CHAPTER II

TOPOLOGIES OF INTEGRAL TRANSFORMS.

The main results of this chapter have been communicated for publication as detailed below:

1. Topologies of Integral Transforms.

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2.1 Introduction:

The integral transform of a real valued function $f$ is defined over an interval $(a,b)$ (finite, semi-infinite or doubly-infinite) by the relation

$$F(\alpha) = \int_{a}^{b} w(x) \cdot k(\alpha,x) \cdot f(x) \, dx$$

where (i) the integral on the right hand-side exists in Lebesgue sense,

(ii) $k(\alpha,x)$ is a Kernel function,

(iii) $w(x)$ is a weight function and

(iv) $F(\alpha)$ is a real or complex valued function.

The weight function $w(x)$ is generally so chosen that the integral converges. This is particularly more necessary when the range of integration is infinite or when the Kernel becomes infinite within the range of integration.

The class of functions $f$ is said to be the object space and the class of functions $F$ is said to be the image space. The theory of integral transforms has been developed by choosing different Kernels $k(\alpha,x)$ and different ranges of integration.

In series of papers [1], [2], [3], [4] and [5] after introducing metrics S. Ganguly
has proved that the set of Laplace transformed functions (Lt-set) is not a linear metric space but it is a complete metric space with respect to the metric

\[ d(f, g) = |S_F - S_G| + d_c(f, g) \]

where

(i) \( d_c(f, g) = \sup_{S > 0} |f(S + S_F + \alpha) - g(S + S_G + \alpha)| \) is a pseudo-metric on Lt-set.

(ii) \( \alpha \) is a positive number however small and fixed once-for all,

(iii) \( S_F, S_G \) are the abscissas of convergence for the Laplace transformable functions \( F \) and \( G \) respectively,

(iv) \( f(S) = \int_0^\infty e^{-Sx} F(x) \, dx \)

and \( g(S) = \int_0^\infty e^{-Sx} G(x) \, dx \)

exist in Lebesgue sense. He has also proved that the set of all Laplace transformable functions is not a linear metric space but it is a complete metric space and disconnected with respect to the metric.
\[ P(F, G) = |S_F - S_G| + \frac{\int_0^\infty e^{-S_F x} F(x) - e^{-S_G x} G(x) \, dx}{1 + \int_0^\infty e^{-S_F x} F(x) - e^{-S_G x} G(x) \, dx} \]

where \( S_F \) and \( S_G \) are abscissas of convergence of the Laplace transformable functions \( F \) and \( G \) respectively.

In this chapter we will generalize the above results for a more general class of integral transforms whose kernels are linear. In order to achieve this we shall first introduce the algebraic structure in the sets \( I_\tau \) and \( D_\tau \), which are defined in article No. 2.2.

2.2 Notations:

Let the image set of integral transform and the object set of integral transform be denoted by the symbols \( I_\tau \) and \( D_\tau \) respectively. We say that \( F \in I_\tau \) if there exist a function \( f \in D_\tau \) and a linear kernel \( k(\alpha, x) \) such that

\[ F(\alpha) = \int_a^b w(x) k(\alpha, x) f(x) \, dx \]

exist in Lebesgue sense where \( w(x) \) is a weight
function.

Similarly by $f \in D_\tau$ we mean that there exists a real or complex valued function $F \in I_\tau$ such that

$$F(\alpha) = \int_a^b w(x) \cdot k(\alpha, x) \cdot f(x) \, dx$$

exists in Lebesgue sense, where $k(\alpha, x)$ and $w(x)$ are linear kernel and a weight function respectively.

2.3 Algebraic Structure on $I_\tau$:

We shall first establish the following theorem.

Theorem 2.3.1:

The set $I_\tau$ is an abelian group to the binary operation as usual addition.

Proof:

(1) If $F_1, F_2 \in I_\tau$, then by definition

$$F_1(\alpha) = \int_a^b w(x) \cdot k(\alpha, x) \cdot f_1(x) \, dx$$
and \( F_2(\alpha) = \int_{a}^{b} w(x) \cdot k(\alpha, x) \cdot f_2(x) \, dx \) exist.

Since \( \int_{a}^{b} w(x) \cdot k(\alpha, x) \cdot \left[ f_1(x) + f_2(x) \right] \, dx \) exists,

\((F_1 + F_2)(\alpha)\), the integral transform of \((f_1 + f_2)(x)\),

will exist. Therefore the set \( I_\tau \) is closed with respect to the addition.

The value of \( \alpha \) to be kept in the final result will depend upon the kernel chosen. For instance, when \( K(\alpha, x) = e^{-\alpha x} \), we have to choose \( \alpha \) to be the maximum of the abscissas of convergence of \( f_1 \) and \( f_2 \) for which the Laplace transform exists. But in the case of Fourier transform this is not necessary. In both cases above \( w(x) \) may be taken as unity.

The commutative and the associative properties are obvious. The identically zero function belonging to \( I_\tau \) is the additive identity for every \( F \in I_\tau \). The \(- F \in I_\tau \) is the additive inverse of \( F \in I_\tau \).

Therefore \( I_\tau \) is an abelian group with respect to the addition as the binary operation.
Theorem 2.3-2:

\[ I_\tau \text{ is a linear space.} \]

Proof:

\((1)\) In theorem 2.3-1 we have proved that \( I_\tau \) is an abelian group with respect to the binary composition as usual addition

\((2)\) If \( \beta \) is any real or complex number and \( F \in I_\tau \), then

\[
\int_a^b w(x) \cdot k(\alpha, x) \cdot (\beta \cdot f)(x) \, dx = \beta \int_a^b w(x) \cdot k(\alpha, x) \cdot f(x) \, dx
\]

= \( \beta \cdot F(\alpha) \) exists.

\[ \therefore \text{If } F \in I_\tau \text{ then } \beta F \in I_\tau \text{ for every } F \in I_\tau. \text{ The scalar multiplication defined as above satisfies the following conditions:} \]

\( \text{If } \beta_1, \beta_2 \text{ are any two real or complex numbers and } F_1, F_2 \in I_\tau, \text{ then} \)

\((3)\) \( (\beta_1 + \beta_2) \cdot F_1(\alpha) = \beta_1 F_1(\alpha) + \beta_2 F_2(\alpha) \)
(iv) \( \beta_1(\beta_2 F)(\alpha) = (\beta_1 \beta_2) F(\alpha) \)

(v) \( \beta_1(F_1 + F_2)(\alpha) = \beta_1 F_1(\alpha) + \beta_2 F_2(\alpha) \)

(vi) \( 1 F(\alpha) = F(\alpha) \)

Where 1 is the multiplicative identity for the set of all non-zero integers. Hence the set \( I_\tau \) is a linear space.

2.4 Topological Structure:

Since the set \( I_\tau \) contains real valued functions as well as complex valued functions, it is easy to see that the set of all real valued functions of \( I_\tau \) forms a linear subspace of the linear space \( I_\tau \). Let it be denoted by \( I'_\tau \). Now we shall discuss the topological structure of \( I'_\tau \) and \( I_\tau \).

Let \( \rho \) be a functional defined on \( I'_\tau \) as follows:

\[ \rho : I'_\tau \times I'_\tau \rightarrow \mathbb{R} \]

\[ (F_1, F_2) \rightarrow \rho(F_1, F_2) \]
where (i) \( R = \{ x \mid -\infty < x < \infty \} \)

(ii) \( I' \times I' \) is the cartesian product of \( I' \) into itself.

(iii) \( \rho(F_1, F_2) = \int_a^b w(x) \cdot k(x, x) \cdot (f_1(x) - f_2(x)) \, dx \)

In order that the metric defined should be unique, the value of \( a \) shall have to be fixed depending upon the restriction necessary for the convergence of the integral. In some cases above distance function defined in (1.4.1) may not be a metric. For instance in case of the set of Laplace transformed functions, S. Ganguly has defined

\[
d_c(F_1, F_2) = \sup_{S \geq 0} \left| \int_0^\infty e^{-sx} [F_1(S + S_{F_1} + \beta) - F_2(S + S_{F_2} + \beta)] \, dx \right|
\]

where \( \beta \) is a positive number however small and fixed once for all and \( S_{F_1}, S_{F_2} \) are the abcissas of convergence for \( F_1 \) and \( F_2 \) respectively, then \( d_c(F_1, F_2) \) is a Pseudo-metric. To transform the Pseudo-metric into a metric he added \( |S_{F_1} - S_{F_2}| \) to \( d_c(F_1, F_2) \) then

\[
\rho(F_1, F_2) = |S_{F_1} - S_{F_2}| + d_c(F_1, F_2)
\]

will be a metric.
As we have seen in chapter 1 that in the case of Fourier transform no restriction on $\alpha$ is necessary as $|e^{i\alpha x}|$ is equal to 1 and the distance function defined in (2.4.1) will be a metric for $w(x) = 1$, $k(\alpha, x) = e^{i\alpha x}$ on the set of all Fourier transformed functions $F$ of $f \in L^1$. Also if we define
\[
P(F) = \int_{-\infty}^{\infty} |e^{i\alpha x} f(x)| \, dx = \int_{-\infty}^{\infty} |f(x)| \, dx
\]
in the set of all Fourier transformed functions $F$ of $f \in L^1$, then $P(F)$ will be a norm and $P(F)$ induces a metric
\[
d(F_1, F_2) = P(F_1 - F_2)
\]
on it.

In general we shall prove the following theorem.

**Theorem 2.4-1:**

The set $I_\tau$ is a metric space with the metric $\rho$ defined on it in article No. 2.4.

**Proof:**

(i) Let $F_1, F_2 \in I_\tau$ such that

$F_1(\alpha) \neq F_2(\alpha)$ for every $\alpha$
then \( F(F_1, F_2) > 0 \) because \( f_1(x) \neq f_2(x) \) for every \( x \in (a, b) \).

If \( F_1(\alpha) = F_2(\alpha) \) then

\[
\int_a^b w(x) \cdot k(\alpha, x) \cdot f_1(x) \, dx = \int_a^b w(x) \cdot k(\alpha, x) \cdot f_2(x) \, dx
\]

\[
\implies f_1(x) = f_2(x) \text{ almost everywhere.}
\]

\[.\implies \int (F_1, F_2) = 0 \]

Conversely

If \( \int (F_1, F_2) = 0 \) then

\[
\int_a^b w(x) \cdot k(\alpha, x) \left| f_1(x) - f_2(x) \right| \, dx = 0
\]

\[
\implies f_1(x) - f_2(x) = 0,
\]

since \( w(x) \cdot k(\alpha, x) \neq 0 \).

\[.\implies f_1(x) = f_2(x) \text{ for all } x \in (a, b) \text{ almost everywhere.}
\]

Hence \( F_1(\alpha) = F_2(\alpha) \)
(ii) If $F_1, F_2 \in I^\prime_{\tau}$ then

\[
\rho(F_1, F_2) = \int_a^b |w(x) k(\alpha, x) [f_1(x) - f_2(x)]| \, dx
\]

\[
= \int_a^b |w(x) k(\alpha, x) [f_2(x) - f_1(x)]| \, dx
\]

\[
= \rho(F_2, F_1)
\]

\[\therefore \rho \text{ is symmetric.}\]

(iii) **Triangle inequality:**

Let $F_1, F_2, F_3 \in I^\prime_{\tau}$ then

\[
\rho(F_1, F_3) = \int_a^b |w(x) k(\alpha, x)[(f_1(x) - f_3(x)]| \, dx
\]

\[
\leq \int_a^b |w(x) k(\alpha, x) [f_1(x) - f_2(x)]| \, dx
\]

\[
+ \int_a^b |w(x) k(\alpha, x) [f_2(x) - f_3(x)]| \, dx
\]

\[\therefore \rho(F_1, F_3) \leq \rho(F_1, F_2) + \rho(F_2, F_3)\]
Therefore \((I_\tau, \rho)\) is a metric space. Hence the metric \(\rho\) induces a topology on \(I_\tau\).

Now we shall consider the set \(I_\tau\) and we shall prove that \(I_\tau\) is also a metric space with a suitable metric defined on it.

**Theorem 2.4-2:**

The set \(I_\tau\) is a metric space.

**Proof:**

Let \(d\) be a real valued functional defined on \(I_\tau\) as follows:

\[
d: I_\tau \times I_\tau \rightarrow R
\]

\[
(F_1, F_2) \rightarrow d(F_1, F_2)
\]

where (i) \(I_\tau \times I_\tau\) is the cartesian product of \(I_\tau\) into itself.

(ii) \(R = \{x| -\infty < x < \infty\}\)

\[
(2.4.-1) (iii) d(F_1, F_2) = \sqrt{\int_a^b \left| w(x) \cdot k(\alpha, x) \cdot [f_1(x) - f_2(x)] \right|^2 dx}
\]

where the complex or real valued functions \(F_1, F_2\) are
integral transforms of the functions \( f_1 \) and \( f_2 \) respectively.

It is obvious to see that \( d \) is a metric on \( I_{\tau} \). Hence \( d \) induces a topology on \( I_{\tau} \).

In order that the metric defined as above should be unique, the value of \( \alpha \) shall have to fixed depending upon the restriction necessary for the convergence of the integral and in some cases the distance function defined in (2.4.2) is not a metric but it will be a pseudo-metric. For instance in the case of Laplace transformed functions if

\[
(2.4.3) \quad \int_{\tau}^{\infty} |e^{-s_f} f_1 - e^{-s_f} f_2|^2 dx
\]

where \( s_{f_1} \) and \( s_{f_2} \) are the abscissas of convergence for \( f_1 \) and \( f_2 \) respectively, then \( \rho \) will be a pseudo-metric but if we add \( |s_{f_1} - s_{f_2}| \) in (2.4.3) then

\[
|s_{f_1} - s_{f_2}| + \int_{\tau}^{\infty} |e^{-s_f} f_1 - e^{-s_f} f_2|^2 dx \text{ will be a metric on } I_{\tau}. \quad \text{But in the case of Fourier}
\]
transform no restriction on \( \alpha \) and no addition in (2.13)
is necessary as \( |e^{i\alpha x}| \) is equal to one and the metric
will be unique.

Since \( I'_\tau \) is a subspace of a topological
space \( I_\tau \), \( I'_\tau \) has a relative topology with respect
to the metric \( d \) defined on \( I_\tau \). The topology (relative topology) generated by a metric \( d \) on \( I'_\tau \) is
not the same as the topology generated by the metric
\( \rho \) defined on \( I'_\tau \). As far as subspace \( I'_\tau \) is concerned
the metrics \( \rho \) and \( d \) are not equivalent.

We shall now see that which of the algebraic
operations are continuous in the metric topologies intro-
duced in \( I'_\tau \) and \( I_\tau \).

By imposing some conditions on \( w(x), k(\alpha, x) \)
in case of Laplace transformed functions, S. Ganguly \[ 1 \]
has proved that in the set of all Laplace transformed
functions the addition and the scalar multiplication
operations are not continuous with respect to the
metric he has defined on it. But in case of the Fourier
integral transform, since \( w(x) = 1, k(\alpha, x) = e^{i\alpha x} \) and
the metric \( \rho \) is as defined in article 2.4, in the set
of all Fourier transformed functions, in chapter 1 we have
proved that the addition and scalar multiplication operations are continuous and also that the sets of all Fourier transformed and transformable are complete. These have been possible because $|e^{iax}| = 1$. In general the space of integral transformed functions with a linear Kernel need not be either complete or a linear topological space.

2.5 Some theorems on the set of integral transformable functions:

In this article we shall prove that set $D_{\tau}$ of all integral transformable functions is a topological space.

Theorem 2.5-1:

The set $D_{\tau}$ is a linear space.

Proof:

(i) Let $f_1, f_2 \in D_{\tau}$ then $\int_a^b w(x) k(\alpha, x) \left[ f_1(x) + f_2(x) \right] dx$ exists in Lebesgue sense. Therefore $(f_1 + f_2) \in D_{\tau}$. 

Hence $D_{\tau}$ is closed with respect to the
binary composition as usual addition.

(i) The associative and commutative properties can be similarly proved.

(iii) The null element and the additive inverse of a function \( f \in D \) are respectively the identically zero function and \(-f \in D\).

Hence the set \( D \) is an abelian group.

(iv) If \( f \in D \), then

\[
\int_{a}^{b} w(x) \cdot k(a, x) \cdot (\beta \cdot f)(x) \, dx = \beta \int_{a}^{b} w(x) \cdot k(a, x) \cdot f(x) \, dx
\]

exists, where \( \beta \) is a scalar.

\[
\beta f \in D.
\]

The scalar multiplication defined as above satisfy the following:

(v) \( \alpha (\beta f)(x) = (\alpha \beta) f(x) \)

(vi) \( (\alpha + \beta) f(x) = \alpha f(x) + \beta f(x) \)

(vii) \( \alpha [f_1(x) + f_2(x)] = \alpha f_1(x) + \alpha f_2(x) \)

(viii) \( 1 \cdot f(x) = f(x) \)
Hence the set \( D^\tau \) is a linear space.

2.6 Topological Structure in \( D^\tau \).

Let \( \rho^\tau \) be a functional defined on \( D^\tau \) as follows:

\[
\rho^\tau : D^\tau \times D^\tau \rightarrow R
\]

\[
(f_1, f_2) \rightarrow \rho^\tau (f_1, f_2)
\]

where

(i) \( R = \{ x | -\infty < x < \infty \} \)

(ii) \( D^\tau \times D^\tau \) is the cartesian product of \( D^\tau \) into itself.

(iii) \[
\rho^\tau (f, g) = \frac{\int_a^b |w(x) \cdot k(\alpha, x) \cdot [f(x) - g(x)]| \, dx}{1 + \int_a^b |w(x) \cdot k(\alpha, x) \cdot [f(x) - g(x)]| \, dx}
\]

where \( f, g \in D^\tau \) and \( \alpha \) has to be kept fixed as in article No. 4.

Theorem 2.6-1:

The set \( D^\tau \) is a metric space with respect to the metric \( \rho^\tau \).

The proof is obvious.
Similarly one can also prove that the \( D_{\tau} \) is also a metric space with respect to the metric

\[
\rho(f, g) = \int_{a}^{b} |w(x) \cdot k(\alpha, x) \cdot [f(x) - g(x)]| \, dx
\]

where \( f, g \in D_{\tau} \) and \( \alpha \) has to be kept fixed as in article No. 2.4. Hence \( \rho_{2} \) and \( \rho_{3} \) induces the topologies on \( D_{\tau} \).

S. Ganguly [2] has proved that the set of Laplace transformable functions is not a linear metric space but it is complete and disconnected with respect to the metric he has defined in it.

In the case of the set of Fourier integral transformable functions \( f \in L^{1} \) if we define a functional on it as follows:

\[
P(f) = \int_{-\infty}^{\infty} |f(x)| \, dx,
\]

then this functional will be a norm on it, which induces a metric \( \rho_{3}(f, g) = P(f-g) \) which is defined in (2.6). Therefore the set of all Fourier integral transformable functions \( f \in L^{1} \) is a normed linear space with respect to the above norm. It is also easy to prove that with respect to the metric \( \rho_{3} \), the set of all Fourier integral transformable functions \( f \in L^{1} \) is a linear metric space (i.e. in the set of all
Fourier integral transformable functions $f \in L^1$, the addition and the scalar multiplication operations are continuous) and complete. Hence the set all Fourier integral transformable functions $f \in L^1$ is a complete linear topological space.
REFERENCES


