Chapter 3

Bifurcations of ion acoustic solitary and periodic waves in an electron-positron-ion plasma through direct approach

3.1 Introduction

During last few decades[1]-[3], the study of nonlinear wave propagations in electron-positron-ion plasma becomes one of the interesting areas of plasma physics. It is well known that when the atoms are interacted by the cosmic ray nuclei[4], positrons are created in the interstellar medium. In 2003, Helander and Ward [5] have shown that due to collisions of runaway electrons with plasma ions or thermal electrons, positrons have been produced in tokamaks. It has been seen that most of the astrophysical plasmas contain ions in addition to electrons and positrons. So the existence of three components electron-positron-ion plasma in nature is proved and it is really very
important to investigate the nonlinear wave phenomena in such plasmas. In 1995, Popel et al. [6] investigated that the presence of positrons in electron-ion plasma affects the linear and nonlinear wave structures. Due to the long lifetime of the positrons it has been proved that most of the astrophysical and laboratory plasmas [7]-[10] become an admixture of electrons, positrons and ions. Hence to understand the behavior of astrophysical and laboratory plasmas, the study of electron-positron-ion plasmas is of great significance. Srinivas et al. [11] studied electrostatic solitons in an electron-positron-plasma with distinct group of positrons and obtained compressive and rarefactive solitons.

It has been observed that astrophysical and space plasmas contain particles having distribution functions which are quasi-Maxwellian up to the mean thermal velocities and gives non-Maxwellian suprathermals tails when the particle gains high velocities and energies [12]-[14]. These types of plasmas are called nonthermal plasmas and they are found in mercury, in the solar wind, saturn and in the magnetospheres of the Earth [14]-[15]. These types of nonthermal populations are well described by the generalized Lorentzian velocity distribution functions or kappa distributions which was shown for the first time by Vasyliunas [16] in 1968.

Recently, Ghosh et al. [17] studied the properties of non-planar ion acoustic solitary waves in an unmagnetized collisionless electron-positron-ion plasma with superthermal electrons and positrons by deriving modified Gardner equation. They have shown that the value of the spectral index significantly affects the properties of non-planar ion acoustic Gardner solitons. Very recently, Rahman et al. [18] studied the dynamics and propagation of ion acoustic waves in an unmagnetized collisionless plasma with relativistically degenerate electrons and positrons in the presence of inertial cold ions. Jain and Mishra [19] studied the large amplitude ion acoustic double layers in a colli-
sionless plasma with isothermal positrons, warm adiabatic ions and two temperature
distribution electrons using pseudo potential approach and they have shown the ef-
effect of various plasma parameters on the characteristics of the double layers in detail.
El-Tantawy et al. [20] investigated arbitrary amplitude ion-acoustic solitary waves
propagating in a magnetized plasma consisting of positive ions, superthermal elec-
trons and positrons. Boubakour et al. [21] studied arbitrary amplitude ion acoustic
solitary waves in a plasma with Boltzmann distributed positrons and superthermal
electrons. El-Awady et al.[22] studied both small and arbitrary amplitude solitary
waves by deriving Kortewegde Vries equation and energy-integral equation, respec-
tively. They have shown the dependence of the solitary excitation characteristics on
the superthermal parameters, the ion temperature, and the Mach number. Chatter-
jee et al. [23] studied the existence of ion acoustic solitary waves in a magnetized
dense electron-positron-ion plasma. They have obtained the pseudopotential directly
from the basic equations including Poissons equation without assuming the quasineu-
trality condition and they have shown the effect of ion temperature on the solitary
waves. Recently, Samanta et.al [24]-[26] studied nonlinear traveling waves in plas-
mas in the frameworks of KP and ZK equations obtained by reductive perturbation
technique (RPT) by using bifurcation theory of planar dynamical systems. Very re-
cently, Saha and Chatterjee [27] studied nonlinear electron acoustic traveling waves
in an unmagnetized quantum plasma in the framework of KdV equation obtained
by reductive perturbation technique (RPT) by using bifurcation theory of planar dy-
amical systems. Saha and Chatterjee [28] studied dust ion acoustic traveling waves
in a magnetized dusty plasma in the framework of MKP equation obtained by reduc-
tive perturbation technique (RPT) by using bifurcation theory of planar dynamical
systems.
3.2 Model equations

In this work, we consider a three component collisionless unmagnetized plasma whose constituents are inertial ions, and superthermal (\(\kappa\)-distributed) electrons and positrons. In equilibrium, the charge neutrality condition is \(n_{e0} = n_{p0} + n_0\), where \(n_{e0}\), \(n_{p0}\) and \(n_0\) are the unperturbed number densities of electron, positron and ion, respectively. The dynamics of nonlinear ion acoustic waves in such plasma is governed by the following normalized equations:

\[
\frac{\partial n}{\partial t} + \frac{\partial(nu)}{\partial x} = 0, \quad (3.1)
\]

\[
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{\partial \phi}{\partial x}, \quad (3.2)
\]

\[
\frac{\partial^2 \phi}{\partial x^2} = n_e - n_p - n, \quad (3.3)
\]

where \(n_e\), \(n_p\), and \(n\) are the number densities of electrons, positrons and ions, respectively, normalized by their unperturbed densities. In this case, \(u\) and \(\phi\) are the ion fluid velocity and electrostatic potential, respectively, normalized by the ion acoustic speed \(c = (T_e/m)^{1/2}\), and \(T_e/e\), where \(e\) is the electron charge and \(m\) is the mass of ions. The time variable is normalized by the ion plasma frequency \(\omega = (4\pi n_0 e^2 / m)^{1/2}\) and the space variable is normalized by the Debye length \(= (T_e/4\pi n_0 e^2)^{1/2}\), respectively. Here \(p = n_{p0}/n_{e0}\), and \(\sigma = T_e/T_p\).

The density of the \(\kappa\)-distributed electrons and positrons are given by,

\[
n_e = \frac{1}{1 - p}(1 - \frac{\phi}{\kappa})^{-\kappa+1/2}, \quad (3.4)
\]

\[
n_p = \frac{p}{1 - p}(1 + \frac{\phi}{\kappa})^{-\kappa+1/2}. \quad (3.5)
\]

The cause for considering \(\kappa\)-distributed electrons and positrons is discussed in detail in subsection 1.5.1, chapter 1 (see for details).
To find all traveling wave solutions of our system, we consider a new variable \( \xi = x - vt \), where \( v \) is the speed of the traveling wave. Substituting the new variable \( \xi \) into Eqs. (3.1) and (3.2) and applying the initial condition \( u = 0, n = 1 \) and \( \phi = 0 \), we obtain

\[
n = \frac{v}{\sqrt{v^2 - 2\phi}}. \tag{3.6}
\]

Substituting Eqs.(3.4), (3.5) and (3.6) into Eq.(3.3) and considering the terms involving \( \phi \) up to second degree, we have

\[
\frac{d^2 \phi}{d\xi^2} = \frac{(\kappa^2 - 1/4)(1 - p\sigma^2)}{2(1 - p)\kappa^2} - \frac{3}{2v^4}\phi^2 + \left[ \frac{(\kappa - 1/2)(1 + p\sigma)}{(1 - p)\kappa} - \frac{1}{v^2} \right] \phi. \tag{3.7}
\]

Then Eq. (3.7) is equivalent to the following travelling wave system:

\[
\begin{cases}
\frac{d\phi}{d\xi} = z, \\
\frac{dz}{d\xi} = [(\kappa^2 - 1/4)(1 - p\sigma^2)] - \frac{3}{2v^4}\phi^2 + \left[ \frac{(\kappa - 1/2)(1 + p\sigma)}{(1 - p)\kappa} - \frac{1}{v^2} \right] \phi.
\end{cases} \tag{3.8}
\]

The system Eq.(3.8) is a planar Hamiltonian system with Hamiltonian function:

\[
H(\phi, z) = \frac{z^2}{2} - \frac{(\kappa^2 - 1/4)(1 - p\sigma^2)}{2(1 - p)\kappa^2} \phi^3 - \frac{(\kappa - 1/2)(1 + p\sigma)}{(1 - p)\kappa} \phi^2 - \frac{1}{v^2} \phi^2 = h. \tag{3.9}
\]

The system Eq. (3.8) is a planar dynamical system with parameters \( \kappa, p, \sigma \) and \( v \). It is clear that the phase orbits defined by the vector fields of Eq.(3.8) will determine all traveling wave solutions of Eq.(3.7). We will study the bifurcations of phase portraits of Eq.(3.8) in the \( (\phi, z) \) phase plane depending on the parameters. A homoclinic orbit of Eq.(3.8) gives a solitary wave solution of Eq.(3.7). Similarly, a periodic orbit of Eq.(3.8) gives a periodic traveling wave solution of Eq.(3.7).
3.3 Bifurcations of phase portraits

In this section, we study bifurcations of phase portraits of Eq.(3.8). To do that, we need to find all possible periodic orbits and homoclinic orbits of the Hamiltonian system given by Eq.(3.8) when the parameters \( \kappa, p, \sigma \) and \( v \) change. When \( a \neq 0 \) and \( b \neq 0 \), then there exist two equilibrium points \( E_0(\phi_0, 0) \) and \( E_1(\phi_1, 0) \), where \( \phi_0 = 0 \) and \( \phi_1 = -\frac{b}{a} \), where \( a = \frac{(\kappa^2-1/4)(1-p\sigma^2)}{2(1-p)\kappa^2} - \frac{3}{2v^2} \) and \( b = \frac{(\kappa-1/2)(1+p\sigma)}{(1-p)\kappa} - \frac{1}{v^2} \). We consider \( M(\phi_i, 0) \) be the coefficient matrix of the linearized system of Eq.(3.8) at an equilibrium point \( E_i(\phi_i, 0) \). Then one can obtain

\[
J = \text{det} M(\phi_i, 0) = -b - 2a\phi_i. \tag{3.10}
\]

Depending on the theory of planar dynamical systems ([29]-[30]), we obtain the different phase portraits of Eq.(3.8) through the systematic analysis of the system parameters, shown in figures 3.1-3.2.

![Figure 3.1: Phase portrait of the Hamiltonian system Eq.(3.8) for \( \kappa = 1.1, p = 0.52, \sigma = 0.478 \) and \( v = 1.1 \).](image)

![Figure 3.2: Phase portrait of the Hamiltonian system Eq.(3.8) for \( \kappa = 0.56, p = 0.22, \sigma = 0.67 \) and \( v = 1.95 \).](image)

For the phase portrait given by figure 3.1, the parameters \( \kappa, p, \sigma, \) and \( v \) satisfy the relations \( \frac{(\kappa^2-1/4)(1-p\sigma^2)}{2(1-p)\kappa^2} < \frac{3}{2v^2} \) and \( \frac{(\kappa-1/2)(1+p\sigma)}{(1-p)\kappa} > \frac{1}{v^2} \). Here \( E_0(\phi_0, 0) \) is a saddle point.
and $E_1(\phi_1,0)$ is a center. We get homoclinic orbit at the equilibrium point $E_0(\phi_0,0)$ and a family of periodic orbits around the equilibrium point $E_1(\phi_1,0)$.

For the phase portrait given by figure 3.2, the parameters $\kappa, p, \sigma,$ and $v$ satisfy the relations $\frac{(\kappa^2-1/4)(1-p\sigma^2)}{2(1-p)p^2} > \frac{3}{2v^2}$ and $\frac{(\kappa-1/2)(1+p\sigma)}{(1-p)p^2} < \frac{1}{v^2}$. Here $E_0(\phi_0,0)$ is a center and $E_1(\phi_1,0)$ is a saddle point. We get homoclinic orbit at the equilibrium point $E_1(\phi_1,0)$ and a family of periodic orbits around the equilibrium point $E_0(\phi_0,0)$.

### 3.4 Analytical traveling wave solutions

In this section, with the help of the planar dynamical system Eq. (3.8) and the Hamiltonian function Eq. (3.9) with $h = 0$, we present solitary wave solution and periodic traveling wave solution of Eq. (3.7) depending on different parametric conditions.

1. When $\frac{(\kappa^2-1/4)(1-p\sigma^2)}{2(1-p)p^2} < \frac{3}{2v^2}$ and $\frac{(\kappa-1/2)(1+p\sigma)}{(1-p)p^2} > \frac{1}{v^2}$ (see Figures 3.1, 3.3, 3.4, 3.5, and 3.6), the system Eq.(3.7) has compressive solitary wave solution of the form

$$\phi = -\frac{3[(\kappa-1/2)(1+p\sigma)]}{2[2(\kappa^2-1/4)(1-pp^2)/2] - \frac{3}{2v^2}} \sech^2 \left( \frac{1}{2}\sqrt{\frac{(\kappa - 1/2)(1 + p\sigma)}{(1 - p)p^2} - \frac{1}{v^2}}\xi \right). \quad (3.11)$$

The derivation of the solution (3.11) is same as the solution (2.36) in subsection 2.2.5, chapter 2 (see for details).

2. When $\frac{(\kappa^2-1/4)(1-p\sigma^2)}{2(1-p)p^2} > \frac{3}{2v^2}$ and $\frac{(\kappa-1/2)(1+p\sigma)}{(1-p)p^2} < \frac{1}{v^2}$ (see Figures 3.2, 3.7, 3.8, 3.9 and 3.10), the system Eq.(3.7) has the periodic traveling wave solution of the form

$$\phi = -\frac{3[(\kappa-1/2)(1+p\sigma)]}{2[2(\kappa^2-1/4)(1-pp^2)/2] - \frac{3}{2v^2}} \sec^2 \left( \frac{1}{2}\sqrt{\frac{-(\kappa - 1/2)(1 + p\sigma)}{(1 - p)p^2} - \frac{1}{v^2}}\xi \right). \quad (3.12)$$

The derivation of the solution (3.12) is same as the solution (2.35) in subsection 2.2.5, chapter 2 (see for details).
3.5 Parametric analysis

In this section, we study the combined effects of different physical parameters on the characteristics of ion acoustic solitary waves and periodic waves.

In figure 3.3, we have shown the variation of solitary wave profile for different values of spectral index $\kappa$ and fixed values of $\sigma, p$ and $v$. The amplitude of the solitary waves increase and width of the solitary waves decrease with increase in spectral index $\kappa$.

In figure 3.4, we have shown the variation of solitary wave profile for different density ratios $p$ of positrons and electrons and fixed values of $\sigma, \kappa$ and $v$. The amplitude of solitary waves increase and width of solitary waves decrease with increase in density ratio $p$ of positrons and electrons.

![Figure 3.3: Solitary wave of Eq.(3.7) has been plotted for different spectral index $\kappa = 1.1$, (black long dashed), $\kappa = 2.1$ (blue solid), $\kappa = 3.1$ (red dashed), with $\sigma = 0.478$, $p = 0.52$ and $v = 1.1$.](image1)

![Figure 3.4: Solitary wave of Eq.(3.7) has been plotted for different density ratio of positrons to electrons $p = 0.38$ (black long dashed), $p = 0.46$ (blue solid), $p = 0.54$ (red dashed), with $\kappa = 1.1, \sigma = 0.478$, and $v = 1.1$.](image2)
Figure 3.5: Solitary wave of Eq. (3.7) has been plotted for different ratio of electrons temperature to positrons temperature $\sigma = 0.478$ (black long dashed), $\sigma = 0.678$ (blue solid), $\sigma = 0.878$ (red dashed), with $\kappa = 1.1, p = 0.52$ and $v = 1.1$.

Figure 3.6: Solitary wave of Eq. (3.7) has been plotted for different values of $v = 1$ (black long dashed), $v = 1.05$ (blue solid), $v = 1.1$ (red dashed), with $\kappa = 1.1, p = 0.52$ and $\sigma = 0.478$.

Figure 3.5 shows the variation of solitary wave profile for different ratios $\sigma$ of electron temperature and positron temperature and fixed values of $\kappa, p$ and $v$. The amplitude and width of the solitary waves decrease with increase in temperature ratio $\sigma$.

In figure 3.6, we have presented the variation of solitary wave profile for different values of $v$ and fixed values of $\kappa, p$ and $\sigma$. In this case, the amplitude and width of solitary waves increase with increase in $v$.

In Figure 3.7, we have shown the variation of periodic wave profile for different values of spectral index $\kappa$ and fixed values of $\sigma, p$ and $v$. The amplitude and width of the periodic waves increase with increase in spectral index $\kappa$.

In figure 3.8, we have shown the variation of periodic wave profile for different density
Figure 3.7: Periodic wave of Eq.(3.7) has been plotted for different spectral index $\kappa = 0.56$, (black long dashed), $\kappa = 0.57$ (blue solid), $\kappa = 0.58$ (red dashed), with $\sigma = 0.67$, $p = 0.22$ and $v = 1.95$.

Figure 3.8: Periodic wave of Eq.(3.7) has been plotted for different density ratio of positrons to electrons $p = 0.22$ (black long dashed), $p = 0.25$ (blue solid), $p = 0.28$ (red dashed), with $\kappa = 0.56$, $\sigma = 0.67$, and $v = 1.95$.

Figure 3.9: Periodic wave of Eq.(3.7) has been plotted for different ratio of electrons temperature to positrons temperature $\sigma = 0.61$ (black long dashed), $\sigma = 0.67$ (blue solid), $\sigma = 0.73$ (red dashed), with $\kappa = 0.56$, $p = 0.22$ and $v = 1.95$.

Figure 3.10: Periodic wave of Eq.(3.7) has been plotted for different values of $v = 1.91$ (black long dashed), $v = 1.95$ (blue solid), $v = 1.99$ (red dashed), with $\kappa = 0.56$, $p = 0.22$ and $\sigma = 0.67$.  

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ratios $p$ of positrons and electrons and fixed values of $\sigma, \kappa$ and $v$. The amplitude of and width of periodic waves increase with increase in density ratio $p$ of positrons and electrons.

Figure 3.9 shows the variation of periodic wave profile for different ratios $\sigma$ of electron temperature and positron temperature and fixed values of $\kappa, p$ and $v$. The amplitude of and width of periodic waves decrease with increase in temperature ratio $\sigma$.

In figure 3.10, we have presented the variation of periodic wave profile for different values of $v$ and fixed values of $\kappa, p$ and $\sigma$. In this case, The amplitude and width of periodic wave increase with increase in $v$.

El-Awady et al. [22] investigated nonlinear ion acoustic solitary waves in electron-positron-ion plasmas with electrons and positrons (represented by kappa distribution). They have obtained both small and arbitrary amplitude solitary waves by reductive perturbation technique and Sagdeev-like pseudopotential approach, respectively. They have established the existence regions of the solitary pulses. Furthermore, numerical calculations reveal that only supersonic pulses may exist. In this case, both small and arbitrary amplitude solitary waves are of compressive type. Thus, the results of this work [22] are in agreement with our results.

3.6 Conclusions

We have investigated ion acoustic solitary waves and periodic waves in an unmagnetized plasma with kappa distributed electrons and positrons through non-perturbative approach. By using the bifurcations of phase portrait analysis, we have proved that
our model has solitary wave solutions and periodic traveling wave solutions. We have derived two analytical solutions for solitary wave and periodic wave using planar dynamical system (3.8) and hamiltonian function given by Eq.(3.9). From these analytical traveling wave solutions, the effects of spectral index \((\kappa)\), density ratio \((p)\), temperature ratio \((\sigma)\) of electrons and positrons and speed of the traveling wave \(v\) are shown on characteristics of ion acoustic solitary waves and periodic waves. The spectral index \((\kappa)\), density ratio \((p)\) of positrons and electrons and temperature ratio \((\sigma)\) significantly affect the characteristics of ion acoustic solitary, blow up solitary and periodic structures.
Bibliography


