CHAPTER IV

AXISYMMETRIC DIRICHLET PROBLEM FOR A THICK PLATE WITH BOUNDARY CONDITIONS AS DISTRIBUTIONS.

The main results of this chapter have been communicated for publication as detailed below:

1. Axisymmetric Dirichlet problem for a thick plate with boundary conditions as distributions.

4.1 Introduction:

The formulations of boundary-value problems are generally characterized by the fact that their initial and boundary conditions are sufficiently smooth. In many practical problems there may be singularities in the initial and boundary conditions. Such problems can not be solved with the help of conventional functions. These problems can be correctly specified with the help of generalized functions, of which the distributions are particular cases.

In this chapter we shall solve the axisymmetric Dirichlet problem for a thick plate with boundary conditions as distributions by the application of Hankel Transform of generalized functions.

4.2 Definitions:

A continuous linear functionals \( < f, o > \) on some fundamental space are called generalized functions [1,p.82]. Hence in contrast to the conventional functions, generalized functions are not defined in themselves but depend on a selected space.

Let \( I = (0, \infty) \) and \( x \in I \). \( H_n \) denotes the
linear space consisting of all smooth complex-valued
functions \( \phi(x) \) on \( I \) such that, for every pair of
non-negative integers \( m \) and \( k \), the numbers

\[
(4.2-1) \quad \gamma_{m,k}^\mu (\phi) = \sup_{\alpha \in I} |x^m (x^{-1}D)^k x^{-\mu - \frac{1}{2}} \phi(x)|
\]

exist. The set of semi-norms \( \{\gamma_{m,k}^\mu\} \) generates the
topology of \( H_\mu \). The dual space \( H'_\mu \) of \( H_\mu \) is
the space of generalized functions defined on \( H_\mu \). \( \mathcal{E}(I) \)
denotes the space of all complex valued smooth functions
defined on \( I \). Its dual \( \mathcal{E}'(I) \) is the space of distribu-
tions with compact support on \( I \) \([2,p.36]\). The ordi-
nary Hankel transformation \( h_\mu \) is an automorphism on
\( H_\mu \) whenever \( u > - \frac{1}{2} \). The generalized Hankel transfor-
mation \( h'_\mu \) on \( H'_\mu \) is defined by

\[
(4.2-2) \quad < h'_\mu f, \phi > = < f, h_\mu \phi >
\]

where \( \phi, h_\mu \phi \in H_\mu \) and \( f \in H'_\mu \) \([2,p.141]\). When
\( f \in \mathcal{E}'(I) \) and \( u > - \frac{1}{2} \) the Hankel transform \( F = h'_\mu f \)
of \( f \) takes on the form

\[
(4.2.3) \quad F(y) = < f, \sqrt{xy} \cdot J_\mu(xy) >
\]
4.3 Statement of the Problem:

The problem we have considered is the problem of axisymmetric Dirichlet problem for a thick plate with boundary conditions as distributions. The problem can be stated as follows:

Find the harmonic function \( u(\rho, z) \) in the thick plate \( |z| \leq b \) on the domain

\[
\mathcal{C} = \{ (\rho, z) \mid 0 < \rho < \infty, |z| \leq b \}
\]

which satisfies the equation in cylindrical co-ordinates

\[
(4.3-1) \quad \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \rho} + \frac{\partial^2 u}{\partial z^2} = 0
\]

with the following boundary conditions:

(a) \( u(\rho, z) \) converges in some generalized sense to the distribution \( f(\rho) \in \mathcal{C}'(I) \) as \( z \to b \).

(b) \( u(\rho, z) \) converges in some generalized sense to the distribution \( g(\rho) \in \mathcal{C}'(I) \) as \( z \to -b \).

4.4 Solution of the Problem:

Let \( \omega(\rho, z) = \frac{1}{\rho} u(\rho, z) \), then we can write (4.3-1) as follows:
\[(4.4-1) \quad M_0 N_0 \omega + \frac{\partial^2 \omega}{\partial z^2} = 0\]

where \([2, p.154]\) \(M_0 N_0 \omega = \int \frac{1}{p} \frac{\partial}{\partial \rho} \frac{\partial}{\partial \rho} \int \frac{1}{p} \omega.\)

By applying Hankel transformation of order zero to \((4.4-1)\) and formally inter-changing \(h_0\) with \(\frac{\partial^2}{\partial z^2}\) we get

\[(4.4-2) \quad -\int \frac{1}{2} U(\xi, z) + \frac{\partial^2}{\partial z^2} U(\xi, z) = 0\]

where \(U(\xi, z) = h_0 \omega(\rho, z)\). The appropriate solution of equation \((4.4-2)\) is

\[(4.4-3) \quad U(\xi, z) = A(\xi) \sin h\left[\frac{\xi}{2} (b+z)\right] + B(\xi) \sin h\left[\frac{\xi}{2} (b-z)\right]\]

In view of boundary conditions \((a)\) and \((b)\) we have the additional conditions:

\[(4.4-4) \quad h_0 \omega(\rho, Z) \rightarrow h_0 \frac{1}{\rho} f(\rho) \text{ as } z \rightarrow b \quad \text{and}\]

\[(4.4-5) \quad h_0 \omega(\rho, z) \rightarrow h_0 \frac{1}{\rho} g(\rho) \text{ as } z \rightarrow -b \quad \text{as } z \rightarrow -b\]

Then by \((4.4-4)\) and \((4.4-5)\) we have respectively

\(U(\xi, b) = A(\xi) \sinh \left[2 \frac{\xi}{2} b\right]\) and

\(U(\xi, -b) = B(\xi) \sin h \left[2 \frac{\xi}{2} b\right].\)
\[ A(\xi) = U(\xi, b) \cdot \text{Cosech}(2\xi b) \quad \text{and} \]
\[ B(\xi) = U(\xi, -b) \cdot \text{Cosech}(2\xi b) \]

Hence by theorem 5.6-3 [2, p.147] we have

\[ U(\xi, b) = \langle \sqrt{p} f(p), \sqrt{p} J_0(\xi p) \rangle \quad \text{and} \]
\[ U(\xi, -b) = \langle \sqrt{p} g(p), \sqrt{p} J_0(\xi p) \rangle \]

Hence (4.4-3) becomes

\[ U(\xi, z) = \left[ \sqrt{p} f(p), \sqrt{p} J_0(\xi p) \right] \cdot \text{Sinh} \left[ \xi (b+z) \right] \]
\[ + \left[ \sqrt{p} g(p), \sqrt{p} J_0(\xi p) \right] \cdot \text{Sinh} \left[ \xi (b-z) \right] \cdot \text{Cosech}(2\xi b) \]

For each fixed \( z \), by theorems 5.6-1 and 5.6-2 [2, p.146], \( U(\xi, z) \) is a smooth function of \( \xi \) in \( L_1(0, \infty) \).

Therefore we may apply the conventional inverse Hankel transformation to get our formal solution

\[ (4.4-6) \quad \mathcal{U}(p, z) = \int_0^{\infty} U(\xi, z) \cdot \sqrt{p} J_0(\xi p) d\xi \]

defined on \(-\infty, \infty\).

Now we see that (4.4-6) is the solution of (4.4-1).
We know that \( J_0(\frac{\rho z}{\mathcal{q}}) \) and \( J_1(\frac{\rho z}{\mathcal{q}}) \) are bounded on \( 0 < \rho < \mathcal{q} \) and by theorems 5.6-1 and 5.6-2, we may interchange the differentiation in (4.4-1) with the integration in (4.4-6) since at every step the resulting integral converges uniformly on every compact subset of the domain \( \mathcal{q} \). Since the integrand in (4.4-6) satisfies (4.4-1) we conclude that \( \psi(\rho, z) \) satisfies the equation (4.4-1).

For, from (4.4-6) we have

\[
\frac{\partial \psi}{\partial z} = \int_0^\infty \left[ \text{coth} \left( \frac{z b}{2} \right) \left( \cosh \left( \frac{z b}{2} + \frac{z^2}{2} \right) \cdot \frac{z}{2} + \text{sh} \left( \frac{z b - z^2}{2} \right) \cdot \frac{z}{2} \right) \right] \frac{1}{\sqrt{z^2 + \mathcal{q}^2}} J_0(\mathcal{q} z) \, dz.
\]

(4.4-7) \[
\frac{\partial^2 \psi}{\partial z^2} = \int_0^\infty \left[ \text{coth} \left( \frac{z b}{2} \right) \left( \sinh \left( \frac{z b}{2} + \frac{z^2}{2} \right) \cdot \frac{z}{2} + \text{sh} \left( \frac{z b - z^2}{2} \right) \cdot \frac{z}{2} \right) \right] \frac{1}{\sqrt{z^2 + \mathcal{q}^2}} J_0(\mathcal{q} z) \, dz.
\]

(4.4-8) \[
\left( \frac{\partial^2 \psi}{\partial z^2} \right) = -\int_0^\infty \left[ \text{coth} \left( \frac{z b}{2} \right) \left( \sinh \left( \frac{z b}{2} + \frac{z^2}{2} \right) + \text{sh} \left( \frac{z b - z^2}{2} \right) \right) \cdot \frac{z^2}{\sqrt{z^2 + \mathcal{q}^2}} J_0(\mathcal{q} z) \, dz \right]
\]
By adding (4.4-7) and (4.4-8) we get

\[ M_0 N_0 \omega + \frac{\partial^2 \omega}{\partial z^2} = 0 \]

Hence the result. Therefore \( u(p,t) \) satisfies (4.3-1).

We now prove that the initial condition (a) is satisfied. As a function of \( p \), for fixed \( z \), \( U(\xi, z) \) is smooth and for each fixed \( z \) such that \(|z| \leq b\), it is absolutely integrable by virtue of theorems 5.6-1 and 5.6-2 [2,p.146]. Therefore \( U(\xi, z) \) satisfies the conditions under which the ordinary Hankel transformation \( h_0 \) is a special case of generalized Hankel transformation \( h_0 \). According to (4.4-6) its Hankel transformation is \( \omega(p, z) \), so that, for any \( \phi \in H_0 \) and \( \mathcal{D} = h_0 \phi \), by the definition of generalized Hankel transformation we have

\[ \langle \omega(p, z), \phi(z) \rangle \]

\[ = \int_0^\infty U(\xi, z) \cdot \mathcal{D}(\xi) \, d\xi \]

\[ = \int_0^\infty \left[ \left( \langle \sqrt{p} \cdot \mathcal{D}(\xi), \sqrt{p} \xi \cdot \mathcal{D}(\xi) \rangle \cdot \sinh (\xi(b+z)) \right) + \langle \sqrt{p} \cdot \mathcal{D}(\xi), \sqrt{p} \xi \cdot \mathcal{D}(\xi) \rangle \cdot \sinh (\xi(b-z)) \cdot \coth \left( \frac{\xi b}{2} \right) \right] \mathcal{D}(\xi) \, d\xi. \]
Since both the terms under integration are bounded the integral on the right hand side converges uniformly on $0 \leq \rho \leq \infty$. Thus we may interchange the limiting process as $z \to b$ with the integration to get

\begin{equation}
(4.4-10) \quad \lim_{z \to b} \left< U(\rho, z), \phi(z) \right> = \int_0^\infty \left< \sqrt{\rho} f(p), \sqrt{\rho} \cdot J_0(\xi p) \right> \cdot \Xi(\xi) d\xi
\end{equation}

\[ = \left< \sqrt{\rho} f(p), \phi(z) \right> \text{ by virtue of sec. 5.5 equation (4) [2, p.143].} \]

\[ \therefore U(\rho, z) \to \sqrt{\rho} f(p) \text{ in the sense of convergence in } H_0. \]

Hence $U(\rho, z) \to f(\rho)$ in a generalized sense. Similarly $U(\rho, z) \to g(\rho)$ in a generalized sense.

From (4.4-6) we easily derive the formula
\[(4.4-11) \quad \mathcal{U}(\rho, \alpha) = \int_0^\infty \left[ \langle \sqrt{p} \ f(p), \sqrt{p} \ \mathcal{J}_0(\sqrt{p}) \rangle \\
+ \langle \sqrt{p} \ g(p), \sqrt{p} \ \mathcal{J}_0(\sqrt{p}) \rangle \right] \text{Sech} \ \alpha \ \mathcal{J}_0(\alpha \ \sqrt{p}) \ dp \\
= h_0 \left[ \frac{1}{2} \left( \langle \sqrt{p} \ f(p), \sqrt{p} \ \mathcal{J}_0(\sqrt{p}) \rangle \\
+ \langle \sqrt{p} \ g(p), \sqrt{p} \ \mathcal{J}_0(\sqrt{p}) \rangle \right) \right] \text{Sech} \ \alpha \quad \text{and} \quad \text{coSech} \ \alpha \]
and \[ \frac{\partial (\varphi, 0)}{\partial z} = 0 \]

4.5 Concluding Remarks:

The choice of \( f(\rho) \) and \( g(\rho) \in \mathcal{E}'(I) \) as the boundary temperature distributions at the free surfaces is quite justifiable, because, for instance the temperature distributions are caused due to sudden application of impulsive sources, then the distributions of temperature of free surfaces at the moment can not be correctly specified by conventional functions. We may be able to know about \( f(\rho) \) and \( g(\rho) \) from the effect they later on produce. It is such situations which are depicted by generalized functions since in our choice \( f(\rho) \) and \( g(\rho) \) has a compact support, it justifies the phenomena that beyond certain bounded and closed domain the temperature distribution of the free surface is zero, because however large the magnitude of the impulsive sources may be, its effect beyond certain closed and bounded domain will be zero provided that are visualising that the expanse of free surfaces is fairly large.
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