The main results established in this chapter have been published as detailed below:

1. Topology of Laplace transformable functions of two variables.

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3.1 Introduction:

Let $f(x,y)$ be a real or complex valued function of two variables, defined on the region $R^*(0 \leq x \leq \infty, 0 \leq y \leq \infty)$ and integrable in the sense of Lebesgue over an arbitrary finite rectangle $R_{a,b}(0 \leq x \leq a, 0 \leq y \leq b)$. If $P \equiv \sigma + i \mu$, $Q = \sigma + i \nu$ and the integral

$$
(3.1-1) \quad F(P,q,a,b) = \int_0^a \int_0^b e^{-(px+qy)} f(x,y) \, dx \, dy
$$

satisfy the following conditions:

(i) the integral (3.1-1) is bounded at the point $(P,q)$ with respect to the variables $a$ and $b$ i.e. $F(P,q : a,b) < M(P,q)$ for all $a > 0$ and $b > 0$, where $M(P,q)$ is a positive constant independent of $a$ and $b$.

(ii) at the point $(P,q)$

$$
\text{limit } F(P,q,a,b) = F(P,q) \text{ exists,}
$$

\[ a \to -\infty \]

\[ b \to -\infty \]
where

\[(3.1-2) \quad F(P,q) = \int_0^\infty \int_0^\infty e^{-(Px+qy)} f(x,y) \, dx \, dy\]

then the integral (3.1-2) is called the two dimensional Laplace transform of the function \(f(x,y)\) \([5, P.4]\). If the conditions (i) and (ii) are satisfied simultaneously, then the integral (3.1-2) is said to converge boundedly in at least one point \((P,q) \in D\), the region of convergence of the Laplace integral \([5, P.8]\).

Ganguly and Mukherjee \([1]\), Dutta and Ganguly \([2,6]\) have proved that the set of all Laplace transformable functions of one variable is metrizable and that it is not necessarily a linear metric space. It has been already shown by Doetsch (1943) that the set of Laplace transformable functions is not a Hilbert space. According to Doetsch the object spaces are defined as spaces of functions for which the Laplace transforms exist. Dutta \([3]\) has revealed some interesting features of the abstract structure of the integral transform theory and for simplicity and convenience he has taken the object space and the image space to be the Hilbert spaces.
In this chapter we shall extend S.Ganguly's results to the Laplace transform of two variables. We shall consider the set $S$ of all Laplace transformable functions of two variables for which the integral $(3.1-2)$ is boundedly convergent. We shall find out some suitable topology which can be introduced in the space of Laplace transformable function of two variables for which the integral $(3.1-2)$ converges boundedly. In order to achieve this we shall first introduce the algebraic structure in the set $S$. We further prove that this set is a linear system and it is a Pseudometric space with a Pseudometric defined in article No. 3.3. It has been also shown that with a suitable equivalence relation $R$ defined on $S$, the Pseudometric becomes a metric on $S/R$ (Quotient set).

3.2 Algebraic Structure:

In this article we shall prove two theorems.

Theorem (3.2-1):

The set $S$ is an abelian group with respect to the binary operation as usual addition.
Proof:

Let \( f_1, f_2 \in S \), then there exist points \( (P_1, q_1), (P_2, q_2) \in D \) such that \( \int \int e^{-((P_1 x + q_1 y))} f(x, y) \, dx \, dy \)

and \( \int \int e^{-((P_2 x + q_2 y))} f_2(x, y) \, dx \, dy \) converges boundedly in \( D \) in Lebesgue sense. Let \( (P, q) \) be any point of \( D \) such that \( \Re(P-P_1) > 0 \), \( \Re(q-q_1) > 0 \), \( \Re(P-P_2) > 0 \)

and \( \Re(q-q_2) > 0 \). Then we define uniquely the sum of the two Laplace transformable functions of two variables as follows:

\[
\int \int e^{-(P x + q y)} \left[ f_1(x, y) + f_2(x, y) \right] \, dx \, dy
\]

where the integral converges boundedly in \( D \) hence \( (f_1 + f_2) \in S \).

Now it is easy to see that the set \( S \) is closed with respect to the binary operation as usual addition. The associative and commutative properties are obvious. Since \( f \in S \), then \(-f \in S\). Therefore the additive inverse of \( f \) is \(-f\) and the additive
identity is identically zero function. Thus the set \( S \) is an abelian group.

**Theorem (3.2-2):**

The set \( S \) is a linear system.

**Proof:**

(i) In theorem (3.2-1) we have proved that \( S \) is an abelian group with respect to the binary operation '0' defined on it as follows:

\[
0 : S \times S \rightarrow S
\]

\[
(f_1, f_2) \rightarrow 0(f_1, f_2)
\]

where \( 0(f_1, f_2) = f_1 \odot f_2 = f_1 + f_2 \)

(ii) If \( f \in S \) and \( \alpha \) be any scalar (real or complex) then for \( (P, q) \in D \) we have

\[
\iiint e^{-(Px + qy)} \cdot (\alpha f)(x, y) \, dx \, dy
\]

\[
= \iiint e^{-(Px + qy)} \cdot \alpha \cdot f(x, y) \, dx \, dy
\]

\[
= \alpha \iiint e^{-(Px + qy)} \cdot f(x, y) \, dx \, dy
\]
Hence $\alpha f \in S$ for a real or complex $\alpha$. It is obvious to see that the following conditions are easily satisfied.

(iii) $\alpha(\beta f) = (\alpha \beta) f$ for every $f \in S$, where "$\cdot$" denotes the usual multiplication in the set of real or complex numbers and $\alpha, \beta$ are scalars.

(iv) $(\alpha + \beta) f = \alpha f + \beta f$ for every $f \in S$.

\[
\alpha(f_1 + f_2) = \alpha f_1 + \alpha f_2 \text{ for every } f_1, f_2 \in S.
\]

(v) $1 \cdot f = f$, where 1 is the multiplicative identity element in the field $F$ of real numbers.

(vi) $0 \cdot f = 0$, where 0 is the null element for both $S$ and the field $F$.

(vii) If $\alpha f = 0$, then $\alpha = 0$ if $f \not= 0$

Thus the set $S$ is a linear system.

3.3 Metrization:

In this article we shall need the following theorem [5, page 21].
Theorem:

In order that \( f(x,y) \) shall belong to the class \( S \), it is necessary and sufficient that the inequality

\[
(3.3-1) \quad \left| \int_0^a \int_0^b f(x,y) \, dx \, dy \right| < M
\]

holds for one pair of values \((a, \beta)\), \( a > 0, \beta > 0 \), where \( a > 0 \) and \( b > 0 \).

Define

\[
\alpha_0 = \inf \{ \alpha \mid \alpha > 0 \text{ and (3.3-1) holds} \}
\]

\[
\beta_0 = \inf \{ \beta \mid \beta > 0 \text{ and (3.3-1) holds} \}
\]

Now \( \alpha_0 > 0 \) and \( \beta_0 > 0 \). Let \( C > 0 \) be a fixed constant. Define \( P_f \), \( q_f \) as follows:

\[
R \, P_f = \alpha_0 + C, \quad R \, q_f = \beta_0 + C
\]

Let \( f, g \in S \), we define a functional \( d \) on \( S \) as follows:

\[
d : S \times S \to \mathbb{R}
\]

\[
(f, g) \mapsto d(f, g)
\]
where (i) \( R = \{ z \mid -\infty < z < \infty \} \), \( x, y \in R \)

\[ (ii) \quad d(f, g) = \int_0^\infty \left( e^{-Pf x + q_f y} f(x, y) - e^{-(Pg x + q_g y)} g(x, y) \right) dx \, dy \]

Now we prove the following theorem for \( d(f, g) \).

**Theorem (3.3-1):**

\[ d \text{ is a Pseudometric on } S. \]

**Proof:**

(i) for \( f, g \in S \), if \( f(x, y) = g(x, y) \) for every \( x, y \in R \), then \( Pf = Pg \), \( q_f = q_g \).

Therefore \( d(f, g) = 0 \)

Conversely if \( d(f, g) = 0 \) for every \( f, g \in S \) then it is not necessary that \( f = g \).

(ii) \( d \) is symmetric.

Let \( f, g \in S \), then
\[ d(f,g) = \left| \int_0^\infty \int_0^\infty \left( e^{- (P_f x + q_f y)} f(x,y) - e^{- (P_g x + q_g y)} g(x,y) \right) \, dx \, dy \right| \]

\[ = \left| \int_0^\infty \int_0^\infty \left( e^{- (P_g x + q_g y)} g(x,y) - e^{- (P_f x + q_f y)} f(x,y) \right) \, dx \, dy \right| \]

\[ = d(g,f). \]

(iii) \( d \) satisfies triangle inequality.

Let \( f, g, h \in S \), then

\[ d(f,h) = \left| \int_0^\infty \int_0^\infty \left( e^{- (P_f x + q_f y)} f(x,y) - e^{- (P_h x + q_h y)} h(x,y) \right) \, dx \, dy \right| \]

\[ \leq \left| \int_0^\infty \int_0^\infty \left( e^{- (P_f x + q_f y)} f(x,y) - e^{- (P_g x + q_g y)} g(x,y) \right) \, dx \, dy \right| \]

\[ + \left| \int_0^\infty \int_0^\infty \left( e^{- (P_g x + q_g y)} g(x,y) - e^{- (P_h x + q_h y)} h(x,y) \right) \, dx \, dy \right| \]

Therefore \( d(f,h) \leq d(f,g) + d(g,h) \) hence \( d \) is a \textit{Pseudometric} on \( S \) and it generates a \textit{Pseudometric topology} on \( S \).

Since \( d \) is a \textit{Pseudometric} on \( S \),

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to introduce a metric on $S$, we shall define the following relation on $S$. We say that $f \mathrel{R} g$, if $d(f, g) = 0$.

It is obvious to see that the above relation defined on $S$ is an equivalence relation on $S$. Therefore this equivalence relation partitions the set $S$ into equivalence classes. Each class consists of all functions which are equivalent to a given one. If $F, G \in S/R$ (Quotient set) then for $f \in F, g \in G$, we define $d(F, G) = d(f, g)$. Now it is easy to see that $d(F, G)$ is well-defined and the set $S/R$ is a metric space.

If $f \mathrel{R} f_1$ and $g \mathrel{R} g_1$ then we have

$$(f + g) \mathrel{R} (f_1 + g_1) \quad \text{and} \quad (\alpha f) \mathrel{R} (\alpha f_1)$$

where $\alpha$ is a scalar. Therefore the set $S/R$ of all equivalence classes is also a linear space.

It will be observed that it is easy to prove that the scalar multiplication for the Pseudometric defined in (3.3) is continuous.
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