CHAPTER 5
STATE FEEDBACK CONTROL OF NANOPOSITIONING SYSTEM

In Chapter 4, the characteristics and performance of the nanopositioning system using conventional controllers have been discussed. In this chapter, another kind of controller i.e. state feedback controller designed by pole placement technique and linear quadratic regulator (LQR) has been addressed. To implement a closed-loop state feedback controller, information about system states has to be available. In present case only some of the states are measurable, so, a state observer has been designed which delivers the information about the states so that they can be used for control. Observer based state feedback controller has also proposed. Analysis has been presented for the conventional and state space feedback controllers. It has also been shown that state space feedback controllers perform better than conventional controllers.

5.1 INTRODUCTION
The main disadvantage of the conventional controllers such as PI, PID and PII controllers is that poles cannot be placed arbitrarily by choosing any value of controller gain $K_p$ and $K_i$. Generally the conventional control design approach consists of varying the controller’s transfer function until a desired closed loop performance is achieved. The major limitation of the classical design approach is that for a larger order system, variation of a limited number of constants in the controller transfer function can vary in a specified manner the location of only a few of the closed loop poles, but not all of them. So, a higher order controller is required to have full freedom for arbitrarily pole placement [161].

Further system design according to classical control theory based upon the trial and error, does not generate an optimum control system. Conversely, design of system according to the modern control theory using state space approach allows designer to design a system having a desired characteristic equation or an optimum
control for a given performance index. However, to design a system according to modern control theory using state space approach, mathematical representation of system’s dynamic characteristic is essential. In classical scheme, mathematical representation of system’s dynamic is not required. [185-186]. To implement a closed-loop state feedback controller, information about system states has to be available. If all the states of the system are not measurable, an estimator/observer must be designed which delivers information about the states so that they can be used for control. In this chapter observer based state feedback controller is presented.

5.2 STATE FEEDBACK CONTROLLER

System’s dynamic response in terms of rise time, settling time, maximum overshoot, stability margins and other measures of transient and frequency responses can be changed by using a closed loop system in which a controller is designed to place the closed loop poles at the desired location. Placing poles at specified location is desirable because the location of the poles corresponds directly to the eigenvalues of the system, which control the characteristics of the response of the system. The advantage of this approach is that the stability of the closed loop system is guaranteed by placing the closed loop poles in the stable region i.e left half of s-plane for continuous time system. State feedback is a method employed in feedback control system to place the closed-loop poles of a system at predetermined locations in the s-plane. State feedback controllers can place all poles of the closed loop system at desired location arbitrarily to achieve a satisfactory dynamic response and develops the control law for the closed-loop system that corresponds to satisfactory dynamic response. The state space approach using a full state feedback provides sufficient number of controller design parameters to move all the closed loop poles independently of each other [161-162,186].

The dynamics of the system can be represented by state and output equation as

\[ \dot{x} = Ax + Bu \]  \hspace{1cm} (5.1)

\[ y = Cx \]  \hspace{1cm} (5.2)
Where \( x \) is state vector of the system, \( u \) is control signal, \( y \) is output signal, \( A \) is \( n \times n \) state matrix, \( B \) is \( n \times r \) input matrix, \( C \) is \( r \times n \) output matrix, \( n \) is the order and \( r \) is the number of inputs of the system.

In state feedback controller input vector ‘\( u \)’ can be generated according to a control law [161]

\[
u = K[x_d - x] - K_d x_d
\]  

(5.3)

where \( x \) is the state vector of the plant and \( x_d \) is the desired state vector. \( K \) and \( K_d \) are the controller gain matrices and are the design parameters of the control system. The aim of controller is to achieve the desired state vector \( x_d \) in the steady state and eliminate the effect of noise.

To design a regulator, desired state vector \( x_d = 0 \) and assume that all the undesired inputs to the system i.e. noise are zero.

Then equation 5.3 becomes

\[
u = -Kx = [K_1 \ K_2 \ldots \ldots \ldots \ K_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
\]  

(5.4)

where \( K \) is state feedback gain matrix. For a system of the order of \( n \) with \( r \) inputs, the size of \( K \) will be \( (r \times n) \). Since there are \( n \) roots of the system, there is enough degree of freedom to select arbitrarily any desired roots location by choosing the proper value of \( K \). Schematic diagram of full state feedback regulator is shown in figure 5.1.
Substituting the feedback law given by equation 5.1, the state and output equations becomes
\[ \dot{x} = (A - BK)x \]  
\[ y = Cx \]  
(5.5)

The regulator is homogeneous system and is described by the closed loop state coefficient matrices
\[ A_{cl} = A - BK, B_{cl} = 0, C_{cl} = C \text{ and } D_{cl} = 0 \]  
(5.7)

The characteristics equation of the closed loop system will be
\[ \text{det}[sI - (A - BK)] = 0 \]  
(5.8)

The solution of equation (5.5) is
\[ x = e^{A-BK}x(0) \]  
(5.9)

Where \( x(0) \) is the initial state conditions caused by external disturbances. The stability and time response characteristics of the closed loop system are determined by the regulator poles i.e eigenvalues of the closed loop matrix \((A-BK)\).

Matrix \( K \) is not unique for a given system but it depends upon the selection of the desired closed loop poles locations. By the proper choice of \( K \), the matrix \( A-BK \) can be made asymptotically stable matrix and \( x(t) \) can approach to zero as \( t \) approaches infinity for all \( x(0) \neq 0 \). When the system is disturbed by a non-zero initial conditions, performance specification of the system such as settling time, rise time and maximum overshoot are determined by the location of closed loop poles.

### 5.3 State Feedback Controller Using Pole Placement Technique

Design of a controller using pole placement technique is to design regulator gain matrix \( K \) by selecting the desired closed loop pole location. To design regulator gain matrix \( K \), plant must be in controllable canonical form. Let a plant of the order \( n \) is described by its controllable canonical form as
Where $a_0, a_1, \ldots, a_{n-1}$ are the coefficients of plant's characteristics polynomial

$$|sI - A| = s^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0 \quad (5.11)$$

The full state feedback gain matrix is a row vector of $n$ unknown parameters given as

$$K = [K_1 \ K_2 \ K_3 \ldots \ldots \ K_n] \quad (5.12)$$

The closed loop poles are placed so that the characteristics polynomial of the closed loop system is

$$|sI - A_{cl}| = |sI - (A - BK)| = 0 \quad (5.13)$$

$$= s^n + \alpha_{n-1}s^{n-1} + \alpha_{n-2}s^{n-2} \ldots + \alpha_1s + \alpha_0 = 0 \quad (5.14)$$

The closed loop state matrix $A_{cl} = (A-BK)$ is given as

$$A_{cl} = \begin{bmatrix} -a_{n-1} & -a_{n-2} & -a_{n-3} & \ldots & -a_1 & -a_0 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} - \begin{bmatrix} \vdots \\ K_1 \ K_2 \ K_3 \ldots \ K_n \vdots \end{bmatrix}$$

or

$$A_{cl} = \begin{bmatrix} -a_{n-1} - K_1 & -a_{n-2} - K_2 & -a_{n-3} - K_3 & \ldots & -a_1 - K_{n-1} & -a_0 - K_n \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix}$$

(5.15)

The coefficients of closed loop characteristics polynomials are
\[
\begin{align*}
\alpha_{n-1} &= a_{n-1} + K_1 \\
\alpha_{n-2} &= a_{n-2} + K_2 \\
\alpha_{n-3} &= a_{n-3} + K_3 \\
&\quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
\alpha_1 &= a_1 + K_{n-1} \\
\alpha_0 &= a_1 + K_{n-1}
\end{align*}
\] (5.16)

The feedback gain matrix can be calculated as

\[ K = \alpha - a \] (5.17)

Where

\[
\alpha = \begin{bmatrix} a_{n-1} & a_{n-2} & a_{n-3} & \ldots & a_1 & a_0 \end{bmatrix}
\]

\[
a = \begin{bmatrix} a_{n-1} & a_{n-2} & a_{n-3} & \ldots & a & a_0 \end{bmatrix}
\] (5.18)

### 5.4 INTRODUCING REFERENCE INPUT TO THE STATE FEEDBACK REGULATOR

To get rid of steady state error and to obtain the transient response of the pole placement designs to any input signal it is necessary to introduce the reference input to the system [165-166]. In other design methods, we feedback the output signal and compare it with the reference signal to compute the error, but in state feedback controller, all the states are feedback to the system. So it is necessary to measure steady state value of the states, multiply this value by the feedback gain K and use the new value as a reference for computing the input. This can be done by adding constant gain \( N \) after the reference. The value of this constant gain \( N \) depends upon the pole location and can be calculated as:

Let with the reference input ‘r’ the control signal ‘u’ becomes

\[ u = -Kx + r \] (5.19)

Now the system will have non-zero steady state error to a step input. To correct this problem, compute steady state values of the states and control input that will result in zero steady state error and then force them to take these values. The block diagram of a closed loop system using reference input is shown in figure 5.2.
Let the desired final values of the states and control input are $x_{ss}$ and $u_{ss}$ respectively, then the feedback law is

$$u = u_{ss} - K(x - x_{ss})$$  \hspace{1cm} (5.20)

If there is no error then $x = x_{ss}$ and $u = u_{ss}$. The system must have zero steady state error to any constant input. In the constant steady state, the state and output equations becomes

$$0 = Ax_{ss} + Bu_{ss}$$  \hspace{1cm} (5.21)

$$y_{ss} = Cx_{ss} + Du_{ss}$$  \hspace{1cm} (5.22)

The actual output of the system $y_{ss}$ must be equal to the reference input $r_{ss}$ for any value of $r_{ss}$. For this let $x_{ss} = N_x r_{ss}$ and $u_{ss} = N_u r_{ss}$. By substituting these values, equations (5.21) and (5.22) can be written as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$ \hspace{1cm} (5.23)

The solution for $N_x$ and $N_u$ can be

$$\begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$ \hspace{1cm} (5.24)

To obtain zero steady state error to a step input, the control signal $u$ will be
\[ u = N_u r - K(x - N_x r) \quad \text{(5.25)} \]
\[ u = -Kx + (N_u + KN_x) r \quad \text{(5.26)} \]
If \((N_u + KN_x) = \bar{N}\) then
\[ u = -Kx + \bar{N} r \quad \text{(5.28)} \]

With reference input, the control signal depends upon feedback gain \(K\) as well as on the dynamics of the open loop system. The block diagram of the system using each of the feedback gain is shown in figure 5.3.

![Block diagram of the system using each feedback gain](image)

**Figure 5.3** State feedback controller with individual gain

With the reference input, the closed loop system has input \(r\) and output \(y\). The poles of the closed loop system are at eigenvalues of closed loop system i.e. \(A - BK\). In order to find the step response of the system, it is also necessary to know the closed loop zeros of the system. The zeros of the closed loop system are the values of \(s\) such that

\[
\begin{vmatrix}
    sI - (A - Bk) & -\bar{N}B \\
    C & 0
\end{vmatrix} = 0
\quad \text{(5.29)}
\]

Although the poles of closed loop system are placed at desired position, the zeros of the closed loop system remains unchanged, when full state feedback is used.
5.5 CONTROLLABILITY AND OBSERVABILITY OF THE NANOPOSITIONING SYSTEM

Condition of controllability and observability governs the existence of a complete solution to the control system design problem. The first step in designing a feedback controller using state space technique is to check controllability and observability of open loop system.

5.5.1 Controllability

A system is said to be state controllable if it is possible by means of unconstrained input vector \( u(t) \) (has no limits on the amplitudes of \( u(t) \)), to take a system from any initial state \( x(t_0) \) to any final state \( x(t_f) \) in a finite time \( (t_f - t_0) \) where \( t_0 \leq t \leq t_f \). For a completely controllable system every state must be controllable [162, 186].

A system described by equations 5.1 and 5.2 is said to be completely controllable if and only if the rank of controllability matrix \( C_t \) is equals to the order of system.

\[
C_t = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}
\]  

A system is said to be stabilizable if matrices \( A \) and \( B \) are controllable. The systems having rank of controllability matrix less than the order of system are known as rank deficient system and from the controllability test theorem it is observed that the present system is partially controllable. Suitable feedback technique has to be used to improve system characteristics.

5.5.2 Observability

An unforced system (input vector \( u(t) = 0 \)) is said to be completely observable if any initial state \( x(t_0) \) can be determined by the observation of output \( y(t) \) over a finite interval \( t_0 \leq t \leq t_1 \). Sometimes all state variables are not accessible for direct measurement, in such situations the concept of observability is very useful to reconstruct un-measurable state variables from the measurable variables [161].
A system described by equations 5.1 and 5.2 is said to be completely observable if and only if the rank of observability test matrix $O_t$ is equals to $n$, the order of system.

$$O_t = [C^T : A^T C^T : (A^T)^2 C^T : \ldots \ldots \ldots (A^T)^{n-1} C^T]$$  

For the nanopositioning system considered for this research work (as given by equation 3.34), the rank of observability matrix is 3 which is less than order of system hence from the observability test theorem the system is partially observable. So, an estimator/observer must be designed to measure all the state variables.

### 5.6 ESTIMATOR FOR STATE FEEDBACK REGULATOR

A closed loop system using full state feedback can be designed when it is possible to measure and feedback all the state variables of the system. But in practice, it is rarely possible to measure all the state variables and sometimes some state variables are not physical quantity. Even in such cases when all the state variables are physical quantities, accurate sensors may not be available, or they can be too expensive to measure all the state variables. In nutshell, it can be concluded that only some of the state variables are easily accessible and in general only the outputs of the system are measurable. Therefore, when feedback from the state variables is required, it is necessary to observe the states from the information contained in the output and input variables. The mathematical model of the system that performs the observation of the state variable from the measured output and the known input is known as an observer or state estimator. The block diagram of such system is given in figure 5.5.
As shown in figure 5.4, when feedback from the state variables is required, it may be possible to measure or observe the states from the information contained in the output as well as input variables based on the information received from the measurement. An observer is an essential part of modern control theory and is very useful in estimating the system’s dynamic from a limited number of outputs and is used for parameter estimation, fault detection and other similar applications. For a control system, observers are used in feeding the estimated state vectors to a controller for generating the input signals for the plant.

An observer which estimates all the state variables is known as full order state observer. However, the State variables that can be measured need not to be estimated and can be directly derived from the output. The observer which observes only the unmeasured state variables is known as reduced order observer.

5.6.1 Design of Full Order Observer
If the systems / plants for which the observer is designed are linear time invariant then the dynamics of the observer will also be described by the linear state equations. The full order observer can be designed by two methods:

- Open loop method
- Closed loop method

5.6.1.a Open Loop Method
To design an observer, an obvious method is to select an extra set of the state space model of the original plant model (A,B,C,D). When these two models are subjected to the same input signal, the state variables can be created or observed from the input signal alone [165-166]. These states should be exactly the same as the states of the original model. The block diagram for such method is given by figure 5.5. Let \( \hat{x} \) is the estimated values of actual states \( x \).

![Figure 5.5 Open loop observer](image)

A state observer can be defined by state and output equations as

\[
\dot{\hat{x}} = A\hat{x} + Bu \tag{5.32}
\]

\[
\hat{y} = C\hat{x} + Du \tag{5.33}
\]

The error between the actual state vector and estimated states dynamics can be described as

\[
e = \hat{x} - x \tag{5.34}
\]

By knowing A, B and u, the estimator states will track the true state exactly if it is possible to obtain correct initial condition \( x(0) \) and set \( \hat{x}(0) \) equal to it. Poor estimation of the initial conditions cause the continuously increase in estimated error. The initial state must be estimated each time we use the observer. Moreover, this error converges to zero for a stable system but if A is unstable then the error between \( x \) and \( \hat{x} \) grows with time. In general, open-loop observer is not satisfactory.

5.6.1.b Closed Loop Method

The dynamics of the closed loop observer can be defined by the state equation as
\[ \dot{x} = A\dot{x} + Bu + K_e(y - C\dot{x}) = (A - K_e C)\dot{x} + Bu + K_e y \] (5.35)

And output equation as
\[ \hat{y} = Cy \] (5.36)

\( \hat{y} \) should be viewed as the estimated value of \( y \) and \( K_e \) is the observer gain matrix of \( n \times r \) order which must be selected to make \( (A - K_e C) \) stable. It is a weighting matrix to minimize the estimated error, difference between measured output \( y \) and estimated output \( C\dot{x} \), to zero in the steady state. This term continuously corrects the model outputs and improves the performance of the observer. The inputs to the observer are the output \( y \) and the control input \( u \) as shown in figure 5.6. The change in estimator error can be obtained as

\[ \dot{e} = \dot{x} - \dot{\hat{x}} = Ax - A\dot{x} - K_e(Cx - C\dot{x}) = (A - K_e C)(x - \dot{x}) = (A - K_e C) e \] (5.37)

Equation (5.37) depicts that the error vector dynamics is determined by the eigenvalues of matrix \( (A - K_e C) \). If \( (A - K_e C) \) matrix is stable then the error vector will converge to zero for any initial error vector \( e(0) \) or \( \dot{x} \rightarrow x \) eventually. Even if there is an initial large error \( e(0) \), \( e(t) \) still goes zero as \( t \rightarrow \infty \) provided \( (A - K_e C) \) is stable. Moreover in closed loop observer there is no need to compute the initial estimate \( \dot{x}(0) \). The block diagram of full state observer using closed loop technique is shown in figure 5.6.

![Figure 5.6 Full state closed loop observer](image)

The designing of full state observer requires that the eigenvalues of \( (A - K_e C) \) must be chosen such that the dynamic behavior of the error vector is asymptotically stable and is adequately fast, then the error vector will tend to zero with an adequate speed. So far, it is assumed that matrices \( A, B \) and \( C \) in the observer are same as
those of physical plant. But if there are any discrepancies in A, B and C matrices of the observer and the actual plant, the dynamics of the observer error will not be governed by equation 5.34 and the error may not approach to zero as expected. Therefore observer gain matrix $K_e$ must be chosen such that observer is stable and the error remains acceptably small in the presence of small model uncertainty and disturbing inputs.

### 5.6.1.c State Feedback and Observer Based Controller

The performance of the observer can be tested by adding the actual state feedback controller with observer to the plant and then calculating the estimator error. Having designed a suitable state observer, the closed loop system with state feedback controller can be designed as shown in figure 5.7.

![Observer based state feedback control system](image)

Figure 5.7 Observer based state feedback control system

By combining the equations 5.5 of the system with state feedback and equation 5.35 of the system with full state observer, the dynamics of the observer based full state feedback control system can be written as

\[ u = r - K\hat{x} \]  \hspace{1cm} (5.38)

the state equations of the closed loop system become
\begin{align}
\dot{x} &= Ax - BKx + Br \\
\hat{x} &= A\hat{x} + Bu + K_e(y - C\hat{x}) = (A - K_eC)\hat{x} + B(r - K\hat{x}) + K_eCx
\end{align}

In matrix form the dynamics of observer based state feedback control system can be written as

\[
\begin{bmatrix}
\dot{\hat{x}} \\
\dot{\hat{\hat{x}}}
\end{bmatrix} = 
\begin{bmatrix}
A & -BK \\
K_eC & A - K_eC - BK
\end{bmatrix}
\begin{bmatrix}
\hat{x} \\
\hat{\hat{x}}
\end{bmatrix} + 
\begin{bmatrix}
B \\
B
\end{bmatrix} r
\tag{5.41}
\]

\[
y = \begin{bmatrix}
C & 0
\end{bmatrix}
\begin{bmatrix}
\hat{x} \\
\hat{\hat{x}}
\end{bmatrix}
\tag{5.42}
\]

\[
\hat{y} = \begin{bmatrix}
C & -C
\end{bmatrix}
\begin{bmatrix}
\hat{x} \\
\hat{\hat{x}}
\end{bmatrix}
\tag{5.43}
\]

The closed loop poles of the observer based state feedback control system consist of the poles due to pole placement design alone and the poles due to observer design alone. So, the pole placement regulator and observer can be designed independently. For \(n^{th}\) order plant using full state observer, the characteristic equation of the observer based closed loop system will be of order \(2n\).

### 5.7 STATE FEEDBACK REGULATOR FOR NANOPositionING SYSTEM

The performance characteristics of a system can be changed by using closed loop control system in which controller is designed to place the closed loop poles at the desired position. Such technique to design a controller is known as pole placement technique. The pole placement method is very much similar to root locus method. The basic difference is that in root locus only dominant closed loop poles are placed at the desired locations while in pole placement design all closed loop poles are placed at the desired locations. For the design of pole placement, assume that all state variables are measurable and are available for feedback. This type of approach is known as full state feedback. This approach provides sufficient number of controller design parameters to move all the closed loop poles independently to each other.

#### 5.7.1 Pole Placement Regulator
Consider the transfer function of the nanopositioning system described by equation 3.34 (rewritten here)

\[ G(s) = \frac{9.7 \times 10^4(s - (7.2 \pm 7.4i) \times 10^3)}{(s + (1.9 \pm 4.5i) \times 10^3)(s + (1.2 \pm 15.2i) \times 10^2)} \]

The magnitude of the control input is determined by the poles in the left half plane which are farthest away from the imaginary axis and the speed of the system is given by the poles which are closer to the imaginary axis in the left half plane called dominant poles.

The damping ratio for the dominant pole of the open loop nanopositioning system is very low i.e 0.0787 which makes system very oscillatory and slow. To improve the performance specifications of the device, let us place closed loop poles arbitrarily at

\[ p_{1,2} = -(2.0 \pm 3.5i) \times 10^3; \quad p_{3,4} = -(2.5 \pm 3.5i) \times 10^2; \]

The regulator gain matrix ‘K’ can be calculated as

\[ K = [4.6 \times 10^2 \quad -8.66 \times 10^6 \quad -5.69 \times 10^9 \quad -5.24 \times 10^{13}] \] (5.44)

The closed loop state dynamic matrix \( A_{cl} \) for system using pole placement regulators is given as

\[ A_{cl} = \begin{bmatrix} -4.5 \times 10^3 & -1.84 \times 10^7 & -8.86 \times 10^9 & -3.0 \times 10^{12} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \] (5.45)

The transfer function of the closed loop system is given as

\[ G_{cl}(s) = \frac{-4.547 \times 10^{-12}s^3 + 9.7 \times 10^4s^2 - 1.397 \times 10^9s + 1.034 \times 10^{13}}{s^4 + 4500 \times s^3 + 1.843 \times 10^7s^2 + 8.865 \times 10^9s + 3.006 \times 10^{12}} \] (5.46)

Closed loop poles, corresponding feedback gain, damping ratio and frequency corresponding to these arbitrary poles are given in table 5.1.
Change in location of desired closed loop poles will change the behavior of the closed loop system. As seen from the table 5.1, by placing the poles at desired location, the pole placement regulator improves damping ratio of the systems and hence the system performance. Further improvement in system performance can be achieved by placing closed loop poles deeper inside the left half plane by decreasing real part of the arbitrary closed loop poles. Let new desired closed loop poles location are

\[ P_{1,2} = -(2.5 \pm 4.0i) \times 10^3 \quad \text{and} \quad P_{3,4} = -(3.0 \pm 3.5i) \times 10^2 \]

which produce regulator gain matrix as

\[
K = \begin{bmatrix}
1.56 \times 10^3 & -1.63 \times 10^6 \\
-1.48 \times 10^8 & -5.07 \times 10^{13}
\end{bmatrix}
\]

(5.47)

The closed loop state matrix \( A_{cl} \) of the system is given as

\[
A_{cl} = \begin{bmatrix}
-5.59 \times 10^3 & -2.55 \times 10^7 & -1.44 \times 10^{10} & -4.73 \times 10^{12} \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

(5.48)

The transfer function of the closed loop system is given as

\[
P_{1,2} = -(2.5 \pm 4.0i) \times 10^3 \\
P_{3,4} = -(3.0 \pm 3.5i) \times 10^2
\]

\[
\begin{bmatrix}
1.56 \times 10^3 & -1.63 \times 10^6 \\
-1.48 \times 10^8 & -5.07 \times 10^{13}
\end{bmatrix}
\]

(5.47)

The closed loop state matrix \( A_{cl} \) of the system is given as

\[
A_{cl} = \begin{bmatrix}
-5.59 \times 10^3 & -2.55 \times 10^7 & -1.44 \times 10^{10} & -4.73 \times 10^{12} \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

(5.48)

The transfer function of the closed loop system is given as

\[
P_{1,2} = -(2.5 \pm 4.0i) \times 10^3 \\
P_{3,4} = -(3.0 \pm 3.5i) \times 10^2
\]
\[ G_{cl}(s) = \frac{5.457 \times 10^{-12}s^3 + 9.7 \times 10^4s^2 - 1.397 \times 10^9s + 1.034 \times 10^{13}}{s^4 + 5600xs^3 + 2.546 \times 10^7s^2 + 1.441 \times 10^{10}s + 4.728 \times 10^{12}} \]  \hspace{1cm} (5.49)

The closed loop step response and frequency response of the nanopositioning system for different values of closed loop poles location are plotted in figure 5.8 and 5.9 respectively.

![Time response of system using pole-placement regulator for different values of closed loop poles](image1)

**Figure 5.8** Time response of closed loop system using pole placement regulator

![Frequency response of system for poles at p1,2=-(2.0\pm3.6)i\times10^3, p3,4=-(2.6\pm3.6)i\times10^2](image2)

**Figure 5.9** Frequency response of closed loop system using pole placement regulator
Figure 5.9 Frequency response of closed loop system using pole placement regulator

The summary for the performance specifications of the closed loop systems observed from the step and frequency response of the closed loop system is tabulated in table 5.2.

<table>
<thead>
<tr>
<th>Arbitrarily closed loop poles</th>
<th>Rise time (msec.)</th>
<th>Settling time (sec.)</th>
<th>$M_p$ (%)</th>
<th>Gain margin (dB)</th>
<th>Phase margin (degree)</th>
<th>Bandwidth (Hz)</th>
</tr>
</thead>
</table>
| $P_{1,2} = -(2.0 \pm 3.5i) \times 10^3$  
$P_{3,4} = -(2.5 \pm 3.5i) \times 10^2$ | 4.18 | 0.0142 | 10.67 | 5.9 | 21 | 506 |
| $P_{1,2} = -(2.5 \pm 4.0i) \times 10^3$  
$P_{3,4} = -(3.0 \pm 3.5i) \times 10^2$ | 4.20 | 0.0134 | 6.7 | 10.89 | 45 | 498 |
| $P_{1,2} = -(2.0 \pm 4.5i) \times 10^3$  
$P_{3,4} = -(3.5 \pm 3.5i) \times 10^2$ | 4.31 | 0.0123 | 4.34 | 14.15 | 65 | 497 |
| $P_{1,2} = -(2.25 \pm 4.5i) \times 10^3$  
$P_{3,4} = -(4.0 \pm 3.0i) \times 10^2$ | 4.91 | 0.0078 | 1.52 | 15.43 | 79 | 437 |
| $P_{1,2} = -(2.65 \pm 4.5i) \times 10^3$  
$P_{3,4} = -(4.0 \pm 3.0i) \times 10^2$ | 4.92 | 0.00783 | 1.52 | 15.87 | 84 | 436 |
| $P_{1,2} = -(3.25 \pm 4.5i) \times 10^3$  
$P_{3,4} = -(4.5 \pm 3.0i) \times 10^2$ | 4.78 | 0.00779 | 0.9 | 17.80 | 121 | 448 |

Table 5.2 Performance characteristics of closed loop nanopositioning system using pole placement regulator for different value of closed loop poles

By referring to the figures 5.8, 5.9 and table 5.2, the analysis clearly demonstrate that Pole placement regulator has the faster response with very small values of settling time and rise time. Further it can be seen that by placing poles deeply into the left half plane, the rise time, settling time and maximum overshoot decreases more rapidly. Frequency response shows noticeable improvement in stability margins i.e gain, phase margin and bandwidth of the system using pole placement regulator. The optimal value of phase margin is achieved by suitably placing the closed loop poles.

5.7.2 Pole Placement Regulator with Reference Input
In this section, a reference input $r$ is introduced to the pole placement regulator and it introduces steady state error to the system. The value of the state gain $N_u$ and control gain $N_x$ depends upon the dynamics of the system. The value of $N_x$ and $N_u$ for the system represented by equation 3.34 will be calculated as

$$N_x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 9.67 \times 10^{-14} \end{bmatrix} \text{ and } N_u = 5.365$$

The composite value of gain $\bar{N}$ depends on the pole locations of closed loop system. For closed loop poles $P_{1,2} = -(2.0 \pm 3.5i) \times 10^3$ and $P_{3,4} = -(2.5 \pm 3.5i) \times 10^2$, the value of necessary gain $\bar{N}$ for zero steady state error to a step input will be

$$\bar{N} = 2.91 \times 10^{-1}$$

With this scaled reference input to make steady state error zero and for closed loop pole locations $P_{1,2} = -(2.0 \pm 3.5i) \times 10^3$; $P_{3,4} = -(2.5 \pm 3.5i) \times 10^2$, the resultant transfer function of the closed loop system is given as

$$G_{cl} = \frac{2.73 \times 10^{-12}s^3 + 2.82 \times 10^4s^2 - 4.061 \times 10^8s + 3.006 \times 10^{12}}{s^3 + 4500s^3 + 21.843 \times 10^7s^2 + 8.865 \times 10^9s + 3.006 \times 10^{12}}$$

(5.50)

The step response and frequency response of the nanopositioning system to scaled step input signal for different values of closed loop pole locations are given in figures 5.10 and 5.11 respectively.
The observations based on analysis of time and frequency response of nanopositioning system for different values of pole locations and with reference input is tabulated in table 5.3.
<table>
<thead>
<tr>
<th>Arbitrarily closed loop poles</th>
<th>Reference input gain (N)</th>
<th>Rise time (msec.)</th>
<th>Settling time (msec.)</th>
<th>$M_o$ (%)</th>
<th>Gain margin (dB)</th>
<th>Phase margin (degree)</th>
<th>Bandwidth (Hz)</th>
</tr>
</thead>
</table>
| $P_{1,2} = -(2.0 \pm 3.5i) \times 10^3$  
$P_{3,4} = -(2.5 \pm 3.5i) \times 10^2$ | 0.291 | 4.16 | 14.1 | 10.67 | 16.64 | 102 | 506 |
| $P_{1,2} = -(2.5 \pm 4.0i) \times 10^3$  
$P_{3,4} = -(3.0 \pm 3.5i) \times 10^2$ | 0.457 | 4.28 | 13.38 | 6.80 | 17.68 | 127 | 498 |
| $P_{1,2} = -(2.0 \pm 4.5i) \times 10^3$  
$P_{3,4} = -(3.5 \pm 3.5i) \times 10^2$ | 0.575 | 4.30 | 11.23 | 4.35 | 18.97 | 167 | 497 |
| $P_{1,2} = -(2.25 \pm 4.5i) \times 10^3$  
$P_{3,4} = -(4.0 \pm 3.0i) \times 10^2$ | 0.612 | 4.90 | 7.80 | 1.52 | 19.77 | 178 | 437 |
| $P_{1,2} = -(2.65 \pm 4.5i) \times 10^3$  
$P_{3,4} = -(4.0 \pm 3.0i) \times 10^2$ | 0.659 | 4.91 | 7.82 | 1.52 | 19.48 | 179 | 436 |
| $P_{1,2} = -(3.25 \pm 4.5i) \times 10^3$  
$P_{3,4} = -(4.5 \pm 3.0i) \times 10^2$ | 0.872 | 4.77 | 7.77 | 0.91 | 18.99 | 179 | 448 |

Table 5.3 Performance characteristics of nanopositioning system using pole placement regulator with reference input

As seen from figures 5.10 and 5.11, the steady state error is completely removed by the addition of reference scaled input to the pole placement regulator. The analysis of table 5.3 shows that there is hardly any change in settling time, rise time and maximum overshoot of the system as compared to the closed loop system without reference input. But analysis of frequency response of closed loop system, it can be observed that there is improvement in frequency response characteristics in terms of gain margin, phase margin and bandwidth of the system.

5.7.3 Observer Based Pole Placement Controller

A full state feedback can be designed when it is possible to measure and feedback all the state variables of the system. But in practice, only some of the variables can be measured and available for feedback. When feedback from the state variables is required, the state vector can be estimated by using estimator or
observer. Observer observes the output of the system for a known input and for finite time interval and reconstruct the state vector from the record of the output. In this section, a state observer will be designed and then performance of the observer based controller will be analyzed on the basis of time and frequency response.

Let us consider that the poles of the pole placement regulators are at
\[ P_1, P_2 = -(3.56 \pm 5i) \times 10^5 \quad P_3, P_4 = -(3.75 \pm 5i) \times 10^3 \]  
(5.51)

The gain of feedback controller for these pole locations will be
\[ K = [7.165 \times 10^5 \quad 3.824 \times 10^{11} \quad 2.856 \times 10^{15} \quad 1.473 \times 10^{19}] \]  
(5.52)

Let the poles for the observer are located at three different positions. The first, second and third locations of the observer’s poles are PO1, PO2 and PO3.

Let the poles PO1 for the observer are located at positions PO1, PO2, PO3 and PO4 as given below
\[ PO_1 = 5 \times P_1 \times 10^1 ; \quad PO_2 = 5 \times P_2 \times 10^1 ; \quad PO_3 = 115 \times P_3 \times 10^{-2} \quad \text{and} \quad PO_4 = 115 \times P_4 \times 10^{-2} \]

The observer’s gain \( K_e \) for these values will be
\[ K_e = [-1.053 \times 10^9 \quad 3.545 \times 10^5 \quad 1.489 \times 10^2 \quad 1.679 \times 10^{-2}] \]  
(5.53)

The transfer function of the observer based controller will be
\[ G_{el} = \frac{8.074 \times 10^{17} s^3 + 6.05 \times 10^{21} s^2 + 3.245 \times 10^{25} s + 6.372 \times 10^{28}}{s^4 + 3.80 \times 10^7 s^3 + 9.693 \times 10^{14} s^2 + 7.762 \times 10^{22} s + 1.22 \times 10^{27}} \]  
(5.54)

Different values of observer’s pole locations and the time response characteristics of the observer based state feedback controller are tabulated in table 5.4:  

<table>
<thead>
<tr>
<th>Observer poles</th>
<th>Observer Gain</th>
<th>Rise time (sec)</th>
<th>Settling time (sec)</th>
<th>( M_p ) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>PO1 = [5 \times P_1 \times 10^1 \quad 5 \times P_2 \times 10^1 \quad 115 \times P_3 \times 10^{-2} \quad 115 \times P_4 \times 10^{-2}]</td>
<td>K_e = [-1.053 \times 10^9 \quad 3.545 \times 10^5 \quad 1.489 \times 10^2 \quad 1.679 \times 10^{-2}]</td>
<td>1.37 \times 10^{-7}</td>
<td>1.68 \times 10^{-7}</td>
<td>1.93</td>
</tr>
</tbody>
</table>
$$PO2 = \begin{bmatrix} 7\times P1 \times 10^1 \\ 190 \times P3 \times 10^{-2} \end{bmatrix} \begin{bmatrix} 7\times P1 \times 10^1 \\ 190 \times P4 \times 10^{-2} \end{bmatrix}$$

$$K_e = \begin{bmatrix} -9.385 \times 10^7 & 2.551 \times 10^6 \\ 5.241 \times 10^2 & 6.688 \times 10^2 \end{bmatrix}$$

$$Ke = 1.09 \times 10^{-7} \quad 1.34 \times 10^{-7} \quad 1.06$$

$$PO3 = \begin{bmatrix} 5\times P1 \times 10^2 \\ 165 \times P3 \times 10^{-2} \end{bmatrix} \begin{bmatrix} 5\times P1 \times 10^2 \\ 165 \times P4 \times 10^{-2} \end{bmatrix}$$

$$K_e = \begin{bmatrix} -5.550 \times 10^{10} & 9.540 \times 10^7 \\ 2.339 \times 10^4 & 2.139 \times 10^6 \end{bmatrix}$$

$$Ke = 8.36 \times 10^{-11} \quad 1.02 \times 10^{-10} \quad 0.466$$

Table 5.4 Performance characteristics of nanopositioning system using observer based pole placement controller

From the analysis of table 5.4, it can be concluded that there is significant improvement in the time response characteristics of the nanopositioning system using observer based pole placement controller. The rise time, settling time and maximum overshoot of the system decrease drastically.

5.8 COMPARISON OF DIFFERENT SCHEMES OF POLE PLACEMENT CONTROLLER

In above sections, pole placement controllers with / without reference input, with and without observer have been discussed. The comparison of performance characteristics of nanopositioning system, nanopositioning system with pole-placement controller and nanopositioning system observer based pole placement controller is shown in figure 5.12.
From the analysis of figure 5.12 it has been concluded that the time response characteristics of nanopositioning system with observer based pole placement controller enhanced significantly. The settling time of the nanopositioning system with observer is just $1.02 \times 10^{-7}$ sec. as compared to the 0.00779 sec. for pole placement controller with reference input without observer and 0.035 sec. for open loop nanopositioning system. The problem of maximum overshoot is almost negligible with pole placement controller whereas in open loop system maximum overshoot is about 83%. The conclusion drawn is that observer based pole placement regulator significantly improves the system performance.

### 5.9 LQR OPTIMAL CONTROLLER

State feedback regulator designed using pole placement technique requires the change in pole locations according to the requirements of system performance. Choosing best closed-loop pole locations is very difficult because in this method we analyze the transient and frequency responses of the system, identify the
various pole locations and change the pole locations that are undesired. Further larger movement of the pole requires the greater amount of feedback gain $K$ which leads to cost and saturation problem. By using pole placement regulator, a desired close-loop behavior is satisfactory but it makes the use of large amount of input. Optimal control can be used to take best trade-off between closed-loop system performance and input energy. So, an alternative approach to pole placement regulator i.e. Linear Quadratic Regulator (LQR) theory, which not only place the poles at arbitrary locations, but also guarantees stable closed-loop poles with more desirable robustness properties. The performance of the closed loop system is specified in terms of a cost, which is the integral of a weighted quadratic function of the system state and control inputs which must be minimized by the optimal controller. The values of the state weighting matrix, $Q$, and the control cost matrix, $R$, must be selected properly so that the transient response of the closed-loop system can be improved.

In general, control systems are designed with basic objectives: terminal control and tracking control [187]. Terminal control is to guide a system from a given initial state to a desired final state in a prescribed time. On the other hand, tracking control is to maintain the plant’s states in a close proximity of a reference state. Let the plant is described by state equation

$$\dot{x} = f[x(t)u(t), t]$$  \hspace{1cm} (5.55)

A terminal control system has to change the plant’s states $x(t)$ from an initial states $x_i = x(t_i)$ at a given initial time, $t_i$, to a final state $x_f = x(t_f)$, at a specified time, $t_f$, by applying control input $u(t)$ in a fixed control interval, $(t_i \leq t \leq t_f)$.

Tracking control problems requires reduction of all errors to zero when time becomes large compared to the time scale of the plant dynamics. If control interval is large, then it is not important to consider relatively much faster variations in the plant dynamics which can be averaged out over a long period of time. These variations can be approximated by a linear time invariant system where plant coefficient matrices $A$, $B$, $C$ and $D$ are constants. Such types of problems with the objective of zero steady state error are known as Infinite
horizon control [188]. The design of tracking controller is carried out such that the plant’s state follows the nominal trajectory, $x_d(t)$, which satisfy the equation

$$x_d = f[x_d(t), u_d(t), t]$$  \hspace{1cm} (5.56)

Where $u_d(t)$ is the nominal control input.

Optimal control theory offers a powerful method to drive both terminal and tracking controllers and provide a systematic framework for modern control design methods. Optimal control refers to the time history of the control input that take the system from the initial state to the desired state and minimize the objective function. The evolution of the system from the initial state to final state under the application of optimal control input is known as optimal trajectory. To find optimal trajectory between the two states is the primary task of the terminal control and is carried out in open loop. A small variation in the initial conditions or the presence of external disturbances causes a perturbation of the system from its optimal trajectory, generally in the closed loop. Thus, open loop optimal control yields a terminal control input on which tracking control input calculated by the optimal feedback control law is superimposed.

5.9.1 Design of LQR Controller

LQR controllers are optimal controllers which uses the state—space approach to analyze a system. Optimal Controller provides an efficient way to find a controller that provides the best possible performance with respect to some given measure of performance. The cost of controlling a plant is a function of the largest control input magnitude. If control input is large, higher would be the energy required by the actuator in generating the control input and hence higher the cost of control. The control effort required in controlling a plant can be minimizes by imposing a limitation on the regulator or controller gain matrix $K$. Optimal control system minimizes the cost associated with generating control inputs. Desired performance objectives $J$ with minimum control energy can be obtained by formulating an objective function which must be minimized in the design process [161]. Objective function for the optimal control system can be found by time integral of sum of transient and control energy. Transient energy
specifies the acceptable values of maximum overshoot and settling time while control energy specifies the required input signal. LQR is a control strategy that provides the best possible performance with respect to some given measure of performance. In this method a feedback gain matrix $K$ is designed which minimize the objective function in order to achieve some compromise between the control efforts used and speed of the response that will guarantee a stable system.

The optimal control regulator consists of finding the feedback gain matrix $K$ of the optimal control vector so as to minimize the objective function or performance index $J$ given below [161-162, 188-189].

$$J = \frac{1}{2} \int_{0}^{\infty} [(x^T Q x + u^T R u)] dt$$  \hspace{1cm} (5.57)

First term of equation (5.57) represents the transition energy of the state where $Q$ is $(n \times n)$ positive definite or positive semi-definite square symmetric state weighting matrix and second term of equation 5.57 represents the control energy where $R$ is $r \times r$ optimal or cost control matrix which must be real symmetric where $r$ is number of inputs to the system [161-162, 187-188]. The weighting matrices, $Q$ and $R$, control the characteristics of system. Appropriate values of $Q$ and $R$ must be chosen to obtain acceptable levels of device performance characteristics. Optimization of cost function provides a design method for the system to operate with maximum efficiency and performance. The objective of optimal controller design is to select the feedback gain matrix that minimizes the performance index $J$. Making $J$ small means to keeps total energy of the closed loop system very small. Since both states $x(t)$ and control inputs $u(t)$ of the system are weighted in $J$ so if $J$ is small then neither $x(t)$ nor $u(t)$ can be too large. The linear feedback gain matrix $K$ or Kalman gain matrix in terms of $R$ can be obtained as

$$K = R^{-1}B^T M_0$$ \hspace{1cm} (5.58)

The matrix $P$ must satisfy the reduced matrix riccati equation

$$A^T M_0 + M_0 A - M_0 B R^{-1} B^T M_0 + Q = 0$$ \hspace{1cm} (5.59)
Where $M_0$ is the solution to the algebraic reccati equation. For a system having $n$ states or order and $r$ inputs, the size of $K$ is $r \times n$. Deciding the state feedback gain matrix $K$ is to design the controller. When the system is disturbed by a non-zero initial conditions, performance specification of the system such as settling time, rise time and maximum overshoot are determined by the location of closed loop poles. Existence of unique positive definite solution to reccati algebraic equation will be guaranteed if plant is stabilizable and detectable with the output $y = Cx$ where $C^TC = Q$ is positive semi-definite and $R$ is symmetric positive definite matrices [190].

### 5.9.2 Weight Selection of Q and R

Matrix $K$ is not unique for a given system but it depends upon the selection of the $Q$ and $R$ matrices. By the proper choice of $Q$ and $R$ and hence $K$, the matrix $A-BK$ can be made asymptotically stable matrix and $x(t)$ can approach to zero as $t$ approaches infinity for all $x(0) \neq 0$. Different approaches have been suggested for the selection of matrices $Q$ and $R$. The various techniques to choose the elements of weighting matrices $Q$ and $R$ are:

- At the beginning of the optimal control design, the weight matrices should be chosen as
  \[ Q = C^TC \text{ and } R = BB^T \]  
  \[ (5.60) \]

  If the performance characteristics of the system are not satisfactory, then these matrices can be redesigned to meet the desired specification.

- Bryson and Ho [190] suggested that an appropriate choice of these matrices should be made to obtain the acceptable level of state and control, such as
  \[ Q_{ii} = \left[ (t_f - t_0) \times \max_i |x_i(t_f)|^2 \right]^{-1} \]  
  \[ R_{ii} = \left[ (t_f - t_0) \times \max_i |u_i(t_f)|^2 \right]^{-1} \]  
  \[ (5.61) \]
  \[ (5.62) \]

- Anderson and Moore [191] suggest the selection of the $Q$ and $R$ matrices on the basis of properties of closed loop system such as eigenvalues, robustness, uncertainties of the plant parameters. However this criterion does not gives
the direct relationship between these matrices and the aforementioned characteristics.

5.10 LQR FOR NANOPositionING SYSTEM

To design a LQR controller the first requirement is the availability of entire state vector for measurement and feedback. Optimality of LQR controller totally depends upon the choice of Q and R matrices. If selected Q and R are not proper then the designed controller will not meet the specified performance of the system and positive definite solution of the algebraic reccati equation will not exist. There is no particular method for the selection of Q and R, the current approach to choose Q and R is simulation and trial. Generally Q and R can be chosen as diagonal matrices or identity matrices. Second choice is to consider Q as \( C^T C \). To have smaller value of input vector, R must be large. But to have smaller value of input to the states, larger value of corresponding column of Q is needed [161,192]. Q and R must be chosen in such a way so that positive definite solution to the algebraic reccati equation exists even if the plant is uncontrollable because positive definite solution to the algebraic reccati equation results in an asymptotically stable control system.

Now consider the nanopositioning system described by equation 3.34 rewritten here

\[
G(s) = \frac{9.7 \times 10^4 (s - (7.2 \pm 7.4i) \times 10^3)}{(s + (1.9 \pm 4.5i) \times 10^3)(s + (1.2 \pm 15.2i) \times 10^2)}
\]

Q and R matrices are the design parameters to get appropriate value of feedback gain K which can satisfy the system performance. In this section, the effect of matrix Q and R independently on the system performance has been observed.

5.10.1 Effect of State Weighting Matrix Q on the System Performance

Design LQR regulator to optimize the objective function with weighting matrices as

\[
Q = 0.1(C^T \times C) \text{ to } 100(C^T \times C)
\]

and keeping R fixed at 0.1 value.
For \( Q=0.1(C^T \times C) \) and \( R=0.1 \), the solution \( M_1 \) of algebraic reccati equation is given as

\[
M_1 = \begin{bmatrix}
1.45 \times 10^1 & 5.99 \times 10^4 & 3.64 \times 10^8 & 9.55 \times 10^{10} \\
5.99 \times 10^4 & 2.81 \times 10^8 & 1.64 \times 10^{12} & 1.20 \times 10^{15} \\
3.64 \times 10^8 & 1.64 \times 10^{12} & 9.73 \times 10^{15} & 5.87 \times 10^{18} \\
9.55 \times 10^{10} & 1.20 \times 10^{15} & 5.87 \times 10^{18} & 2.34 \times 10^{22}
\end{bmatrix}
\]

The closed loop state matrix of the system using LQR controller is given as

\[
A_{cl} = \begin{bmatrix}
-4.18 \times 10^3 & -2.77 \times 10^7 & -1.82 \times 10^{10} & -5.64 \times 10^{13} \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

The closed loop transfer function of the nanopositioning system becomes

\[
G_{cl} = \frac{10^{-11}s^3 + 9.7 \times 10^4 s^2 - 1.397 \times 10^9 s + 1.034 \times 10^{13}}{s^4 + 4186s^3 + 2.77 \times 10^7 s^2 + 1.82 \times 10^{10} s + 5.64 \times 10^{13}}
\] (5.63)

The solution of algebraic reccati equation for \( Q = C^T \times C \) and \( R= 0.1 \) is given as

\[
M_1 = \begin{bmatrix}
7.13 \times 10^1 & 3.13 \times 10^5 & 1.86 \times 10^9 & 8.92 \times 10^{11} \\
3.13 \times 10^5 & 1.56 \times 10^9 & 8.97 \times 10^{12} & 8.19 \times 10^{15} \\
1.86 \times 10^9 & 8.97 \times 10^{12} & 5.27 \times 10^{16} & 4.33 \times 10^{19} \\
8.92 \times 10^{11} & 8.19 \times 10^{15} & 4.33 \times 10^{19} & 1.47 \times 10^{23}
\end{bmatrix}
\]

Corresponding value of closed loop state matrix for LQR controller using these parameters will be

\[
A_{cl} = \begin{bmatrix}
-4.75 \times 10^3 & -3.023 \times 10^7 & -3.31 \times 10^{10} & -6.44 \times 10^{13} \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

The closed loop transfer function of the nanopositioning system will be

\[
G_{cl} = \frac{9.095 \times 10^{-13}s^3 + 9.7 \times 10^4 s^2 - 1.397 \times 10^9 s + 1.034 \times 10^{13}}{s^4 + 4753s^3 + 3.023 \times 10^7 s^2 + 3.31 \times 10^{10} s + 6.44 \times 10^{13}}
\] (5.63)

<table>
<thead>
<tr>
<th>State Weighting</th>
<th>Feedback gain ( K )</th>
<th>Eigenvalues</th>
<th>Damping</th>
<th>Frequency (rad/s)</th>
</tr>
</thead>
</table>
Corresponding values of feedback gain $K$, eigenvalues, damping factor and frequency of the closed loop system using different values of $Q$ matrix and fixed $R$ are given below in table 5.5.

<table>
<thead>
<tr>
<th>Matrix</th>
<th>Eigenvalues</th>
<th>Damping</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q = 0.1(C^T \times C)$</td>
<td>$[1.457 \times 10^2, 5.99 \times 10^3]$</td>
<td>$3.64 \times 10^9, 9.55 \times 10^{11}$</td>
<td>$\text{P1,2} = -(1.90 \times 10^3 \pm 4.50 \times 10^3 i)$; $\text{P3,4} = -(1.93 \times 10^2 \pm 1.53 \times 10^2 i)$</td>
</tr>
<tr>
<td>$Q = (C^T \times C)$</td>
<td>$[7.128 \times 10^2, 3.13 \times 10^6]$</td>
<td>$1.85 \times 10^3, 8.92 \times 10^{12}$</td>
<td>$\text{P1,2} = -(1.90 \times 10^3 \pm 4.51 \times 10^3 i)$; $\text{P3,4} = -(4.80 \times 10^2 \pm 1.57 \times 10^2 i)$</td>
</tr>
<tr>
<td>$Q = 10(C^T \times C)$</td>
<td>$[2.21 \times 10^3, 1.14 \times 10^7]$</td>
<td>$6.44 \times 10^3, 6.18 \times 10^{13}$</td>
<td>$\text{P1,2} = -(1.87 \times 10^3 \pm 4.56 \times 10^3 i)$; $\text{P3,4} = -(1.25 \times 10^3 \pm 1.81 \times 10^3 i)$</td>
</tr>
<tr>
<td>$Q = 100(C^T \times C)$</td>
<td>$[4.92 \times 10^3, 3.19 \times 10^7]$</td>
<td>$1.76 \times 10^3, 2.76 \times 10^{14}$</td>
<td>$\text{P1,2} = -(1.83 \times 10^3 \pm 4.93 \times 10^3 i)$; $\text{P3,4} = -(2.65 \times 10^3 \pm 2.23 \times 10^3 i)$</td>
</tr>
</tbody>
</table>

Table 5.5 Feedback Gain $K$, eigenvalues, damping and frequency of closed loop system for different values of $Q$.

As indicated by equation 5.59 and proved by table 5.5 that the feedback gain $K$ is proportional to the solution of Algebraic Reccati Equation and hence weight of $Q$ and $R$. If the states are heavily weighted i.e. $Q$ is high, then feedback gain increases and the closed loop poles are placed more deeply into the left half plane which makes system more stable. Moreover as weight of $Q$ increases, the damping factor of the dominant poles of the system approaches towards critical system and hence oscillations decrease which makes system stable in small amount of time and hence system becomes speedy. Table 5.5 also shows that the natural frequency of the closed loop system designed using LQR controller is better than open loop system.

For the values of $Q$ as given in table 5.5, time and frequency responses of the closed loop nanopositioning system are shown in figures 5.13 and 5.14 respectively. These plots illustrate the correlation between the weighting matrices $Q$ and $R$ and performance specifications i.e. settling time, maximum overshoot, gain margin, phase margin and bandwidth of the closed loop system.
Figure 5.13  Closed loop time response of system for different values of state weighting matrix Q

Figure 5.14  Closed loop frequency response of nanopositioning system for different values of Q
The summary for the corresponding performance specifications of the closed loop systems using LQR regulator for different values of state weighting matrix $Q$ obtained by step and frequency response shown in figures 5.13 and 5.14 are given in table 5.6.

<table>
<thead>
<tr>
<th>State weighting matrix $Q$</th>
<th>Rise time (msec.)</th>
<th>Settling time (sec.)</th>
<th>$M_p$ (%)</th>
<th>Gain margin (dB)</th>
<th>Phase margin (degree)</th>
<th>Bandwidth (KHz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q= 0.1 \times (C^T \times C)$</td>
<td>6.7681</td>
<td>0.0210</td>
<td>72.15</td>
<td>8.6648</td>
<td>$\infty$</td>
<td>2.4954</td>
</tr>
<tr>
<td>$Q= (C^T \times C)$</td>
<td>0.72175</td>
<td>0.0086</td>
<td>41.90</td>
<td>16.175</td>
<td>$\infty$</td>
<td>2.5606</td>
</tr>
<tr>
<td>$Q= 10(C^T \times C)$</td>
<td>0.68988</td>
<td>0.0029</td>
<td>14.55</td>
<td>23.992</td>
<td>$\infty$</td>
<td>3.0399</td>
</tr>
<tr>
<td>$Q=100 (C^T \times C)$</td>
<td>0.52859</td>
<td>0.0018</td>
<td>10.52</td>
<td>31.646</td>
<td>$\infty$</td>
<td>4.6318</td>
</tr>
</tbody>
</table>

Table 5.6 Performance characteristics of the closed loop nanopositioning system for different values of state matrix $Q$ and fixed $R$ matrix

Table 5.6 depicts that the performance characteristics of closed loop system using LQR regulator are superior than that of open loop system. When $Q$ increases, penalty on the states increases i.e the magnitude of the state responses decreases significantly which in turn makes the system’s output signal smaller. The speed of the system reaching to the final value depends upon the value of $Q$. Increase in weight of state matrix, improves the time response of the system by giving very small values of settling time and rise time. Further it can be seen that placing poles deeply into the left half plane by increasing weight of $Q$ exhibits decrease in the rise time, settling time and maximum overshoot of the system. Frequency response given in figure 5.14 shows noticeable improvement in stability margins i.e gain, phase margin and bandwidth of the system using LQR control technique. For any input signal, choosing higher value of $Q$ gives faster response and better stability of the system.
5.10.2 Effect of Cost Control Matrix R on the System Performance

Now consider that state weighting matrix Q is fixed at $C^T \times C$ and control cost matrix R is varied from 2.6 to 0.001.

Corresponding values of feedback gain K, eigenvalues, damping factor and frequency of the closed loop system using different values of Q and R matrices are given below in table 5.7.

<table>
<thead>
<tr>
<th>Control Cost Matrix R</th>
<th>Feedback Gain K</th>
<th>Eigenvalues</th>
<th>Damping</th>
<th>Frequency (rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.6</td>
<td>[6.47×10^1 2.63×10^5 3.69×10^{12}]</td>
<td>P1,2 = -(1.90×10^3±4.50×10^3)i P3,4 = -(1.53×10^3±1.52×10^3)i</td>
<td>0.389 0.0997</td>
<td>4.88×10^3 1.53×10^3</td>
</tr>
<tr>
<td>1</td>
<td>[1.45×10^2 5.99×10^5 9.55×10^{11}]</td>
<td>P1,2 = -(1.90×10^3±4.50×10^3)i P3,4 = -(1.93×10^2±1.53×10^3)i</td>
<td>0.389 0.126</td>
<td>4.89×10^3 1.54×10^3</td>
</tr>
<tr>
<td>0.1</td>
<td>[7.13×10^2 3.14×10^6 8.92×10^{12}]</td>
<td>P1,2 = -(1.90×10^3±4.51×10^3)i P3,4 = -(4.80×10^3±1.57×10^3)i</td>
<td>0.388 0.292</td>
<td>4.89×10^3 1.64×10^3</td>
</tr>
<tr>
<td>0.01</td>
<td>[2.21×10^3 1.14×10^7 6.18×10^{13}]</td>
<td>P1,2 = -(1.87×10^3±4.56×10^3)i P3,4 = -(1.25×10^3±1.80×10^3)i</td>
<td>0.380 0.570</td>
<td>4.92×10^3 2.20×10^3</td>
</tr>
<tr>
<td>0.001</td>
<td>[4.92×10^3 3.19×10^7 2.76×10^{14}]</td>
<td>P1,2 = -(1.83×10^3±4.93×10^3)i P3,4 = -(2.65×10^3±2.23×10^3)i</td>
<td>0.348 0.766</td>
<td>5.26×10^3 3.46×10^3</td>
</tr>
</tbody>
</table>

Table 5.7 Feedback Gain K, eigenvalues, damping and frequency of closed loop system for different values of Q and fixed R matrices

As observed from table 5.7, with the decrease in weight of control cost matrix R, the feedback gain K is increasing and poles are more deeply penetrate into the left half plane which is desirable condition for stability and to increase speed of system. Further by decreasing R, increase in natural frequency and damping factor of the closed loop system can also be observed which makes the system less oscillatory and speedy.

Time and frequency responses of the closed loop system for different values of cost control matrix R and fixed value of state weighting matrix Q are shown in figures 5.15 and 5.16 respectively.
The figures 5.15 and 5.16 illustrate the effect of change in weight of cost control matrix $R$ on the time and frequency response specifications of the system. As seen from time response shown in figure 5.15, the performance characteristics of the closed loop nanopositioning system for fixed value of state weighting matrix...
Q has been improved significantly i.e. reducing settling time, rise time, overshoot and improving stability margins as value of cost control matrix $R$ decreases.

The performance characteristics of the system by analyzing time and frequency response of closed loop nanopositioning system for different values of $R$ and fixed value of $Q$ are tabulated in table 5.8.

<table>
<thead>
<tr>
<th>Cost control matrix $R$</th>
<th>Rise time (msec.)</th>
<th>Settling time (sec.)</th>
<th>$M_p$ (%)</th>
<th>Gain margin (dB)</th>
<th>phase margin (degree)</th>
<th>Bandwidth (KHz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100000</td>
<td>0.6599</td>
<td>0.0335</td>
<td>83.06</td>
<td>4.6342</td>
<td>27.18</td>
<td>2.4878</td>
</tr>
<tr>
<td>2.75</td>
<td>0.6673</td>
<td>0.0255</td>
<td>78.54</td>
<td>6.5774</td>
<td>60.59</td>
<td>2.4905</td>
</tr>
<tr>
<td>2.65</td>
<td>0.6675</td>
<td>0.0255</td>
<td>78.38</td>
<td>6.6355</td>
<td>66.07</td>
<td>2.4906</td>
</tr>
<tr>
<td>2.6</td>
<td>0.6677</td>
<td>0.0255</td>
<td>78.29</td>
<td>6.6659</td>
<td>$\infty$</td>
<td>2.4907</td>
</tr>
<tr>
<td>1</td>
<td>0.6768</td>
<td>0.0210</td>
<td>72.15</td>
<td>8.6648</td>
<td>$\infty$</td>
<td>2.4954</td>
</tr>
<tr>
<td>0.1</td>
<td>0.7217</td>
<td>0.0086</td>
<td>41.9035</td>
<td>16.1753</td>
<td>$\infty$</td>
<td>2.5606</td>
</tr>
<tr>
<td>0.01</td>
<td>0.6898</td>
<td>0.0029</td>
<td>14.5459</td>
<td>23.9928</td>
<td>$\infty$</td>
<td>3.0399</td>
</tr>
<tr>
<td>0.001</td>
<td>0.5285</td>
<td>0.0018</td>
<td>10.5229</td>
<td>31.6462</td>
<td>$\infty$</td>
<td>4.6318</td>
</tr>
</tbody>
</table>

Table 5.8   Performance characteristics of the closed loop system for different values of $R$ and fixed value of $Q$ matrix

As depicted from table 5.8, when state weighting matrix $Q$ is fixed and decrease in value of $R$ gives a significant improvement in the time and frequency response characteristics of closed loop system designed using LQR regulator. A large improvement can be observed in the settling time and rise time of closed loop system as compared to open loop system. For $R=0.001$ the settling time and overshoot of closed loop system is 0.0018 seconds and 10.5229% as compared to 0.0335 and 83.6061% respectively of open loop system. Phase margin is infinity.
for all values of R less than 2.6. Bandwidth and gain margin improved very much with the decrease of weight of R.

From the analysis of LQR controller for different values of state weighting matrix Q and cost control matrix R, it can also be observed that increase in value of R deflate the performance of LQR controller. At $Q = C^T \times C$ and $R = 10^5$, the performance characteristics such as settling time, maximum overshoot, gain margin, phase margin and bandwidth of closed loop system using LQR regulator are almost same as that of open loop system. Further performance characteristics of the system for time and frequency response are identical if R and Q are increased by the same ratio i.e characteristics at $Q = 10 \times C^T \times C$ and $R = 0.001$ are same as that for $Q = 10 \times C^T \times C$ and $R = 0.01$.

5.11 LQR WITH REFERENCE INPUT

As in pole placement regulator, to reduce steady state error, introduce an input $\bar{N}$ after the reference input. The step response of nanopositioning system using LQR for different values of cost control matrix R with reference input is shown in figure 5.17.

![Figure 5.17 Closed loop time response of nanopositioning system using LQR with reference input](image-url)
The effect of weight of individual matrix on the time and frequency response of closed loop system using LQR control strategy has been analyzed. Now effect of both Q and R on the input or control vector will be analyzed. Consider two optimal designs i.e.

1st design: \( Q = C^T \times C ; \quad R = 0.1 \) and 2nd design: \( Q = 100 \times C^T \times C ; \quad R = 0.01 \).

The time response plots for open loop and these two optimal control designs are given in figure 5.18.

![Step response of Nanopositioning system using two different LQR controller with reference input](image)

Figure 5.18  Closed loop time response of nanopositioning system for different values of Q and R

From figure 5.18, it can be seen that response of closed loop nanopositioning system using LQR with reference input is much better than open loop system. The closed loop eigenvalues of the 2nd design are further inside the left half plane than those of 1st design, which indicates that the 2nd design would have lower value of settling time i.e faster response and lower value of input is required as compared to the 1st design. The second design gives settling time of only 0.0017 sec and maximum overshoot of 11.95 as compared to the settling time of 0.0335 sec. and maximum overshoot of 83.6% of open loop system.

### 5.12 COMPARISON OF CONVENTIONAL AND STATE FEEDBACK CONTROLLERS

As observed from the table 3.1, the dynamic characteristics of the open loop nanopositioning system are very poor especially in terms of maximum overshoot and
phase margin. In chapter 4, conventional controllers have been designed and simulated results show that dynamic characteristics of the closed loop nanopositioning system significantly improved by using P, PI, PII and PID controller. Set point tracking and output disturbance rejection has been effectively achieved by using IMC technique.

The main disadvantage of the conventional controllers such as P, PI, PID and PII controllers is that poles cannot be placed arbitrarily by choosing any value of controller gain $K_p$, $K_i$ and $K_{ii}$. Generally the conventional control design approach consists of varying the controller’s transfer function until a desired closed loop performance is achieved. Conversely, using state space approach, the closed loop performance of the system can be changed by placing all the poles arbitrarily at the desired location. Design of system according to the modern control theory using state space approach allows designer to design a system having a desired characteristic equation or an optimum control for a given performance index. Simulated results have been shown that the performance characteristics of the nanopositioning device using pole placement regulator are much better than conventional controller. Further improvement in system performance and time response of the system to any input can be observed by adding a constant gain after the reference input.

State feedback regulator designed using pole placement technique requires the change in pole locations according to the requirements of system performance. Choosing best closed-loop pole locations is very difficult because in this method we analyze the transient and frequency responses of the system, identify the various pole locations and change the pole locations that we don’t like. Further larger movement of the pole requires the greater amount of feedback gain $K$ which leads to cost and saturation problem. By using pole placement regulator, a desired close-loop behavior is satisfactory but it makes the use of large amount of input. Optimal control can be used to take best trade-off between closed-loop system performance and input energy. So, an alternative approach to pole placement regulator is Linear Quadratic Regulator (LQR) theory, which does not only place the
poles at arbitrary locations, but also guarantees stable closed-loop poles with more desirable robustness properties.

Comparative simulated results of the conventional controllers and state feedback controllers are shown in figure 5.19 and summary of performance characteristics of the nanopositioning system using these controllers is given in table 5.9.

<table>
<thead>
<tr>
<th>Type of controller</th>
<th>Rise time (sec.)</th>
<th>Settling time (sec.)</th>
<th>$M_p$ (%)</th>
<th>Gain margin (dB)</th>
<th>Phase margin (degree)</th>
<th>Bandwidth (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Open loop system</td>
<td>0.0066</td>
<td>0.0335</td>
<td>83.061</td>
<td>4.6342</td>
<td>27.181</td>
<td>2.49K</td>
</tr>
<tr>
<td>PI Controller</td>
<td>0.0300</td>
<td>0.0549</td>
<td>0</td>
<td>7.3</td>
<td>-180</td>
<td>71.92</td>
</tr>
<tr>
<td>IMC</td>
<td>0.0058</td>
<td>0.0102</td>
<td>0.009</td>
<td>25.67</td>
<td>179</td>
<td>373</td>
</tr>
</tbody>
</table>

Figure 5.19  Comparison of step response of the nanopositioning system using different types of controllers
As depicted from Table 5.9 and Figure 5.19, the open loop characteristics of the nanopositioning system are very poor and need to be improved. The problem of overshoot of open loop system has been overcome by using PI controller at the cost of increase in settling time. Set point tracking, disturbance rejection and improvement in system performance have been achieved by using IMC. Stability and time response characteristics of the system have been further enhanced by using modern state space controller.

Design and analysis of pole placement regulator have shown that system performance has been enhanced by using pole placement regulator. Furthermore, by moving closed loop poles deeply into the left half plane, the decrease in rise time, settling time and maximum overshoot of the system has been observed. But at the same time control input to the system will increase. The reduction in steady state error and improvement in performance characteristics of the system have been obtained by adding a constant gain after the reference input to the system.

Optimal controller LQR has been successfully designed and closed loop system has been simulated using LQR controller. The performance of the system for different values of the performance index matrices Q and R has also been analyzed. Based upon results and analysis, a conclusion has been drawn that an increase in weighting matrix Q results in non-linear decline in the settling and rise time of the system which makes system speedy and stable. Stability margins and bandwidth have also been improved by increasing Q. Increase in Q gives decrease in the cost of system by decreasing the required amount of control input. Decrease in the R causes the increases in the maximum feedback gain and decrease in the required control input. This causes an enhancement in the time response and frequency response characteristics of the system. The correlation between Q and R conclude that device performances can be improved by increasing weight of Q and
decreasing weight of $R$ matrices. Table 5.9 shows that the LQR controller gives optimal control and better performance characteristics of the system as compared to the pole placement controller.

5.13 CONCLUSION

Performance characteristics i.e settling time, rise time, overshoot, stability margins and bandwidth of the system for open loop and closed loop configuration of nanopositioning system have been analyzed. Conventional PID and modern controller using state space methodology have been designed and simulated using MATLAB. The system performance using conventional controllers is satisfactory but practically is not robust. State space controller using pole placement and LQR techniques have been successfully designed, simulated and implemented for nanopositioning system. These controllers not only meet the design requirements but also robust to the parameter variation.

In chapters 4 and 5, it was assumed that accurate plant is available. But in actual practice, model is essentially an approximation of the actual physical plant. There may exist modeling uncertainties i.e difference between the model and the actual plant properties. The problem of guaranteeing good control performance in the presence of plant uncertainties plays a fundamental role in all controller design. A successfully designed control system must be able to maintain stability margins and performance level even in the presence of uncertainties/ nonlinearities in system dynamics and/or in the working environment to a certain degree. If the designed controller tolerates the model mismatch, the controller is known as robust controller. The next chapter (chapter 6) synthesizes robust controllers for nanopositioning system which will improve the performance characteristics of the system in the presence of uncertainties.