Chapter 8

Propagation and interaction of ion-acoustic multi-solitons in electron-positron-ion magnetoplasma

8.1 Introduction

Recently, Sahu et al. [1] studied propagation of the two-soliton for ion acoustic waves (IAWs) in an unmagnetized plasma consisting of superthermal electrons and warm ions in bounded non-planar geometry. They showed that the presence of superthermally distributed electrons influences propagation of the two-soliton in non-planar geometry. Ref. [2] studied propagation of two-soliton for electron acoustic waves in an unmagnetized, collision-less plasma consisting of a cold-electron fluid and hot electrons obeying $\kappa$ velocity distribution, and stationary ions in non-planar geometry. The author found that the propagation of the two-soliton wave is affected by
superthermal hot electrons and other plasma parameters. Very recently, ref. [3] studied the propagation of ion acoustic two-soliton interaction in a three component collision-less unmagnetized plasma in the framework of the KdV equation. They have demonstrated the effect of the parameter $q$ on the profiles of two-soliton structures.

Ref. [4] investigated nonautonomous matter waves with time-dependent modulation in a one-dimensional trapped spin-1 Bose-Einstein condensate with the help of the generalized three-coupled Gross-Pitaevskii equations by using the Hirota’s bilinear method. Ref. [5] studied three-coupled KdV equations corresponding to the Neumann system of the fourth order eigenvalue problem through the dependent variable transformations. They obtained bilinear forms of these equations to obtain the multi-soliton solutions and their interaction. Ref. [6] investigated the coupled higher-order nonlinear Schrödinger equations with variable coefficients, which represent the propagation of femtosecond soliton pulses comprising two fields with the left and right polarization in the inhomogeneous optical fiber media. They derived one-soliton and two-soliton-like solutions and they also studied head-on and overtaking elastic interactions decided by the directions of the velocities. Ref. [7] obtained double-Wronskian soliton solutions by virtue of some double Wronskian identities and they applied asymptotic analysis to investigate the interaction between the two solitons. They also constructed first- and second-order rogue-wave solutions via a generalized Darboux transformation. Ref. [8] investigated breakup and switching of the Manakov-typed bound vector solitons (BVSs) induced by two types of stochastic perturbations: the homogenous and nonhomogenous. They discovered symmetry-recovering for the asymmetrical homogenous case, while soliton switching was found to be related to the perturbation amplitude and soliton coherence.
8.2 Basic Equations

We consider the same model equations as in section 5.2, chapter 5. Detailed descriptions of the model equations are given in section 5.2. The normalized continuity, momentum and Poisson’s equations are, respectively, given by:

\[
\frac{\partial n}{\partial t} + \nabla (n \vec{U}) = 0, \quad (8.1)
\]

\[
\frac{\partial \vec{U}}{\partial t} + (\vec{U} \cdot \nabla) \vec{U} = -\nabla \phi + \vec{U} \times \hat{x}, \quad (8.2)
\]

\[
\nabla^2 \phi = \alpha (n_e - n_p - n), \quad (8.3)
\]

The normalized electron and positron densities are given by

\[n_e = (1 + p)(1 - \frac{\phi}{\kappa-3/2})^{-(\kappa-1/2)} \quad \text{and} \quad n_p = p(1 + \frac{\delta \phi}{\kappa-3/2})^{-(\kappa-1/2)},\]

where \(p = \frac{n_{po}}{n_{so}}, \alpha = \frac{r^2}{\lambda^2}, \quad r = \frac{C_s}{\Omega} \quad \text{and} \quad \delta = \frac{T_e}{T_p}.

The cause for considering \(\kappa\) distributed electrons and positrons is described in detail in subsection 1.5.1, chapter 1 (see for details).

8.3 KP equation

Applying the RPT one can obtain the KP equation given by

\[
\frac{\partial}{\partial \eta} \left[ \partial \phi_1 + A \phi_1 \frac{\partial \phi_1}{\partial \eta} + B \frac{\partial^3 \phi_1}{\partial \eta^3} \right] + C \frac{\partial^2 \phi_1}{\partial Y^2} = 0, \quad (8.4)
\]

where

\[A = \frac{1}{3W} \left[ 3 - \frac{(\kappa+1) (1+p-p \delta^2)}{(\kappa-\frac{3}{2}) (1+p+p \delta^2)} \right], \quad B = \frac{W}{2 \alpha (1+p+p \delta) (\kappa-\frac{3}{2})}, \quad \text{and} \quad C = \frac{V}{\eta}.
\]

The KP equation (8.4) is derived in section 5.3, chapter 5 (see for details).
8.4 Two-soliton and three-soliton solutions of the KP equation

Let us replace \( \eta \) by \( \bar{\eta} \), \( \phi_1 \) by \( -6\bar{\phi}_1 B/A \), \( \tau \) by \( \bar{\tau}/B \), and \( Y \) by \( \sqrt{\frac{C}{D}} \bar{Y} \), then the equation (8.4) can be transformed to the following standard KP equation

\[
\frac{\partial}{\partial \bar{\eta}} \left( \frac{\partial \bar{\phi}_1}{\partial \tau} - 6 \bar{\phi}_1 \frac{\partial \bar{\phi}_1}{\partial \bar{\eta}} + \frac{\partial^3 \bar{\phi}_1}{\partial \bar{\eta}^3} \right) + 3 \frac{\partial^2 \bar{\phi}_1}{\partial \bar{Y}^2} = 0,
\]

Now using the transformation \( \bar{\phi}_1 = -2(\log f) \bar{\eta} \), we obtain the Hirota bilinear form of the KP equation (8.5) as

\[
(D_\bar{\tau} D_\bar{\eta} + D_\bar{\eta}^4 + 3D_\bar{Y}^2) \{ f.f \} = 0.
\]

Here \( D \) is the Hirota’s operator. Now in order to obtain the two-soliton solution, we insert \( f = 1 + \varepsilon(e^{\tilde{\theta}_1} + e^{\tilde{\theta}_2}) + \varepsilon^2 f_2 \) into the Eq.(8.6), where \( \tilde{\theta}_i = k_i \bar{\eta} + \omega_i \bar{\tau} + l_i \bar{Y} + \beta_i, \ i=1, 2 \). The coefficients of different powers of \( \varepsilon \) give \( \omega_i = -\frac{k_i^4 + 3l_i^2}{k_i^2} \), \( f_2 = a_{12}e^{\tilde{\theta}_1 + \tilde{\theta}_2} \) with the phase shift as

\[
a_{12} = \frac{(k_1 \omega_2 + k_2 \omega_1 + 4k_1^3k_2 - 6k_1^2k_2^2 + 4k_1k_2^3 + 6l_1l_2)}{(k_1 \omega_2 + k_2 \omega_1 + 4k_1^3k_2 - 6k_1^2k_2^2 + 4k_1k_2^3 + 6l_1l_2)}.
\]

Finally considering \( \varepsilon = 1 \), we obtain the two-soliton solution of the KP equation (8.5) as

\[
\bar{\phi}_1 = -2k_1^2e^{\tilde{\theta}_1} + k_2^2e^{\tilde{\theta}_2} + ((k_1 - k_2)^2 + a_{12}((k_1 + k_2)^2 + k_2^2e^{\tilde{\theta}_1} + k_1^2e^{\tilde{\theta}_2}))e^{\tilde{\theta}_1 + \tilde{\theta}_2}.
\]

Thus, the two-soliton solution of the KP equation (8.4) is given by

\[
\phi_1 = \frac{12Bk_1^2e^{\theta_1} + k_2^2e^{\theta_2} + ((k_1 - k_2)^2 + a_{12}((k_1 + k_2)^2 + k_2^2e^{\theta_1} + k_1^2e^{\theta_2}))e^{\theta_1 + \theta_2}}{A(1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1 + \theta_2})^2}.
\]
where \( \theta_i = k_i \eta - \frac{(k_i^4 + 3l_i^2)}{k_i} B \tau + \sqrt{\frac{3E}{C}} Y l_i + \beta_i, \quad i = 1, 2. \)

Similarly, the three-soliton solution of the KP equation (8.4) has the form

\[
\phi_1 = -2 \frac{\partial^2}{\partial \eta^2} (\ln[g(\eta, \tau, Y)]),
\]

(8.10)

where \( g(\eta, \tau, Y) = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + \alpha_{12} e^{\theta_1+\theta_2} + \alpha_{23} e^{\theta_2+\theta_3} + \alpha_{31} e^{\theta_3+\theta_1} + \alpha_{123} e^{\theta_1+\theta_2+\theta_3}, \)

with \( \alpha_{ij} = \frac{(k_i \omega_j + k_j \omega_i + 4k_i^2 k_j^2 - 6k_i^2 k_j^2 + 4k_i k_j^3 + 6l_i l_j)}{(k_i \omega_j + k_j \omega_i + 4k_i^2 k_j^2 + 6k_i^2 k_j^2 + 4k_i k_j^3 + 6l_i l_j)}, \quad (i, j = 1, 2, 3, i < j), \alpha_{123} = \alpha_{12} \alpha_{23} \alpha_{31} \) and

\[
\theta_i = k_i \eta - \frac{(k_i^4 + 3l_i^2)}{k_i} B \tau + \sqrt{\frac{3E}{C}} Y l_i + \beta_i, \quad i = 1, 2, 3.
\]

### 8.5 Numerical simulations and discussions

In this section, with the help of numerical simulations, we present the propagation and interaction of two-solitons and three-solitons. It is important to mention that the two-soliton and the three-soliton solutions can be expressed in particular forms such that their relationship to the solitary waves are clearly apparent, and the utility of this special formulation of the solutions can be demonstrated in analyzing the structures during interactions of the two-soliton and the three-soliton solutions of the KP equation (8.4). The two-soliton (the three-soliton) with different amplitudes will collide when the largest one of them is overtaking the smaller ones.

In figure 8.1, time evolutions of interaction of the compressive two solitons \( \phi_1 \) vs \( \eta \) is plotted for different values of \( \tau \). It is shown that for \( \tau = -5 \) the larger amplitude soliton is behind the smaller amplitude soliton. Then, at \( \tau = -1 \) the two solitons merge and become one soliton at \( \tau = 0 \). But, at \( \tau = 1 \), they separate from each other and then finally they depart from each other when \( \tau = 5 \). The combined profile of the two-soliton is also shown here.
Figure 8.1: Variation of the compressive two-soliton profiles for different values of $\tau$, with $k_1 = 1, k_2 = 2, l_1 = 1, l_2 = 2, Y = 2, \kappa = 0.1, p = 0.8, \beta_1 = 1, \beta_2 = 1$ and $\delta = 1.6$.

In figure 8.2, the variation of the compressive two-soliton profiles for different values of $\kappa$ with $\tau = 5$ is presented with same values of other parameters as figure 8.1. It is clear from the figure 8.2 that when $\kappa$ increases, velocity of the two-soliton increases. Thus, the value of $\kappa$ significantly affects the nature of propagation of the two-soliton in e-p-i magnetoplasmas.
Figure 8.2: Variation of the compressive two-soliton profiles for different values of $\kappa$ with $\tau = 5$, and other parameters are same as figure 8.1.

In figure 8.3, time evolutions of interaction of the compressive three solitons $\phi_1$ vs $\eta$ is plotted for several values of $\tau$ with $k_1 = 1, k_2 = 1.6, k_3 = 2.2, l_1 = 1, l_2 = 2, l_3 = 3, Y = 10, \kappa = 0.1, p = 0.8, \beta_1 = 1, \beta_2 = 1, \beta_3 = 1, \alpha = 0.8$, and $\delta = 1.6$. It is seen that for $\tau = -5$ the larger amplitude soliton is behind the small amplitude solitons. When $\tau = 1$, the three solitons merge and become one soliton at $\tau = 2.5$. But, at $\tau = 4$, they separate from each other, and then finally they depart from each other when $\tau = 10$. The combined picture of the compressive three-soliton profiles is also presented here.

In figure 8.4, the variation of the compressive three-soliton profiles for different values of $\kappa$ with $\tau = 10$ is presented keeping other parameters same as figure 8.3. It is clear from figure 8.4 that when $\kappa$ increases, velocity of three-solitons increase. Thus, the value of $\kappa$ significantly affects the propagation of three-solitons in e-p-i magnetoplasmas. To conclude, the parameter $\kappa$ plays a critical role in determining the nature of propagation of two-soliton and three-soliton in e-p-i magnetoplasmas.
Figure 8.3: Variation of the compressive three-soliton profiles for different values of $\tau$, with $k_1 = 1, k_2 = 1.6, k_3 = 2.2, l_1 = 1, l_2 = 2, l_3 = 3, Y = 10, \kappa = 0.1, p = 0.8, \beta_1 = 1, \beta_2 = 1, \beta_3 = 1, \alpha = 0.8$, and $\delta = 1.6$. 
8.6 Conclusions

The KP equation in electron-positron-ion magnetoplasma with kappa distributed electrons and positrons is derived using RPT. Using the Hirota’s direct method, the propagation of the two-soliton and the three-soliton of the KP equation is studied. The effect of the spectral index $\kappa$ on propagation of the two-soliton and the three-soliton is presented. The spectral index $\kappa$ plays a significant role on the propagation of the multi-soliton of the KP equation.
Bibliography


