-I** is nothing but combination of a pattern akin to the modified Fritzsch-like texture of quark mass matrices \([182, 183]\),

\[
\begin{bmatrix}
0 & 2\mathcal{A} & 0 \\
2\mathcal{A} & 6\mathcal{A} & 4\mathcal{A} \\
0 & 4\mathcal{A} & 5\mathcal{A}
\end{bmatrix}, \quad \text{(3.18)}
\]

and, a \(\mu-\tau\) symmetric texture as shown below,

\[
\begin{bmatrix}
\mathcal{A} & -\mathcal{C} & \mathcal{C} \\
-\mathcal{C} & \mathcal{C} & \mathcal{C} \\
\mathcal{C} & \mathcal{C} & \mathcal{C}
\end{bmatrix}. \quad \text{(3.19)}
\]

Next, we turn towards another possibility, i.e., the **NHQD** scenario which is motivated by \(\eta\)-based parametrization. The matrix elements of the concerned neutrino mass matrix are illustrated in Table.(3.2).
\[ \begin{align*} 
A & \approx b^2 \gamma^2 + \frac{c^2 \eta^2}{3} + \frac{2}{3} c \eta^2 c'^2 \\
B & \approx -\frac{1}{4} \eta \{ b \gamma (c^2 \eta^2 - 3) + 2 b c \gamma \eta^2 c'^2 + \sqrt{2} (c - 1) c \eta \kappa c' \} \\
C & \approx \frac{1}{3} \{ b \gamma \kappa (3 - c^2 \eta^2) - 2 b c \gamma \eta^2 \kappa c'^2 + \sqrt{2} (c - 1) c \eta \kappa c' \} \\
D & \approx \frac{1}{4} \eta^2 \{ c (\sqrt{6} b \gamma \eta c' + \sqrt{3} \kappa c'^2) + 3 (b c \eta - \sqrt{2} \kappa c')^2 + 9 \} \\
E & \approx \frac{1}{4} \eta \{ -2 \eta^2 \kappa (c')^2 (b^2 c^2 - 1) + \kappa (-b^2 \gamma^2 c^2 \eta^2 + c^3 \eta^2 - 3) + \sqrt{2} (c - 1) c \gamma \eta c' (\eta^2 - \kappa^2) \} \\
F & \approx \frac{1}{4} \eta^2 \{ \sqrt{3} c \kappa \gamma + \sqrt{6} \eta c' \}^2 + \frac{1}{6} c \eta^2 \{ \sqrt{3} c \eta - \sqrt{6} b \gamma \kappa c' \}^2 + \kappa^2 
\end{align*} \]

Table 3.2: The elements of the general neutrino mass matrix (NH-QD case)

As before, we encounter the following exact sum rules,

\[
\begin{align*}
\text{Sum rule 1:} & \quad D - F - B = 0, \quad (3.20) \\
\text{Sum rule 2:} & \quad 4E - D = 0, \quad (3.21) \\
\text{Sum rule 3:} & \quad 3C - 2(D - F) = 0, \quad (3.22)
\end{align*}
\]

which prescribe the following texture of the neutrino mass matrix guided by three parameters,

\[
M_\nu = \begin{bmatrix}
A & B & \frac{2}{3} B \\
B & 4E & E \\
\frac{2}{3} B & E & 4E - B
\end{bmatrix} \ m_3, \quad \text{(Texture-II)} \quad (3.23)
\]

with, \(4E > A > E > B\). The above neutrino mass matrix is consistent with the prediction,

\[
\theta_{12} \approx 34.08^\circ, \quad \theta_{23} \approx 49.66^\circ, \quad \theta_{13} \approx 10^\circ. \quad (3.24)
\]

We see that the above framework predicts a reactor angle, lying slightly above the experimental observation. Here also, we experience a \(\mu-\tau\) symmetric texture with a perturbation matrix,

\[
\begin{bmatrix}
A & B & B \\
B & 4E & E \\
B & E & 4E
\end{bmatrix} + \begin{bmatrix}
0 & 0 & \frac{1}{3} B \\
0 & 0 & 0 \\
\frac{1}{3} B & 0 & -B
\end{bmatrix}. \quad (3.25)
\]
A precise value $\theta_{13} \approx 9.3^0$, consistent within $1\sigma$ bound is obtainable at the cost of sacrificing the Sum rule 2 in Eq. (3.21) (See Fig.(3.2)). The other angles remain untouched. And, the corresponding texture of the neutrino mass matrix appears as in the following,

$$M_\nu = \begin{bmatrix} A & B & \frac{2}{3}B \\ B & D & \mathcal{E} \\ \frac{2}{3}B & \mathcal{E} & D - B \end{bmatrix} m_3, \quad \text{(Texture-III)} \quad (3.26)$$

with $D > A > \mathcal{E} > B$. The discussion related to the hidden $\mu$-$\tau$ symmetric texture is similar to that for Texture-II. Similarly for the case of Inverted hierarchy (IH-QD), the matrix elements are shown in Table.(3.3). The realization of the following sum rules:

Sum rule 1: $\mathcal{B} - \mathcal{C} = 0,$ \quad (3.27)

Sum rule 2: $A + \mathcal{E} - D + B = 0,$ \quad (3.28)

promote the neutrino mass matrix, $M_\nu$, to assume the following texture,

$$M_\nu = \begin{bmatrix} A & B & B \\ B & D & D - A - B \\ B & D - A - B & \mathcal{F} \end{bmatrix} m_2, \quad \text{(Texture-IV).} \quad (3.29)$$

with $A > \mathcal{F} > D > |B|$. 
The above sum rules, restricts the reactor angle at,

$$\theta_{13} \approx 7.1^0,$$

(3.30)

Which is little lower than what we observe experimentally. The other predictions are,

$$\theta_{12} \approx 34.84^0, \quad \theta_{23} \approx 50.76^0.$$

(3.31)

which are consistent within 1\(\sigma\). By changing the Sum rules a little, (See Fig.(3.2)) as in the following,

Sum rule 1: \(B - C = 0\),

(3.32)

Sum rule 2: \(A - B - F + E = 0\).

(3.33)

We can set the reactor angle within the 1\(\sigma\) bound \((\theta_{13} \approx 8.5^0)\). The corresponding neutrino mass matrix appears as in the following,

$$M_\nu = \begin{bmatrix} A & B & B \\ B & D & F - A + B & F \\ B & F - A + B & F \end{bmatrix} m_2, \quad (\text{Texture-V})$$  

(3.34)

Where, \(A > F > D > |B|\).
\begin{table}
\centering
\begin{tabular}{|c|c|c|}
\hline
\textbf{Parameters} & \textbf{Prediction} & \textbf{Texture} \\
\hline
\textbf{NH-ND} & $\epsilon = 0.151859,$ & $m_1 = 0.00135518 \text{ eV},$
& $f = 3.53882,$ & $m_2 = 0.00892393 \text{ eV},$
& $d = 4.96783,$ & $\Delta m^2_{21} = 7.78 \times 10^{-5} \text{ eV}^2,$
& $s = 1.21652,$ & $\Delta m^2_{31} = 2.33 \times 10^{-3} \text{ eV}^2,$
& $m_3 = 0.0483055 \text{ eV},$ & $\sin^2 \theta_{12} = 0.289,$
& & $\sin^2 \theta_{23} = 0.569,$
& & $\sin^2 \theta_{13} = 0.0233,$
& & $\Sigma m_i = 0.0585846 \text{ eV}.$
\hline
\textbf{NH-QD} & $\eta = 0.762386,$ & $m_1 = 0.0330417 \text{ eV},$
& $b = 1.56568,$ & $m_2 = 0.0341791 \text{ eV},$
& $c = 0.96663,$ & $\Delta m^2_{21} = 7.65 \times 10^{-5} \text{ eV}^2,$
& $m_3 = 0.0588 \text{ eV},$ & $\Delta m^2_{31} = 2.37 \times 10^{-3} \text{ eV}^2,$
& & $\sin^2 \theta_{12} = 0.315,$
& & $\sin^2 \theta_{23} = 0.581,$
& & $\sin^2 \theta_{13} = 0.0319,$
& & $\Sigma m_i = 0.126036 \text{ eV}.$
\hline
\textbf{IH-QD} & $\eta = 0.756875,$ & $m_1 = 0.0325638 \text{ eV},$
& $b = 1.4945,$ & $m_2 = 0.0336859 \text{ eV},$
& $c = 0.96663,$ & $\Delta m^2_{21} = 7.43 \times 10^{-5} \text{ eV}^2,$
& $m_3 = 0.0588 \text{ eV},$ & $\Delta m^2_{31} = 2.40 \times 10^{-3} \text{ eV}^2,$
& & $\sin^2 \theta_{12} = 0.314,$
& & $\sin^2 \theta_{23} = 0.573,$
& & $\sin^2 \theta_{13} = 0.0259,$
& & $\Sigma m_i = 0.12506 \text{ eV}.$
\hline
\end{tabular}
\caption{The summary of parametrization and the neutrino mass matrix texture}
\end{table}
3.5 Discussion

In the above discussion, for all the five textures, the parameters are chosen in such a way that the two observational mass parameters can lie always within the $1\sigma$ boundary of experimental data. Also, it is seen that the number of independent matrix elements is 2, 3 or 4. Needless to say, that textures we have displayed are hierarchy dependent and exact. Also, one interesting fact that we see is the existence of a $\mu$-$\tau$ symmetric or partially broken $\mu$-$\tau$ symmetric textures is unavoidable in the above patterns. But this does not allow us to take the initial choice of $M_\nu$ to be $\mu$-$\tau$ symmetric. Since, both $\epsilon$-based or $\eta$-based parametrization are reluctant to assume a vanishing $\theta_{13}$ (which is one of the important traces of the $\mu$-$\tau$ symmetry), certainly, we have to look into some other possibilities.

Trying to understand the underlying flavor symmetry groups or to unveil the first principle working behind the above textures will be a fascinating exercise for the model builders. We hope that Refs. [184–187], can provide some important clue in this line. In order to keep the discussion simple, all the parameters are treated as real. However, we have stressed on different phenomenological facets when it is possible to express at least one of the mixing angles in terms of the mass ratio.