Chapter II

Optimal Replacement Policy for General Reliability Systems Exposed to Dependent Stochastic Failures

1. Introduction

Optimisation methods based primarily on mean time to failure (MTTF) performances in addition to cost considerations is attracting much attention from researchers in recent times (see [8]). Nakagawa [1] developed an optimal replacement policy for example, (ORP) for a parallel system comprised of IDENTICAL UNITS operating in a random environment, the individual units being assumed to fail independently (following Rade [3]). He further improved the policy for systems exposed to dependent stochastic failures (see also [2]) with replacement costs structured as a linear function of the number of failed components. He observed that the stochastic dependency in failure patterns is more reasonable, for the damage to the component caused by shocks would cumulate and hence the failure rate increases the ageing factor sets in.

Relaxing the restrictive assumption of IDENTICAL UNITS leads to many operational advantages. For example, in real life situations as obtained due to 'fluctuating market conditions' and/or 'technological progress' (specially, for example, in such cases as micro-electronic components), it may not always be
possible to replace failed components with the original /identical units. Thus there arises the valid reason to develop optimisation methods to systems that are comprised of not necessarily identical units (in the sense that the failure probabilities of individual units may not be the same).

We call these as "general systems", that is, parallel reliability systems comprised of units not necessarily identical (see [4-6]). We now describe an improved version of ours as compared to the ORP in [6]. The improvement essentially consists in the practically useful ramification of 'dependent stochastic failures', in the sense that an operating unit failing by the jth shock depends on the number of shocks already received by the system and the cumulative damage that is built-up into system. For such a system (comprised of say 'n' units) we propose an ORP where the system is replaced after failure of the optimal number of units \( k^* \) \((< n)\) \( k^* \) being identified thus: (i). the system is replaced (or exchanged) once the number of failed units exceed \( k^* \), even before the system's total failure (that is, after the failure of all the n units), and (ii). \( k^* \) is obtained in minimising the expected replacement cost per unit of expected replacement time, corresponding to the steady state case.

In the following sections we state the model and derive results leading to the ORP for the model. Empirical work illustrating the applicational use of the method is provided as a
companion study. Also earlier results in [2] are recovered as special cases.

We first give the nomenclature and notation for ready reference.

**Nomenclature and Notation**

\[ n = \text{Number of units constituting a general parallel system (that is system comprising of non-identical components), } n \text{ (} n \geq 2 \text{).} \]

\[ p_i(j) = \text{Probability of failure of the } i\text{th component in a system, } i = 1,2,\ldots,n, \text{ by the } j\text{th shock, } j = 1,2,\ldots. \]

\[ q_i(j) = [1-p_i(j)], \text{ that is probability of failure-free operation of the } i\text{th component after the } j\text{th shock.} \]

\[ Y_j = \text{Total number of components that failed by the } j\text{th shock, } j = 1,2,\ldots. \]

\[ P_i(j) = \sum_{\ell=1}^{j} p_i(\ell) = \text{Cumulative failure probability of the } i\text{th component up to the } j\text{th shock, that is probability of the component failure due to a shock received at or earlier to the } j\text{th epoch of shock occurrence.} \]

\[ \bar{P}_i(j) = 1-P_i(j) \]

\[ k = \text{The number of components more than which, if failed, the system is exchanged before total failure of all the '} n' \text{ components (that is, the system is exchanged if } (k+1), (k+2),\ldots,(n-1) \text{ components fail). Thus '} k' \text{characterises the replacement policy (combining the two cases of exchange as well as replacement} \]
after total failure) of the general parallel system. Let us call this as the k-replacement policy.

\[ B_n(k) = \text{Probability of replacement of the system consequent upon total failure corresponding to a k-replacement policy.} \]

\[ B_m(k) = \text{Probability of exchange of the system prior to total failure when number of failed components is } m, \]
\[ (k+1) \leq m \leq (n-1) \]

\[ \text{MTR (k)} = \text{Mean time to replacement of the system corresponding to a k-replacement policy.} \]

\[ C_0 = \text{Cost resulting from the failure of a single unit.} \]

\[ nC_0 + C_1 = \text{Total replacement cost of the system, consequent upon the failure of all the 'n' units, } C_1, \text{ a constant characterising some fixed replacement costs.} \]

\[ mC_0 + C_2 = \text{Total exchange cost of the system after 'm' units have failed, } C_2, \text{ a constant characterising fixed exchange costs.} \]

\[ C(k) = \text{Expected replacement cost per unit of expected replacement time, in the steady state, time span being assumed to be infinite.} \]

\[ \text{ORP} = \text{Optimal replacement policy.} \]

\[ r \sum_{i=1}^{n} \frac{r}{i} = \sum_{i=1}^{n} \frac{r}{i}, \quad r \geq 1 \]

\[ = 1, \quad r = 0 \]
2. THE MODEL AND THE ORP

The Model

Let us consider a general parallel system comprised of 'n' (≥ 2) components not necessarily identical in the sense that their failure probabilities may not be the same. The system operates in a random environment which sends out shocks. The epochs of shock occurrences are assumed to form a renewal point process whose interval properties are governed by known random law \( F(.) \) with \( F(0) = 0 \). Let the mean of these random time intervals between two successive shock occurrence be \( \frac{1}{\lambda} \). A shock may or may not produce a component's failure or may also produce more than one (that is, multiple) failures of the components. The components fail consequent upon shocks received as per the following pattern of stochastic dependency. Let the probability that an ith (i = 1, 2, ..., n) operating unit failing by the shock be denoted by \( p_i(j) \), j = 1, 2, ..., which depends on the number of shocks, and let \( P_i(j) = \sum_{\ell=1}^{j} p_i(\ell) \) be the cumulative failure probability.

The system is replaced when all the components fail where a cost \( nC_0 + C_1 \), is assumed to be incurred, \( C_0 \) being the costs resulting from the failure and the replacement of one component and \( C_1 \), being the general fixed costs of replacement of the system. We have several exchange policies corresponding to different \( k's \) \((k = 0, 1, 2, ..., (n-1))\), when the system is exchanged.
before the total failure of the system (that is, when all, the 'n' components fail). A system's replacement policy is thus characterised by $k(n)$, combining both the actions of replacement and exchange.

Now, in the notation as given in nomenclature, the expected replacement cost per unit of expected replacement time (in the steady-state case) corresponding to a $k$-replacement policy is given by:

$$C(k) = \frac{(C_0 n + C_1) B_n(k) + \sum_{m=k+1}^{n-1} (C_0 m + C_2) B_m(k)}{MTR(k)}$$

(1)

The ORP

We are now naturally lead to the ORP in terms of $k^*$ replacement action where $k^*$ corresponds to that number at which $C(k)$ in (1) attains its minimum value. Notice that this minimum value is identified through a perusal of all the 'n' computed values of $C(k)$ for $k = 0, 1, 2, \ldots, (n-1)$.

In order to obtain the ORP so proposed, we need the explicit expressions for $B_n(k)$, $B_m(k)$ and $MTR(k)$. The general as well as other results for the special case of dependent stochastic failures governed by geometric law are presented in the following section.
3. ANALYTIC RESULTS

We now state and establish the following results for the general model (using the notation given in the nomenclature).

Theorem I: The probability $B_m(k)$ is given by:

$$B_m(k) = \sum_{j=1}^{\infty} \sum_{i=1}^{n-m} \binom{n}{m} \binom{r}{m} \frac{k}{\sum_{r=0}^{m-1} \binom{m}{r} u \cdot \binom{m}{v}} \sum_{i=1}^{n-m} \binom{m}{r} u \cdot \binom{m}{v} \prod_{j=1}^{\infty} B_{m,j}(k),$$

(2)

where $B_{m,j}(k) = \sum_{i=1}^{n-m} \binom{n}{m} \binom{m}{r} \cdot \binom{m-\gamma}{r} \prod_{j=1}^{\infty} p_{w}(j) \prod_{j=1}^{\infty} p_{w}(j-1)$,

and $m = (k+1), \ldots, n$.

(2a)

where $p_{w}(j)$'s, $p_{w}(j)$'s and $p_{w}(j)$'s run through the different distinct combinations as indicated in $(.)$, in the $\binom{r}{m}$'s, $w = i,u,v$ and the failure probabilities $p_i$, $p_u$ and $p_v$ correspond to different components.

PROOF: A system's exchange in a $k$-replacement policy brought about due to failure of '$m$' units consequent to the $j$th shock ($j = 1,2,\ldots$), when exactly '$m$' components $[(k+1)_{m \leq n}]$ fail up to the $j$th shock and only '$r$' components ($r = 0,1,2,\ldots,k$) fail up to $(j-1)$th shock so that at the $j$th shock, $(n-m)$ components still operate failure-free. The components being distinct (that is, NON-IDENTICAL). We have $\binom{n}{r}$ and $\binom{n}{m}$ combinations respectively of such events. Thus, we have

$$B_m(k) = \sum_{j=1}^{\infty} \sum_{r=0}^{k} \Pr \left[ y_1 + y_2 + \ldots + y_{j-1} = r \text{ and } y_j = (m-r) \right],$$

(3)

and

$$= \sum_{j=1}^{\infty} B_{m,j}(k).$$

(3a)
Now, using the well known addition and multiplication laws of probabilities to the events in (3) and the arguments given above, we obtain the result in (2).

Now proof is complete.

**Theorem II:** The MTR\((k)\) is given by:

\[
MTR(k) = \sum_{j=1}^{\infty} \frac{1}{\lambda} \sum_{m=k+1}^{n} B_{m,j}(k).
\]

**PROOF:** Using (3a), the probability of exchange/replacement is:

\[
\sum_{m=k+1}^{n} B_{m,j}(k).
\]

The average time upto \(j\)th shock is given by:

\[
\frac{j}{\lambda},
\]

so that:

\[
MTR(k) = \sum_{j=1}^{\infty} \frac{1}{\lambda} \sum_{m=k+1}^{n} B_{m,j}(k).
\]

The proof is complete.

**Theorem III:** The probability \(B_{n}(k)\) is given by:

\[
B_{n}(k) = \sum_{j=1}^{\infty} \sum_{r=0}^{k} \binom{n-r}{r} \sum_{u=1}^{n-r} \sum_{v=1}^{r} p_{u}(j) p_{v}(j-1),
\]

in the notation as explained in (2).

**PROOF:** The proof proceeds on similar lines as in Theorem I from the relation:
\[ B_n(k) = \sum_{j=1}^{\infty} \sum_{r=0}^{n} \Pr[y_1 + y_2 + \ldots + y_{j-1} = r \text{ and } y_j = (n-r)]. \quad (7) \]

Theorem IV: \( C(k) \) is given by:

\[
C_2 + (C_1 - C_2)B_n(k) + C_0 \sum_{m=k+1}^{n} m B_m(k) \]

\[
\frac{\text{MTR}(k)}{MTR(k)}.
\]

\[
(8)
\]

**PROOF:** Using the theorems: I, II and III, the model descriptio-

\[
\text{nal and the replacement policy we have :}
\]

\[
\frac{(C_0 + C_1) B_n(k) + \sum_{m=k+1}^{n-1} (C_0 + C_2) B_m(k)}{MTR(k)}.
\]

\[
C(k) = \frac{\sum_{m=k+1}^{n} B_m(k)}{MTR(k)},
\]

as given in (1)

Noticing that,

\[
B_n(k) = 1 - \sum_{m=k+1}^{n} B_m(k), \quad (9)
\]

After some rearrangement of terms in the numerator of (1)

and using (9), we obtain the result in (8)

The proof is complete.

A Remark: When the units are IDENTICAL, we observe that

the results in Nakagawa [2] follow from our general results in

(2), (4), (6) and (8).

Now, for the purpose of illustrating the operational use

of these analytic results, we specialise to the case of dependent

stochastic failures, governed by the geometric law:
\[ p_i(j) = p_i q_i^{j-1}; \quad p_i + q_i = 1; \quad p_i, q_i > 0, \quad i=1,2,...,n \quad \text{and} \quad j=1,2,... \]  

(10)

Further, for the particular systems corresponding to: \( n=2 \) and 4, explicit expressions are derived and presented in the following:

Numerical work was also later presented for the choices (of values): \( c_0 = 0.05, \quad c_1 = 1 \) and \( c_2 = 10 \) not only for illustrative purposes but also to facilitate comparison with the numerical work corresponding to those in Nakagawa[2].

As \( \lambda \) does not participate in the comparisons of the \( c(k) \) values, no particular numerical value is specified in the empirical work.

Explicit expressions for \( n=3 \) may also be derived on similar lines but these are omitted here not only for the sake of brevity but these are not used in later comparative analysis.

(a) 2-Unit System

\[ B_1(0) = \sum_{j=1}^{\infty} \left\{ q_2^j p_2 q_2^{j-1} + q_1^j p_1 q_1^{j-1} \right\} \]

\[ = \frac{p_1 q_2 + p_2 q_1}{1 - q_1 q_2} \]  

(11)

\[ B_2(0) = \sum_{j=1}^{\infty} (p_1 p_2)(q_1 q_2)^{j-1} = \frac{p_1 p_2}{1-q_1 q_2} \]  

(12)
\[
MTR(0) = \frac{1}{\lambda} \left\{ \sum_{j=1}^{\infty} j (q_1 q_2)^{j-1} \left[ p_1 p_2 + p_1 q_2 + p_2 q_1 \right] \right\} \cdot \frac{\left[ p_1 p_2 + p_1 q_2 + p_2 q_1 \right]}{\lambda (1 - q_1 q_2)^2} . \tag{13}
\]

\[
B_2(1) = \sum_{j=1}^{\infty} (p_1 p_2) (q_1 q_2)^{j-1} + \sum_{j=1}^{\infty} p_1 q_1^{j-1} (1 - q_2^{j-1}) + \sum_{j=1}^{\infty} p_2 q_2^{j-1} (1 - q_1^{j-1}) .
\]

\[
= \frac{p_1 p_2 - p_1 - p_2}{1 - q_1 q_2} + \frac{p_1}{1 - q_1} + \frac{p_2}{1 - q_2} . \tag{14}
\]

\[
MTR(1) = \frac{1}{\lambda} \left\{ \sum_{j=1}^{\infty} j (q_1 q_2)^{j-1} p_1 p_2 + \sum_{j=1}^{\infty} j (q_1)^{j-1} p_1 (1 - q_2^{j-1}) + \sum_{j=1}^{\infty} j (q_2)^{j-1} p_2 (1 - q_1^{j-1}) \right\} ,
\]

\[
= \frac{1}{\lambda} \left\{ \frac{(p_1 p_2 - p_1 - p_2)}{(1 - q_1 q_2)^2} + \frac{p_1}{(1 - q_2)^2} + \frac{p_2}{(1 - q_2)^2} \right\} . \tag{15}
\]
\[
\begin{align*}
B_2(0) &= \sum_{j=1}^{\infty} \left\{ (q_1q_2)^j p_3 p_4 (q_3q_4)^{j-1} + (q_1q_3)^j p_2 p_4 (q_2q_4)^{j-1} + \\
&\quad (q_1q_4)^j p_2 p_3 (q_2q_4)^{j-1} + (q_2q_3)^j p_1 p_4 (q_1q_4)^{j-1} + \\
&\quad (q_2q_4)^j p_1 p_3 (q_1q_3)^{j-1} + (q_3q_4)^j p_1 p_2 (q_1q_2)^{j-1} \right\}, \\
&= \frac{[p_3 p_4 q_1 q_2^* + p_2 p_4 q_1 q_3^* + p_2 p_3 q_1 q_4^* + p_1 p_4 q_2 q_3^* + p_1 p_3 q_2 q_4^* + p_1 p_2 q_3 q_4^*]}{(1 - q_1 q_2 q_3 q_4)}.
\end{align*}
\]

\[
B_3(0) = \sum_{j=1}^{\infty} \left\{ q_1^j p_2 p_3 p_4 (q_2 q_3 q_4)^{j-1} + q_2^j p_1 p_3 p_4 (q_1 q_3 q_4)^{j-1} + \\
&\quad q_3^j p_1 p_2 p_4 (q_1 q_2 q_4)^{j-1} + q_4^j p_1 p_2 p_3 (q_1 q_2 q_3)^{j-1} \right\}, \\
&= \frac{[p_2 p_3 p_4 q_1^* + p_1 p_3 p_4 q_2^* + p_1 p_2 p_4 q_3^* + p_1 p_2 p_3 q_4^*]}{(1 - q_1 q_2 q_3 q_4)}. \\
\]

\[
B_4(0) = \sum_{j=1}^{\infty} (p_1 p_2 p_3 p_4) (q_1 q_2 q_3 q_4)^{j-1}, \\
&= \frac{p_1 p_2 p_3 p_4}{1 - q_1 q_2 q_3 q_4}.
\]
\[ \text{MTR}(O) = \frac{1}{\lambda} \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} j (q_1 q_2 q_3 q_4)^{j-1} \left[ p_1 p_2 p_3 p_4 + p_1 p_2 p_3 q_4 + p_1 p_2 q_3 p_4 + p_1 q_2 p_3 p_4 + q_1 p_2 p_3 p_4 + p_1 q_2 p_3 q_4 + q_1 q_2 p_3 p_4 + q_1 q_2 p_3 q_4 \right]^2 \]

\[ = \left[ p_1 p_2 p_3 p_4 + p_1 p_2 p_3 q_4 + p_1 q_2 p_3 p_4 + q_1 p_2 p_3 q_4 + q_1 q_2 p_3 q_4 \right] \times (1-q_1 q_2 q_3 q_4)^2 \]

\[ b_2(1) = b_2(0) + \sum_{j=1}^{\infty} \left\{ (q_1 q_2)^{j-1} (1-q_4)^{j-1} + (q_1 q_2)^{j-1} (1-q_4)^{j-1} + (q_1 q_3)^{j-1} (1-q_4)^{j-1} + (q_1 q_3)^{j-1} (1-q_4)^{j-1} + (q_2 q_3)^{j-1} (1-q_4)^{j-1} + (q_2 q_3)^{j-1} (1-q_4)^{j-1} + (q_2 q_4)^{j-1} (1-q_4)^{j-1} + (q_2 q_4)^{j-1} (1-q_4)^{j-1} + (q_3 q_4)^{j-1} (1-q_4)^{j-1} + (q_3 q_4)^{j-1} (1-q_4)^{j-1} \right\} \]

\[ = b_2(0) - \frac{\left[ p_1 q_2 q_3 + p_1 q_2 q_4 + p_1 q_2 q_3 + q_1 q_2 q_3 \right]}{(1-q_1 q_2 q_3 q_4)} \]
\[
\begin{align*}
B_3(1) &= B_3(0) + \sum_{j=1}^{\infty} \left\{ q_1^{j-1}(p_2p_3)(q_2q_3)^{j-1}(1-q_4^{j-1}) + q_2^{j-1}(p_3p_4)(q_3q_4)^{j-1}(1-q_1^{j-1}) + \\
&\quad q_3^{j-1}(p_4p_1)(q_4q_1)^{j-1}(1-q_2^{j-1}) + q_4^{j-1}(p_1p_2)(q_1q_2)^{j-1}(1-q_3^{j-1}) +\right. \\
&\quad \left. q_1^{j-1}(p_2p_4)(q_2q_4)^{j-1}(1-q_3^{j-1}) + q_2^{j-1}(p_3p_4)(q_3q_4)^{j-1}(1-q_1^{j-1}) + \\
&\quad q_2^{j-1}(p_1p_3)(q_1q_3)^{j-1}(1-q_4^{j-1}) + q_3^{j-1}(p_4p_1)(q_4q_1)^{j-1}(1-q_2^{j-1}) +\right. \\
&\quad \left. q_3^{j-1}(p_1p_2)(q_1q_2)^{j-1}(1-q_3^{j-1}) \right\} + \\
&= B_3(0) - \frac{\left[ (p_2p_3)q_1+(p_2p_3)q_4+(p_3p_4)q_2+(p_3p_4)q_1+(p_4p_1)q_3+ \\
(p_4p_1)q_2+(p_2p_4)q_1+(p_2p_4)q_3+(p_1p_3)q_2+(p_1p_3)q_4+\right] (p_1p_2)q_3+(p_1p_2)q_4}{(1-q_1q_2q_3q_4)}.
\end{align*}
\]
\[
\frac{[(p_2 p_3) q_1 + (p_1 p_3) q_2 + (p_1 p_2) q_3]}{(1-q_1 q_2 q_3)} + \frac{[(p_4 p_1) q_2 + (p_2 p_4) q_1 + (p_1 p_2) q_4]}{(1-q_1 q_2 q_4)} + \\
\frac{[(p_3 p_4) q_1 + (p_4 p_1) q_3 + (p_1 p_3) q_4]}{(1-q_1 q_3 q_4)} + \frac{[(p_2 p_3) q_4 + (p_3 p_4) q_2 + (p_2 p_4) q_3]}{(1-q_2 q_3 q_4)}, (22)
\]

\[
B_4(1) = \sum_{j=1}^{\infty} \left\{ (p_1 p_2 p_3 p_4) (q_1 q_2 q_3 q_4)^{j-1} (p_1 p_2 p_3) (q_1 q_2 q_3)^{j-1} (1-q_4^{j-1}) + \\
(p_1 p_2 p_4) (q_1 q_2 q_4)^{j-1} (1-q_3^{j-1}) + \\
(p_1 p_3 p_4) (q_1 q_3 q_4)^{j-1} (1-q_2^{j-1}) + \\
(p_2 p_3 p_4) (q_2 q_3 q_4)^{j-1} (1-q_1^{j-1}) \right\}.
\]

\[
= \frac{[(p_1 p_2 p_3 p_4) - (p_1 p_2 p_3) - (p_1 p_2 p_4) - (p_1 p_3 p_4) - (p_2 p_3 p_4)]}{(1-q_1 q_2 q_3 q_4)} + \\
\frac{p_1 p_2 p_3}{1-q_1 q_2 q_3} + \frac{p_1 p_2 p_4}{1-q_1 q_2 q_4} + \frac{p_1 p_3 p_4}{1-q_1 q_3 q_4} + \frac{p_2 p_3 p_4}{1-q_2 q_3 q_4}. (23)
\]
\[ M(1) = \frac{1}{\lambda} \sum_{j=1}^{M} \left( q_1 q_2 q_3 q_4 \right)^{j-1} \left[ p_1 p_2 p_3 p_4 + p_1 p_2 p_3 q_4 + p_1 p_2 q_3 p_4 + q_1 p_2 p_3 p_4 + q_1 p_2 q_3 q_4 + p_1 q_2 p_3 p_4 + p_1 q_2 q_3 q_4 \right] \]
\[ p_1 q_2 q_4 - q_1 p_2 q_4 - q_1 q_2 q_4 - p_1 p_2 p_3 - p_1 p_2 q_3 - p_1 q_2 p_3 - q_1 p_2 p_3 - q_1 q_2 q_3 - q_1 p_2 q_3 - q_1 q_2 p_3 ]^+ \\
\frac{1}{(1-q_1 q_2 q_3)^2} \left[ p_1 p_2 p_3 + p_1 p_2 q_3 + p_1 q_2 p_3 + p_1 q_2 q_3 + q_1 p_2 q_3 + q_1 q_2 p_3 \right]^+ \\
\frac{1}{(1-q_1 q_3 q_4)^2} \left[ p_1 p_3 p_4 + p_1 p_3 q_4 + p_1 q_3 p_4 + p_1 q_3 q_4 + q_1 p_3 q_4 + q_1 q_3 p_4 \right]^+ \\
\frac{1}{(1-q_1 q_2 q_4)^2} \left[ p_1 p_2 p_4 + p_1 p_2 q_4 + p_1 q_2 p_4 + p_1 q_2 q_4 + q_1 p_2 q_4 + q_1 q_2 p_4 \right]^+ \\
\frac{1}{(1-q_2 q_3 q_4)^2} \left[ p_2 p_3 p_4 + p_2 p_3 q_4 + p_2 q_3 p_4 + p_2 q_3 q_4 + q_2 p_3 q_4 + q_2 q_3 p_4 \right]^+.
\]

\[ B_3(2) = B_3(1) + \sum_{j=1}^{\infty} \left\{ q_1^{j-1} p_2 q_2^{j-1} (1-q_3)^{j-1} (1-q_4)^{j-1} + q_2^{j-1} p_1 q_1^{j-1} (1-q_3)^{j-1} (1-q_4)^{j-1} + q_1^{j-1} p_3 q_3^{j-1} (1-q_2)^{j-1} (1-q_4)^{j-1} + d_3^{j-1} p_1 q_1^{j-1} (1-q_2)^{j-1} (1-q_4)^{j-1} \right\} 
\]
\[ a_j p_4 q_4^{-1}(1-q_2^{-1})(1-q_3^{-1}) + q_4^{-1} p_1 q_1^{-1}(1-q_2^{-1})(1-q_3^{-1}) + \]
\[ q_2^{-1} p_3 q_3^{-1}(1-q_1^{-1})(1-q_4^{-1}) + q_3^{-1} p_2 q_2^{-1}(1-q_1^{-1})(1-q_4^{-1}) + \]
\[ q_2^{-1} p_4 q_4^{-1}(1-q_1^{-1})(1-q_3^{-1}) + q_4^{-1} p_2 q_2^{-1}(1-q_1^{-1})(1-q_3^{-1}) + \]
\[ q_3^{-1} p_3 q_3^{-1}(1-q_1^{-1})(1-q_2^{-1}) + q_3^{-1} p_4 q_4^{-1}(1-q_1^{-1})(1-q_2^{-1}) + \]
\[ \left\{ p_2 q_1^{-1} p_1 q_2^{-1} + p_3 q_1^{-1} p_1 q_3^{-1} + p_4 q_1^{-1} p_1 q_4^{-1} + p_3 q_2^{-1} p_2 q_3^{-1} + p_4 q_2^{-1} p_4 q_3^{-1} \right\} + \]
\[ = \mathcal{B}_3(1) \frac{1}{(1-q_1 q_2 q_3 q_4)} \]
\[ \left[ \frac{p_2 q_1^{-1} p_1 q_2^{-1} + p_3 q_1^{-1} p_3 q_2^{-1} + p_2 q_3^{-1}}{(1-q_1 q_2 q_3)} \right] \]
\[ \left[ \frac{p_2 q_1^{-1} p_1 q_2^{-1} + p_4 q_1^{-1} p_4 q_2^{-1} + p_2 q_4^{-1}}{(1-q_1 q_2 q_4)} \right] \]
\[ \left[ \frac{p_3 q_1^{-1} p_1 q_3^{-1} + p_4 q_1^{-1} p_4 q_3^{-1} + p_3 q_4^{-1}}{(1-q_1 q_3 q_4)} \right] \]
\[ \left[ \frac{p_3 q_2^{-1} p_2 q_3^{-1} + p_4 q_2^{-1} p_4 q_3^{-1} + p_3 q_3^{-1}}{(1-q_2 q_3 q_4)} \right] \]
\[ \left[ \frac{p_3 q_2^{-1} p_2 q_3^{-1} + p_4 q_2^{-1} p_4 q_3^{-1} + p_3 q_3^{-1}}{(1-q_2 q_3 q_4)} \right] \]
\[ B_4(2) = B_4(1) + \sum_{j=1}^{\infty} \left\{ (p_1 p_2)(q_1 q_2)^{-1}(1-q_3^{-1})(1-q_4^{-1}) + \right\} + \]
\[ (p_1 p_3)(q_1 q_3)^{-1}(1-q_2^{-1})(1-q_4^{-1}) + \]
\[ (p_1 p_4)(q_1 q_4)^{-1}(1-q_2^{-1})(1-q_3^{-1}) + \]
\[ (p_2 p_3)(q_2 q_3)^{-1}(1-q_1^{-1})(1-q_4^{-1}) + \]
$$B_4(1) + \frac{[(p_1 p_2) + (p_1 p_3) + (p_1 p_4) + (p_2 p_3) + (p_2 p_4) + (p_3 p_4)]}{(1 - q_1 q_2 q_3 q_4)}$$

$$\left[ \frac{[(p_1 p_2) + (p_1 p_3) + (p_2 p_3)]}{(1 - q_1 q_2 q_3)} \right] - \left[ \frac{[(p_1 p_2) + (p_1 p_4) + (p_2 p_4)]}{(1 - q_1 q_2 q_4)} \right]$$

$$\left[ \frac{[(p_1 p_3) + (p_1 p_4) + (p_3 p_4)]}{(1 - q_1 q_3 q_4)} \right] - \left[ \frac{[(p_2 p_3) + (p_2 p_4) + (p_3 p_4)]}{(1 - q_2 q_3 q_4)} \right].$$

(26)

$$MTR(2) = \frac{1}{\lambda} \sum_{j=1}^{N_8} j(q_1 q_2 q_3 q_4)^{j-1} \left[ p_1 p_2 p_3 p_4 + p_1 p_2 p_3 q_4 + p_1 p_2 q_3 p_4 + p_1 q_2 p_3 q_4 + q_1 p_2 p_3 p_4 \right] +$$

$$j(q_2 q_3 q_4)^{j-1} \left[ p_2 p_3 p_4 + p_2 p_3 q_4 + p_2 q_3 p_4 + q_2 p_3 p_4 \right] (1 - q_1^{j-1}) +$$

$$j(q_1 q_3 q_4)^{j-1} \left[ p_1 p_3 p_4 + p_1 p_3 q_4 + p_1 q_3 p_4 + q_1 p_3 p_4 \right] (1 - q_2^{j-1}) +$$

$$j(q_1 q_2 q_4)^{j-1} \left[ p_1 p_2 p_4 + p_1 p_2 q_4 + p_1 q_2 p_4 + q_1 p_2 p_4 \right] (1 - q_3^{j-1}) +$$

$$j(q_1 q_3 q_3)^{j-1} \left[ p_1 p_3 p_3 + p_1 p_3 q_3 + p_1 q_3 p_3 + q_1 p_3 p_3 \right] (1 - q_4^{j-1}) +$$

$$j(q_1 q_2 q_3)^{j-1} \left[ p_1 p_2 p_3 + p_1 p_2 q_3 + p_1 q_2 p_3 + q_1 p_2 p_3 \right] (1 - q_4^{j-1}) +$$

$$j(q_1 q_2)^{j-1} \left[ p_1 p_2 + p_1 q_2 + p_2 q_1 \right] (1 - q_3^{j-1})(1 - q_4^{j-1}) +$$

$$j(q_1 q_3)^{j-1} \left[ p_1 p_3 + p_1 q_3 + p_1 q_3 \right] (1 - q_2^{j-1})(1 - q_4^{j-1}) +$$

$$j(q_1 q_4)^{j-1} \left[ p_1 p_4 + p_1 q_4 + p_1 q_4 \right] (1 - q_2^{j-1})(1 - q_3^{j-1}) +$$

$$j(q_2 q_3)^{j-1} \left[ p_2 p_3 + p_2 q_3 + p_2 q_3 \right] (1 - q_1^{j-1})(1 - q_4^{j-1}) +$$
\begin{align*}
\sum_{j=1}^{s} j(q_2 q_4)^{j-1} \left[ p_2 p_4 + p_2 q_4 + q_2 p_4 \right] (1-q_1^{j-1})(1-q_3^{j-1}) + \\
\sum_{j=1}^{s} j(q_3 q_4)^{j-1} \left[ p_3 p_4 + p_3 q_4 + q_3 p_4 \right] (1-q_1^{j-1})(1-q_2^{j-1})
\end{align*}

\[= \frac{1}{\lambda} \left( \frac{1}{1-q_1 q_2 q_3 q_4} \right)^2 \left[ p_1 p_2 p_3 p_4 + p_1 p_2 p_3 q_4 + p_1 p_2 q_3 p_4 + p_1 q_2 p_3 p_4 + q_1 p_2 p_3 p_4 - p_2 p_3 p_4 - p_2 p_3 q_4 - p_2 q_3 p_4 - q_2 p_3 p_4 - p_1 p_3 p_4 - p_1 p_3 q_4 - p_1 q_3 p_4 - q_1 p_3 p_4 - q_1 p_3 q_4 - p_1 q_3 p_4 - q_1 p_2 p_4 - p_1 q_2 p_4 - p_1 q_2 q_4 + p_2 q_1 + p_1 q_3 + q_1 p_3 + p_1 q_3 + p_1 q_4 + p_1 q_4 + q_1 p_4 + p_2 q_3 + q_2 p_3 + q_2 p_4 + q_2 p_4 + q_2 p_4 + p_3 p_4 + p_3 q_4 + q_3 p_4 \right] + \\
\frac{1}{\left(1-q_1 q_2 q_3\right)^2} \left[ p_1 p_2 p_3 + p_1 p_2 q_3 + p_1 q_2 p_3 + q_1 p_2 p_3 - p_1 p_2 - p_1 q_2 - p_2 q_1 - p_1 q_3 - p_2 q_3 - q_1 p_3 - p_2 p_3 - p_2 q_3 - q_2 p_3 \right] + \\
\frac{1}{\left(1-q_1 q_2 q_4\right)^2} \left[ p_1 p_2 q_4 + p_1 q_2 q_4 + p_1 q_2 p_4 + q_1 p_2 p_4 - p_1 q_2 - p_1 q_2 - p_2 q_1 - p_1 q_4 - q_1 p_4 - p_2 q_4 - p_2 q_4 - q_2 p_4 \right] + \\
\frac{1}{\left(1-q_1 q_3 q_4\right)^2} \left[ p_1 p_3 p_4 + p_1 p_3 q_4 + p_1 q_3 p_4 + q_1 p_3 p_4 - p_1 p_3 - p_1 q_3 - q_1 p_3 - p_1 p_3 - p_1 q_4 - q_1 p_4 - p_3 p_4 - p_3 q_4 - q_3 p_4 \right] +
\end{align*}
\[
\frac{1}{(1-q_2 q_3 q_4)^2} \left[ p_2 p_3 p_4 + p_2 p_3 q_4 + p_2 q_3 p_4 + q_2 p_3 p_4 - p_2 p_3 - p_2 q_3 - q_2 p_3 - p_3 p_4 - p_3 q_4 - q_3 p_4 - p_2 p_4 - p_2 q_4 - q_2 p_4 \right] + \\
\frac{1}{(1-q_1 q_2)^2} \left[ p_1 p_2 + p_1 q_2 + p_1 q_1 \right] + \frac{1}{(1-q_1 q_3)^2} \left[ p_1 p_3 + p_1 q_3 + q_1 p_3 \right] + \\
\frac{1}{(1-q_1 q_4)^2} \left[ p_1 p_4 + p_1 q_4 + q_1 p_4 \right] + \frac{1}{(1-q_3 q_4)^2} \left[ p_2 p_3 + q_2 p_3 \right] + \\
\frac{1}{(1-q_2 q_4)^2} \left[ p_2 p_4 + p_2 q_4 + q_2 p_4 \right] + \frac{1}{(1-q_3 q_4)^2} \left[ p_3 p_4 + p_3 q_4 + q_3 p_4 \right].
\]

(27)

\[
B_4(3) = B_4(2) + \sum_{j=1}^{\infty} \left\{ (q_1)^{j-1} p_1 (1-q_2)^{j-1} (1-q_3)^{j-1} (1-q_4)^{j-1} + \\
(q_2)^{j-1} p_2 (1-q_1)^{j-1} (1-q_3)^{j-1} (1-q_4)^{j-1} + \\
(q_3)^{j-1} p_3 (1-q_1)^{j-1} (1-q_2)^{j-1} (1-q_4)^{j-1} + \\
(q_4)^{j-1} p_4 (1-q_1)^{j-1} (1-q_2)^{j-1} (1-q_3)^{j-1} \right\},
\]

\[
= B(2) - \frac{1}{(1-q_1 q_2 q_3 q_4)^2} \left[ p_1 p_2 + p_1 p_3 + p_4 \right] + \frac{1}{1-q_1 q_2 q_3} \left[ p_1 p_2 + p_3 \right] + \\
\frac{1}{1-q_1 q_2 q_4} \left[ p_1 + p_2 + p_3 + p_4 \right] + \frac{1}{1-q_1 q_3 q_4} \left[ p_1 + p_3 + p_4 \right] + \\
\frac{1}{1-q_2 q_3 q_4} \left[ p_2 + p_3 + p_4 \right] - \frac{1}{1-q_1 q_2} \left[ p_1 + p_2 \right] - \frac{1}{1-q_1 q_3} \left[ p_1 + p_3 \right] - 
\]
\[
\frac{1}{1-q_1q_4} \left[ p_1 + p_4 \right] - \frac{1}{1-q_2q_3} \left[ p_2 + p_3 \right] - \frac{1}{1-q_2q_4} \left[ p_2 + p_4 \right] - \frac{1}{1-q_3q_4} \left[ p_3 + p_4 \right] = \\
\frac{p_1q_1}{1-q_1} + \frac{p_2q_2}{1-q_2} + \frac{p_3q_3}{1-q_3} + \frac{p_4q_4}{1-q_4}.
\]

(28)

\[
\text{MTR}(3) = \frac{1}{\lambda} \sum_{j=1}^{N} \left\{ j(q_1q_2q_3q_4)^{j-1} p_1 p_2 p_3 p_4 \right. + \sum_{j=1}^{\infty} j(q_2q_3q_4)^{j-1} p_2 p_3 p_4(1-q_1^{j-1}) + \\
\sum_{j=1}^{\infty} j(q_1q_2q_3)^{j-1} p_1 p_2 p_3(1-q_4^{j-1}) + \\
\sum_{j=1}^{\infty} j(q_1q_2q_3)^{j-1} p_1 p_2 p_3(1-q_4^{j-1}) + \\
\sum_{j=1}^{\infty} j(q_1q_2q_3)^{j-1} p_1 p_2 p_3(1-q_4^{j-1}) + \\
\sum_{j=1}^{\infty} j(q_1q_2q_3)^{j-1} p_1 p_2 p_3(1-q_4^{j-1}) + \\
\sum_{j=1}^{\infty} j(q_1q_2q_3)^{j-1} p_1 p_2 p_3(1-q_4^{j-1}) + \\
\sum_{j=1}^{\infty} j(q_1q_2q_3)^{j-1} p_1 p_2 p_3(1-q_4^{j-1}) + \\
\sum_{j=1}^{\infty} j(q_1q_2q_3)^{j-1} p_1 p_2 p_3(1-q_4^{j-1}) + \\
\left. \sum_{j=1}^{\infty} j(q_2q_3)^{j-1} p_2 p_3(1-q_1^{j-1}(1-q_4^{j-1}) + \\
\sum_{j=1}^{\infty} j(q_2q_3)^{j-1} p_2 p_3(1-q_1^{j-1}) + \\
\sum_{j=1}^{\infty} j(q_2q_3)^{j-1} p_2 p_3(1-q_3^{j-1}) + \\
\sum_{j=1}^{\infty} j(q_2q_3)^{j-1} p_2 p_3(1-q_3^{j-1}) + \\
\sum_{j=1}^{\infty} j(q_2q_3)^{j-1} p_2 p_3(1-q_3^{j-1}) + \\
\sum_{j=1}^{\infty} j(q_2q_3)^{j-1} p_2 p_3(1-q_3^{j-1}) + \\
\sum_{j=1}^{\infty} j(q_2q_3)^{j-1} p_2 p_3(1-q_3^{j-1}) + \\
\sum_{j=1}^{\infty} j(q_2q_3)^{j-1} p_2 p_3(1-q_3^{j-1}) + \\
\left. \sum_{j=1}^{\infty} j(q_1)^{j-1} p_1(1-q_2^{j-1}(1-q_3^{j-1})(1-q_4^{j-1}) + \\
\sum_{j=1}^{\infty} j(p_2) j^{-1} p_2(1-q_1^{j-1}(1-q_3^{j-1})(1-q_4^{j-1}) + \right\}.
\]
\[ \sum_{j=1}^{8} \left( q_3^{-j} p_3 (1-q_1^{-j}) (1-q_2^{-j}) (1-q_4^{-j}) + \sum_{j=1}^{8} \left( q_4^{-j} p_4 (1-q_1^{-j}) (1-q_2^{-j}) (1-q_3^{-j}) \right) \right) = \frac{1}{\lambda} \left\{ \frac{1}{(1-q_1 q_2 q_4)^2} \left[ p_1 p_2 p_3 p_4 - p_2 p_3 p_4 - p_1 p_3 p_4 - p_1 p_2 p_4 - p_1 p_2 p_3 + p_1 p_2 + p_1 p_3 + p_1 p_4 + p_2 p_3 + p_2 p_4 + p_3 p_4 - p_1 - p_2 - p_3 - p_4 \right] + \frac{1}{(1-q_2 q_3 q_4)^2} \left[ p_1 p_3 p_4 - p_1 p_2 - p_1 p_3 - p_2 p_3 + p_1 + p_2 + p_3 \right] + \frac{1}{(1-q_1 q_4)^2} \left[ p_1 p_2 p_4 - p_1 p_2 - p_1 p_4 + p_2 p_4 + p_1 + p_2 + p_4 \right] + \frac{1}{(1-q_2 q_3)^2} \left[ p_1 p_3 p_4 - p_1 p_4 - p_3 p_4 + p_1 + p_3 + p_4 \right] + \frac{1}{(1-q_1 q_2)^2} \left[ p_1 p_2 - p_1 - p_2 \right] + \frac{1}{(1-q_1 q_3)^2} \left[ p_1 p_3 - p_1 - p_3 \right] + \frac{1}{(1-q_2 q_4)^2} \left[ p_2 p_4 - p_2 - p_4 \right] + \frac{1}{(1-q_3 q_4)^2} \left[ p_3 p_4 - p_3 - p_4 \right] + \frac{p_1}{(1-q_1)} + \frac{p_2}{(1-q_2)^2} + \frac{p_3}{(1-q_3)^2} + \frac{p_4}{(1-q_4)^2} \right\} \]
4. ILLUSTRATIVE EMPIRICAL WORK

We present the illustrative empirical work based on the 2-unit and 4-unit systems for some typical choices for the $p_k$-values. Using the results in (11) through (29), we compute the $C(k)$-values and hence obtain the $k^*$-values. The ORP thus is obtained through the $k^*$'s.

**TABLE 1 : n=2 ; $p_k$'s C(k)'s and $k^*$**

<table>
<thead>
<tr>
<th>S.No. of the system</th>
<th>n</th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>C(O)</th>
<th>C(1)</th>
<th>$k^*$</th>
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<td>2</td>
<td>0.01</td>
<td>0.01</td>
<td>0.02</td>
<td>0.06</td>
<td>0</td>
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<tr>
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<td>0.05</td>
<td>0.12</td>
<td>0.34</td>
<td>0</td>
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<tr>
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<td>0.40</td>
<td>2.32</td>
<td>2.65</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>0.01</td>
<td>0.05</td>
<td>0.06</td>
<td>0.09</td>
<td>0</td>
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<tr>
<td>5</td>
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<td>0.20</td>
<td>0.30</td>
<td>1.00</td>
<td>1.66</td>
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</tbody>
</table>
TABLE 2: n=4; \( p_k \)'s, \( C(k) \)'s and \( k^* \)

<table>
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<th>S.No. of the system</th>
<th>( n )</th>
<th>( p_1 )</th>
<th>( p_2 )</th>
<th>( p_3 )</th>
<th>( p_4 )</th>
<th>( C(0) )</th>
<th>( C(1) )</th>
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<th>( C(3) )</th>
<th>( k^* )</th>
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</table>

DISCUSSION

The illustrative work in tables 1 and 2 reveal the following important and interesting points. (i) the \( k^* \)-replacement policy may not change even though items may not be identical (that is, even though \( p_L(j) \)'s may be different). (ii) the same \( k^* \)-replacement policy could be achieved even though some components may be of inferior quality in terms of their higher \( p_L(j) \) failure probability (notice that, for example, \( k^* \) (see table 2) is the same corresponding to systems with S.Nos. 3 and 7). This point is noteworthy for operational use/decision-making.
(iii) Finally, as already pointed out, the situations resulting from fluctuating market conditions/developmental aspects in system's designing and so on are well covered in the work as is reported in the above when systems comprised of not necessarily identical components are put into operation. (iv) Remark in (iii) above is particularly crucial in the context of "developing countries", like India.

REFERENCES


""