3.1 Introduction

Nonlinear systems in the presence of noise represent a very general problem that occurs in many areas of science and engineering. It underlies not only autonomous system behavior, such as the control of movement and planning of actions of animals and robots, but also for instance the optimization of financial investment policies and control of chemical plants [1]. The problem is simply stated as: given the system, is it possible to find a control that can steer the system from an initial point to reach a goal state at some future time. Mahmudov and Zorlu [88] investigated the approximate and complete controllability of a general nonlinear stochastic system by using the Picard iteration technique.

Consider the linear stochastic integrodifferential system

\[
\begin{align*}
    dx(t) &= \left[ Ax(t) + Bu(t) + \tilde{F} \left( t, \int_0^t f_1(t, s)ds, \int_0^t f_2(t, s)dw(s) \right) \right] dt \\
    &\quad + \tilde{G} \left( t, \int_0^t g_1(t, s)ds, \int_0^t g_2(t, s)dw(s) \right) dw(t) \\
    x(0) &= x_0, \quad t \in [0, T]
\end{align*}
\] (3.1.1)
and the semilinear stochastic integrodifferential system
\[
\begin{align*}
&dx(t) = \left[ Ax(t) + Bu(t) + F \left( t, x(t), \int_0^t f_1(t, s, x(s))ds, \int_0^t f_2(t, s, x(s))dw(s) \right) \right]dt \\
&\quad + G \left( t, x(t), \int_0^t g_1(t, s, x(s))ds, \int_0^t g_2(t, s, x(s))dw(s) \right) dw(t) \\
&x(0) = x_0, \quad t \in [0, T]
\end{align*}
\] (3.1.2)

where \(A\) and \(B\) are matrices of dimension \(n \times n\) and \(n \times m\) respectively and
\[
F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \quad G : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times n}, \\
f_1 : [0, T] \times [0, T] \times \mathbb{R}^n \to \mathbb{R}^n, \quad f_2 : [0, T] \times [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times n}, \\
g_1 : [0, T] \times [0, T] \times \mathbb{R}^n \to \mathbb{R}^n, \quad g_2 : [0, T] \times [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times n},
\]

In this chapter we show the complete controllability of the semilinear stochastic integrodifferential system (3.1.2) under the natural assumption that the associated linear control system is completely controllable. Motivation for this kind of equations can be found in [47, 96, 97].

### 3.2 Preliminaries

Let us introduce the following matrices and sets:

1. The linear bounded operator
\[
L^T_0 \in \mathcal{L}(L^2_{\mathcal{F}}([0, T], \mathbb{R}^n), L^2(\Omega, \mathcal{F}_T, \mathbb{R}^n))
\]
is defined by
\[
L^T_0 u = \int_0^T \Phi(T - s)Bu(s)ds.
\]

2. The adjoint linear bounded operator
\[
(L^T_0)^* : L^2(\Omega, \mathcal{F}_T, \mathbb{R}^n) \to L^2_{\mathcal{F}}([0, T], \mathbb{R}^m)
\]
is defined by
\[
[(L^T_0)^*z](t) = B^*\Phi^*(T - t)E\{z \mid \mathcal{F}_t\}.
\]

3. The set of all states attainable from \(x_0\) in time \(t > 0\) is given by
\[
\mathcal{R}_t(x_0) = \{x(t; x_0, u) : u(\cdot) \in U_{ad}\},
\]
where \(x(t; x_0, u)\) is the solution of (3.1.2) corresponding to \(x_0 \in \mathbb{R}^n, u(\cdot) \in U_{ad}\).
4. The controllability matrix $\Pi^T_0$ associated with (3.1.1) is defined by

$$\Pi^T_0 \{\cdot\} = L^T_0 (L^T_0)^* \{\cdot\} = \int_0^T \Phi(T - t)BB^*\Phi^*(T - t)\mathcal{E}\{\cdot | \mathcal{F}_t\} dt$$

which belongs to $\mathcal{L}(L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n), L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n))$.

5. The controllability matrix $\Gamma^T_s \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ associated with the corresponding deterministic system is defined by

$$\Gamma^T_s = \int_s^T \Phi(T - t)BB^*\Phi^*(T - t)dt, \quad 0 \leq s < T.$$ 

**Definition 3.2.1.** [87] *The stochastic system* (3.1.2) *is completely controllable on* $[0, T]$ *if*

$$\mathcal{R}_T(x_0) = L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$$

*that is, all the points in* $L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$ *can be reached from the point* $x_0$ *at time* $T$.

The solution of the linear stochastic integrodifferential system (3.1.2) can be written as follows:

$$x(t) = \Phi(t)x_0 + \int_0^t \Phi(t - s)Bu(s)ds + \int_0^t \Phi(t - s)\tilde{F}(s)ds$$

$$+ \int_0^t \Phi(t - s)\tilde{G}(s)dw(s) \quad (3.2.1)$$

where

$$\tilde{F}(t) = \tilde{F} \left(t, \int_0^t f_1(t, s)ds, \int_0^t f_2(t, s)dw(s) \right)$$

and

$$\tilde{G}(t) = \tilde{G} \left(t, \int_0^t g_1(t, s)ds, \int_0^t g_2(t, s)dw(s) \right).$$

and $\Phi(t)$ is the fundamental solution matrix of the linear system

$$x'(t) = Ax(t),$$

$$x(0) = x_0$$

The following lemma gives a formula for a control steering the state $x_0$ to an arbitrary final point $x_T$. 

Lemma 3.2.1. Assume that the operator $\Pi_0^T$ is invertible. Then for arbitrary $x_T \in L_2(\Omega, F_T, \mathbb{R}^n)$, $\tilde{F}(\cdot) \in L_2^\mathcal{F}([0,T], \mathbb{R}^n)$ and $\tilde{G}(\cdot) \in L_2^\mathcal{F}([0,T], \mathbb{R}^{n \times n})$, the control

$$u(t) = B^* \Phi^*(T-t) \mathbb{E}\left\{(\Pi_0^T)^{-1} \left(x_T - \Phi(T)x_0 - \int_0^T \Phi(T-s)\tilde{F}(s)ds \right) \bigg| \mathcal{F}_t\right\}$$

(3.2.2)

transfers the system (3.2.1) from $x_0 \in \mathbb{R}^n$ to $x_T$ at time $T$.

Proof. By substituting (3.2.2) in (3.2.1), we obtain

$$x(t) = \Phi(t)x_0 + \int_0^t \Phi(t-s)BB^*\Phi^*(T-s) \mathbb{E}\left\{(\Pi_0^T)^{-1} \left(x_T - \Phi(T)x_0 - \int_0^T \Phi(T-s)\tilde{F}(s)ds \right) \bigg| \mathcal{F}_s\right\}ds$$

$$+ \int_0^t \Phi(t-s)\tilde{F}(s)ds + \int_0^t \Phi(t-s)\tilde{G}(s)dw(s)$$

$$= \Phi(t)x_0 + \Phi_0^t \left[\Phi^*(t-s)(\Pi_0^T)^{-1} \left(x_T - \Phi(T)x_0 - \int_0^T \Phi(T-s)\tilde{F}(s)ds \right) \right]$$

$$+ \int_0^t \Phi(t-s)\tilde{F}(s)ds + \int_0^t \Phi(t-s)\tilde{G}(s)dw(s)$$

(3.2.3)

Writing $t = T$ in (3.2.3), we see that the control $u(\cdot)$ transfers the system (3.2.1) from $x_0$ to $x_T$. \quad \square

3.3 Controllability Results

In this section we derive the sufficient controllability conditions for the semilinear stochastic integrodifferential system (3.1.2) by using the contraction mapping principle.

We impose the following conditions on the data of the problem:

(H1) The functions $F, G, f_i, g_i, i = 1, 2$ satisfy the following Lipschitz condition: there are constants $M_1, M_2, K > 0$ for $x_i, y_i, z_i \in \mathbb{R}^n, i = 1, 2$ and $0 \leq s \leq t \leq T$ such that

$$\|F(t, x_1, y_1, z_1) - F(t, x_2, y_2, z_2)\|^2 \leq M_1 (\|x_1 - x_2\|^2 + \|y_1 - y_2\|^2 + \|z_1 - z_2\|^2)$$

$$\|G(t, x_1, y_1, z_1) - G(t, x_2, y_2, z_2)\|^2 \leq M_2 (\|x_1 - x_2\|^2 + \|x_1 - x_2\|^2 + \|z_1 - z_2\|^2)$$
\[ \|f_i(t, s, x_1(s)) - f_i(t, s, x_2(s))\|^2 + \|g_i(t, s, x_1(s)) - g_i(t, s, x_2(s))\|^2 \leq K\|x_1 - x_2\|^2 \]

**H2** The functions \(F, f_1\) and \(f_2\) are continuous and satisfy the usual linear growth condition, that is there exists a constant \(L > 0\) such that for all \(t \in [0, T]\) and all \(x, y, z \in \mathbb{R}^n\)

\[
\|F(t, x, y, z)\|^2 \leq L(1 + \|x\|^2 + \|y\|^2 + \|z\|^2)
\]

\[
\|f_i(t, s, x)\|^2 \leq L(1 + \|x\|^2), \; i = 1, 2
\]

and analogously for \(G, g_1, g_2\).

By a solution of the system (3.1.2), we mean a solution of the nonlinear integral equation

\[
x(t) = \Phi(t)x_0 + \int_0^t \Phi(t - s)Bu(s)ds + \int_0^t \Phi(t - s)(Fx)(s)ds \\
+ \int_0^t \Phi(t - s)(Gx)(s)dw(s)
\]

where

\[(Fx)(t) = F(t, x(t), \int_0^t f_1(t, s, x(s))ds, \int_0^t f_2(t, s, x(s))dw(s))\]

and

\[(Gx)(t) = G(t, x(t), \int_0^t g_1(t, s, x(s))ds, \int_0^t g_2(t, s, x(s))dw(s))\]

On the basis of the classical theory of stochastic differential equations of the Ito type, one can prove the basic existence and uniqueness theorem, based on the Picard iteration technique [96, 100]. Although it is well known that there exists a unique solution of the stochastic system (3.1.2), we show that the fixed point technique is also suitable to prove the existence and uniqueness of the solution of the stochastic system (3.1.2) and thus the complete controllability of the stochastic system.

Let \(\mathcal{H}_2\) be the Banach space of all square integrable and \(\mathcal{F}_t\)-adapted processes \(\varphi(t)\) with norm

\[\|\varphi\|^2 := \sup_{t \in [0,T]} \mathbb{E}\|\varphi(t)\|^2\]
To apply the contraction mapping principle, we define the nonlinear operator $P$ from $H_2$ to $H_2$ as

$$(Px)(t) = \Phi(t)x_0 + \int_0^t \Phi(t-s)Bu(s)ds + \int_0^t \Phi(t-s)(\tilde{F}x)(s)ds$$

$$+ \int_0^t \Phi(t-s)(\tilde{G}x)(s)dw(s)$$

(3.3.2)

where

$$u(t) = - \Phi^*(T-t)E\{(\Pi_0^T)^{-1}(x_T - \Phi(T)x_0 - \int_0^T \Phi(T-s)(\tilde{F}x)(s)ds$$

$$- \int_0^T \Phi(T-s)(\tilde{G}x)(s)dw(s)\mid \mathcal{F}_t\}.$$  

(3.3.3)

From Lemma 3.2.1, it can be easily seen that the control (3.3.3) transfers the system (3.3.1) from the initial state $x_0$ to the final state $x_T$ provided that the operator $P$ has a fixed point. So, if the operator $P$ has a fixed point then the system (3.3.1) is completely controllable. Now, for convenience, let us introduce the following notations

$$l_1 = \max\{||\Phi(t)||^2 : t \in [0,T]\},$$

$$l_2 = E||x_T||^2,$$

$$M = \max\{||T_s^T||^2 : s \in [0,T]\}.$$  

Lemma 3.3.1. $[87]$. For every $z \in L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$, there exists a process $\varphi(\cdot) \in L_2^2([0,T], \mathbb{R}^n) \times \mathbb{R}^n$ such that

$$z = Ez + \int_0^T \varphi(s)dw(s)$$

$$\Pi_0^T z = \Gamma_0^T Ez + \int_0^T \Gamma_s^T \varphi(s)dw(s).$$

Moreover,

$$E||\Pi_0^T z||^2 \leq ME\{ ||z||^2 \}^2$$

$$\leq M||Ez||^2, \quad \forall z \in L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n).$$

Note that if the system (3.3.1) is completely controllable, then for some $\gamma > 0$,

$$E\langle \Pi_0^T z, z \rangle \geq \gamma E||z||^2, \quad \forall z \in L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$$

and consequently,

$$||\Pi_0^{-1}|| \leq \frac{1}{\gamma} = l_3.$$
Theorem 3.3.1. Assume that the conditions (H1) and (H2) hold and hypotheses of Lemma 3.2.1 hold. Then the nonlinear stochastic system (3.3.1) is completely controllable provided

$$4l_1(1 + Ml_1l_3)(M_1 + M_2)(T + 1)(1 + 2KT)T < 1 \quad (3.3.4)$$

Proof. In order to prove the complete controllability of the stochastic system (3.3.1), it is enough to show that $\mathcal{P}$ has a fixed point in $\mathcal{H}_2$. To do this, we use the contraction mapping principle. To apply the principle, first we show that $\mathcal{P}$ maps $\mathcal{H}_2$ into itself.

In order to estimate the functions, we shall apply the Lipschitz conditions (H1)-(H2) for the functions $f_i$, $g_i$, $i = 1, 2$, Holder inequality and the elementary inequality

$$\left(\sum_{i=1}^{n} a_i^2\right)^{\frac{s}{2}} \leq \left(\sum_{i=1}^{n} a_i\right)^{s-1} \sum_{i=1}^{n} a_i^s.$$

Now, by Lemma 3.2.1, we have

$$4l_1E\|x_0\|^2 + 4Ml_1l_3E\|x_T - \Phi(T)x_0 - \int_{0}^{T} \Phi(T - r)(\widehat{Fx})(r)dr$$

$$- \int_{0}^{T} \Phi(T - r)(\widehat{Gx})(r)dw(r)\| \leq 4l_1E\|x_0\|^2 + 16Ml_1l_3\left\{l_2 + l_1\|x_0\|^2\right\}$$

$$+ (4l_1 + 16Ml_1l_3)(T + 1) \int_{0}^{T} \|\widehat{Fx}(r)\|^2 + \|\widehat{Gx}(r)\|^2 dr$$

$$\leq 4l_1\|x_0\|^2 + 16Ml_1l_3\left\{l_2 + l_1\|x_0\|^2\right\}$$

$$+ (4l_1 + 16Ml_1l_3)(T + 1)(1 + 2LT) \int_{0}^{T} [1 + E\|x(r)\|^2] dr. \quad (3.3.5)$$
It follows from (3.3.5) that there exists \( C_1 > 0 \) such that
\[
E \|(P_x)(t)\|^2 \leq C_1 \left( 1 + \int_0^T E \|x(r)\|^2 dr \right)
\leq C_1 \left( 1 + T \sup_{0 \leq r \leq T} E \|x(r)\|^2 \right)
\]
for all \( t \in [0, T] \). Therefore \( P \) maps \( \mathcal{H}_2 \) into itself.

Next, we show that \( P \) is a contraction mapping.

\[
E \|(P_x)(t) - (P_x)(t)\|^2
= E \left\| \int_0^t \Phi(t-s)[(\tilde{F}x_1)(s) - (\tilde{F}x_2)(s)] ds \right\|
+ \int_0^T \Phi(t-s)[(\tilde{G}x_1)(s) - (\tilde{G}x_2)(s)] dw(s)
+ \Pi_0 \Phi^*(T-t)(\Pi_0^T)^{-1} \left( \int_0^T \Phi(T-s)[(\tilde{F}x_2)(s) - (\tilde{F}x_1)(s)] ds \right)
+ \int_0^T \Phi(T-s)[(\tilde{G}x_2)(s) - (\tilde{G}x_1)(s)] ds \right\|^2
\leq 4l_1(1 + Ml_1 l_3)(T + 1) \times
\left( E \int_0^T \|(\tilde{F}x_1)(s) - (\tilde{F}x_2)(s)\|^2 ds + E \int_0^T \|(\tilde{G}x_1)(s) - (\tilde{G}x_2)(s)\|^2 ds \right)
\leq 4l_1(1 + Ml_1 l_3)(M_1 + M_2)(T + 1)(1 + 2KT) \int_0^T E \|x_1(s) - x_2(s)\|^2 ds.
\]

It results that
\[
\sup_{t \in [0,T]} E \|(P_x)(t) - (P_x)(t)\|^2
\leq 4l_1(1 + Ml_1 l_3)(M_1 + M_2)(T + 1)(1 + 2KT) \sup_{t \in [0,T]} E \|x_1(t) - x_2(t)\|^2.
\]

Therefore by (9), \( P \) is a contraction mapping. Then the mapping \( P \) has a unique fixed point \( x(\cdot) \in \mathcal{H}_2 \) which is the solution of the equation (3.3.1). Thus the stochastic system (3.3.1) is completely controllable. \( \square \)

**Remark 3.3.1.** Obviously hypothesis (3.3.4) is fulfilled if \( M_1 + M_2 \) is sufficiently small.

**Remark 3.3.2.** Consider the time varying semilinear stochastic integrodifferential...
system of the form
\[ dx(t) = \left[ A(t)x(t) + F \left( t, x(t), \int_0^t f_1(t, s, x(s))ds, \int_0^t f_2(t, s, x(s))dw(s) \right) \right] dt \]
\[ + B(t)u(t)dt + G \left( t, x(t), \int_0^t g_1(t, s, x(s))ds, \int_0^t g_2(t, s, x(s))dw(s) \right) dw(t) \]
\[ x(0) = x_0, \quad t \in [0, T] \]  
(3.3.6)

where \( A(t) \) and \( B(t) \) are matrices of dimension \( n \times n \) and \( n \times m \) respectively and \( F, G, f_i, g_i, i = 1, 2 \) are as before.

The solution of the above equation is
\[ x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^t \Phi(t, s)Bu(s)ds + \Phi(t, s)(Fx)(s)ds \]
\[ + \int_{t_0}^t \Phi(t, s)(Gx)(s)dw(s) \]
where \( \Phi(t, t_0) \) is the fundamental matrix of the homogeneous equation \( \dot{x}(t) = A(t)x(t) \) with \( x(t_0) = x_0 \). If the functions \( F, G, f_i, g_i, i = 1, 2 \) satisfy the local Lipschitz condition then a suitable function will steer the system (3.3.6) from \( x_0 \) to \( x_T \) provided the above equation is satisfied.

**Remark 3.3.3.** Consider the stochastic integrodifferential system of the form
\[ dx(t) = \left[ Ax(t) + f_1(t, x(t)) + \int_0^t f_2(t, s, x(s))ds + \int_0^t f_3(t, s, x(s))dw(s) \right] dt \]
\[ + Bu(t)dt + \left[ g_1(t, x(t)) + \int_0^t g_2(t, s, x(s))ds + \int_0^t g_3(t, s, x(s))dw(s) \right] dw(t) \]
\[ x(0) = x_0, \quad t \in [0, T]. \]  
(3.3.7)

where \( A, B, w \) are as before and
\[ f_1 : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad g_1 : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}, \]
\[ f_2 : [0, T] \times [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad g_2 : [0, T] \times [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}, \]
\[ f_3 : [0, T] \times [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}, \quad g_3 : [0, T] \times [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n \times n}. \]

By a solution of the system (3.3.7), we mean a solution of the nonlinear integral equation
\[ x(t) = \Phi(t)x_0 + \int_0^t \Phi(t - s)Bu(s)ds + \int_0^t \Phi(t - s)(Fx)(s)ds \]
\[ + \int_0^t \Phi(t - s)(Gx)(s)dw(s) \]  
(3.3.8)
where

\[(Fx)(t) = f_1(t, x(t)) + \int_0^t f_2(t, s, x(s))ds + \int_0^t f_3(t, s, x(s))dw(s)\]

and

\[(Gx)(t) = g_1(t, x(t)) + \int_0^t g_2(t, s, x(s))ds + \int_0^t g_3(t, s, x(s))dw(s).\]

**H3** The functions \(f_i, g_i, i = 1, 2, 3\) satisfy the Lipschitz condition and there exist constants \(L > 0\) for \(x_1, x_2 \in \mathbb{R}^n\) and \(0 \leq t \leq T\) such that

\[\|f_i(t, x_1) - f_i(t, x_2)\|^2 + \|g_i(t, x_1) - g_i(t, x_2)\|^2 \leq L\|x_1 - x_2\|^2, \quad i = 1, 2, 3\]

**H4** The functions \(f_i, g_i, i = 1, 2, 3\) are continuous and there exists a constant \(L > 0\) such that for all \(t \in [0, T]\) and for all \(x \in \mathbb{R}^n\),

\[\|f_i(t, x)\|^2 + \|g_i(t, x)\|^2 \leq L(\|x\|^2 + 1).\]

Clearly, under the conditions (H3) and (H4), the basic existence and uniqueness theorem of the stochastic system (3.3.7) can be proved by using the Picard iteration technique [97]. Thus for every \(u(\cdot) \in U_{ad}\) the integral equation (3.3.8) has a unique solution in \(\mathcal{H}_2\). To apply the contraction mapping principle, we define the nonlinear operator \(\mathcal{S}\) from \(\mathcal{H}_2\) to \(\mathcal{H}_2\) as follows:

\[(\mathcal{S}x)(t) = \Phi(t)x_0 + \int_0^t \Phi(t-s)Bu(s)ds + \int_0^t \Phi(t-s)(Fx)(s)ds + \int_0^t \Phi(t-s)(Gx)(s)dw(s).\]

Then it is possible to find a suitable control that transfers the system (3.3.8) from the initial state \(x_0\) to the final state \(x_T\) provided that the operator \(\mathcal{S}\) has a fixed point.

**Theorem 3.3.2.** If the conditions (H3)-(H4) hold, then the system (3.3.7) is completely controllable provided

\[12l_1(1 + Ml_1l_3)(T + 1)(1 + 2LT)T < 1.\]  

(3.3.9)

**Proof.** The proof of this theorem is similar to that of Theorem 3.3.1 and hence it is omitted. □
### 3.4 Example

Consider a scalar nonlinear stochastic integrodifferential control system

\[
dx(t) = \left[ -2x(t) + \frac{1}{\sqrt{1 + |x(t)|}} \right] dt + e^{-2t}u(t)dt \\
+ \ln \left( e^{-t} \left| \int_{0}^{t} x(s)dw(s) \right| + 1 \right) dw(t)
\]

\[x(t) = x_{0}.
\] (3.4.1)

Here \( A(t) = -2 \) and \( B(t) = e^{-2t} \). Then, the solution of (3.4.1) is given by

\[
x(t) = \Phi(t,t_{0})x_{0} \int_{t_{0}}^{t} \Phi(t,s)Bu(s)ds + \int_{t_{0}}^{t} \Phi(t,s)Fx(s)ds \\
+ \int_{t_{0}}^{t} \Phi(t,s)Gx(s)dw(s)
\]

for \( t_{1} > t_{0} \), where

\[
Fx(t) = \frac{1}{\sqrt{1 + |x(t)|}}
\]

and \( Gx(t) = \ln \left( e^{-t} \left| \int_{0}^{t} x(r)dw(r) \right| + 1 \right) \).

Obviously \( \Phi(t,s) = e^{-2(t-s)} \), so that

\[
\Gamma_{t_{0}}^{t_{1}} = \int_{t_{0}}^{t_{1}} e^{-4t}ds \\
= e^{-4t_{1}}(t_{1} - t_{0}) > 0 \quad \text{for some } t_{1} > t_{0}.
\]

It can be easily seen that \( Fx(t) \) and \( Gx(t) \) satisfy the hypotheses (H1)-(H2) of Theorem 3.3.1. Hence, the stochastic system (3.4.1) is completely controllable on \([0, T]\), that is, the system (3.4.1) can be steered from \( x_{0} \) to \( x_{T} \).