1. INTRODUCTION

1.1. INTEGRODIFFERENTIAL INCLUSIONS

There are many problems in applied mathematics that lead to the study of dynamical systems having velocities not uniquely determined by the state of the systems, but depending only loosely upon it. In these cases, the classical equation \( \dot{x}(t) = F(t, x(t)) \) describing the dynamics of the system is replaced by a differential inclusion of the form \( \dot{x}(t) \in F(t, x(t)) \) where \( F(t, x(t)) \) is a set-valued map.

The motivation for the study of differential inclusions came from control theory. There we encounter dynamical systems described by an equation of the form \( \dot{x}(t) = f(t, x(t), u(t)) \) where \( u(t) \) is the control parameter. Every solution of this equation also solves the differential inclusion \( \dot{x}(t) \in F(t, x(t)) \) where \( F(t, x(t)) = \{ f(t, x, u) : u \in U \} \) and this formulation has the advantage that the control variables do not appear explicitly. Also implicit differential equations can be viewed as differential inclusions, specifically if we have \( f(t, x, \dot{x}) = 0 \). Then, if we define \( F(t, x) = \{ z : f(t, x, z) = 0 \} \), the implicit differential equations take the form \( \dot{x} \in F(t, x) \).

The study of differential and integrodifferential inclusions in abstract spaces has emerged, in recent years, as an independent area of modern research because of its applications to many fields. A large class of scientific and engineering problems is modelled by partial differential equations, coupled partial differential equations which can be described as differential inclusions in infinite dimensional spaces using the semigroup theory. Integrodifferential inclusions serve as an abstract formulation of many partial integrodifferential equations which arise in problems connected with electric circuits and many other physical phenomena. For these inclusions the existence of solutions is an important problem. In this thesis, we study the existence of solutions of certain integrodifferential inclusions in infinite dimensional spaces.

1.2. MOTIVATION

As the object of the thesis is to study the existence of abstract nonlinear integrodifferential systems of various forms, we shall motivate our study briefly by giving the occurrence of these systems in different fields of study.

(a) Mathematical Model of Transmission Line.

Consider the transmission lines whose electric schemes have the following forms
Figure 1.

Figure 2.

2
Voltamper characteristics of the systems can be presented on the following diagrams

\[ \text{Figure 3.} \quad \text{Figure 4.} \]

It is clear that both the functions \( Q \) and \( S \) can be treated as upper semicontinuous multivalued maps. Here \( \pm v_0 \) and \( \pm i_0 \) are threshold values for a voltage and for a current, respectively, for which the line should be disconnected.

Denoting the current and voltage by \( i(t, x) \) and \( v(t, x) \) respectively, we can describe the mathematical model of the transmission line by the system

\[
L \frac{\partial i(t, x)}{\partial t} = -\frac{\partial v(t, x)}{\partial x} - (R + r(\int_0^t i^2(t, y)dy))i(t, x)
\]

\[
C \frac{\partial v(t, x)}{\partial t} = -\frac{\partial i(t, x)}{\partial x} - (G + g(\int_0^t i^2(t, y)dy))v(t, x).
\]

The line is characterized by the parameters (all per unit length): inductance \( L \), resistance \( R \), capacitance \( C \), and leakage conductance \( G \). The function \( r : R_+ \to R_+ \) expresses the dependence of the resistance (for unit length) of the line on the power of the signal; the function \( g : R_+ \to R_+ \) describes the dependence of the leakage conductance. Both the functions \( r \) and \( g \) are assumed continuous and bounded. In the first case, the boundary conditions are given by the relations

\[
E(t) = v(t, 0) + R_0i(t, 0) \quad \text{at} \quad x = 0,
\]

\[
v(t, l) \in C_1 \frac{\partial}{\partial t} v(t, l) + Q(v(t, l)) \quad \text{at} \quad x = l.
\]

In the second case, the last inclusion has the form

\[
i(t, l) \in L_1 \frac{\partial}{\partial t} i(t, l) + S(i(t, l)) \quad \text{at} \quad x = l.
\]
We suppose that the function $E(t)$ is smooth. The change of variables
\[ I = i, \]
\[ V = v - E, \]
makes the boundary conditions at $x = 0$ homogeneous. In these new variables, we have the system
\[
\frac{\partial I(t, x)}{\partial t} = -\frac{1}{L} \frac{\partial V(t, x)}{\partial x} - \frac{1}{L} \left( R + r \int_0^t I^2(t, y) dy \right) I(t, x)
\]
\[
\frac{\partial V(t, x)}{\partial t} = -\frac{1}{C} \frac{\partial I(t, x)}{\partial x} - \frac{G}{C} V(t, x) - \frac{1}{C} \sigma \int_0^t I^2(t, y) dy \right)[V(t, x) + E(t)]
\]
\[
- \frac{G}{C} E(t) - E'(t)
\]
with boundary conditions:
\[ V(t, 0) + R_0 I(t, 0) = 0 \text{ at } x = 0 \]
and, for the first case,
\[ I(t, l) \in C_1 \frac{\partial}{\partial t} V(t, l) + Q(V(t, l) + E(t)) + C_1 E'(t) \text{ at } x = l; \]
while for the second case, we have
\[ V(t, l) \in L_1 \frac{\partial}{\partial t} I(t, l) + S(I(t, l)) + E(t) \text{ at } x = l. \]

We can interpret both the systems by a semilinear differential inclusion
\[ x'(t) \in A x(t) + F(t, x(t)) \]
in a Hilbert space $E = L^2[0, l] \times L^2[0, l] \times R$ with a scalar product given as
\[
\langle (\phi_1, \psi_1, b_1), (\phi_2, \psi_2, b_2) \rangle = \int_0^l \phi_1 \phi_2 dx + \int_0^l \psi_1 \psi_2 dx + b_1 b_2.
\]
In the first case, the operator $A$ is defined by
\[
A \begin{pmatrix} \phi \\ \psi \\ b \end{pmatrix} = \begin{pmatrix} -R/L & (-1/L)D_x & 0 \\ (-1/C)D_x & -G/C & 0 \\ (1/C_1)\Gamma & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \\ b \end{pmatrix}
\]
where $D_x = \frac{d}{dx}$ and $\Gamma$ is the evaluation operator $\Gamma(w) = w(t)$. The domain of $A$ can be given as

$$D(A) = \{(\phi, \psi, b) \in E : \phi, \psi \in H^1[0,1], \psi(0) + R_0\phi(0) = 0, b = \psi(1)\}.$$ 

In the second case the operator $A$ is defined by

$$A \begin{pmatrix} \phi \\ \psi \\ b \end{pmatrix} = \begin{pmatrix} -R/L & (-1/L)D_x & 0 \\ (-1/C)D_x & -G/C & 0 \\ 0 & (1/L_1)\Gamma & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \\ b \end{pmatrix}$$

with domain

$$D(A) = \{(\phi, \psi, b) \in E : \phi, \psi \in H^1[0,1], \psi(0) + R_0\phi(0) = 0, b = \phi(1)\}.$$ 

The multivalued nonlinearity $F(t,x)$, with $x = (\phi, \psi, b)$ in the first case, has a form

$$F(t,x) = \left[ -\frac{1}{L} r \left( \int_0^t \phi^2(y)dy \right) \phi, -\frac{1}{C} g \left( \int_0^t \phi^2(y)dy \right) \left( \psi + E(t) \right), \frac{-G}{C} E(t) - E'(t), -\frac{1}{C_1} Q(b + E(t)) - E'(t) \right].$$

In the second case the first two components of $F(t,x)$ are the same and the third is equal to $\frac{1}{C_1} S(b) - E(t)$. Note that $A$ is the infinitesimal generator of a $C_0$-semigroup $e^{At}$.

(c) Hybrid Systems

Consider the model illustrated by the figure given below

![Figure 5.](image-url)
A block of mass \( m \) lies on a rough surface and is connected to a fixed support by a spring. We assume that the spring moves in an anisotropic environment in presence of dry friction on the block. To get the semilinear inclusion as a model, consider the Banach space

\[ E = \{ x = (\phi, \psi, b_1, b_2) \in H^1[0,1] \times L^2[0,1] \times \mathbb{R}^2 : \phi(0) = 0, \phi(1) = b_1 \} \]

with the norm given by

\[ \| x \| = \left( \alpha \int_0^1 \phi' dx + \alpha \int_0^1 \psi^2 dx + \beta b_1^2 + b_2^2 \right)^{1/2} \]

where \( \alpha > 0, \beta > 0 \). Now set \( x(t) = (v(t,-), v(t,:), z(t), z'(t)) \) where \( v(t,x) \) is the elongation of the spring and \( z(t) \) denotes the position of the block. The system can be interpreted as a semilinear differential inclusion of the form

\[ x'(t) \in A(t)x(t) + F(t, x(t)) \]

where

\[
A \begin{pmatrix} \phi \\ \psi \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{Dz}{Dz} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha \Gamma Dz & 0 & -\beta & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \\ b_1 \\ b_2 \end{pmatrix}
\]

and \( Dz = \frac{d}{dz} \Gamma(w) = w(1) \). The domain of \( A \) can be given as

\[ D(A) = \{(\phi, \psi, b_1, b_2) \in E : \phi \in H^2[0,1], \psi \in H^1[0,1], \psi(0) = 0, \psi(1) = b_2 \}. \]

It is not difficult to see that \( A \) generates a \( C_0 \)-semigroup. The multivalued nonlinear term \( F \) is defined as

\[ F(t, x) = (0, \psi W(\phi), 0, e(t) + G(b_2)). \]

Here \( W: C[0,1] \rightarrow \nu(C[0,1]) \) is a bounded upper semicontinuous multioperator modelling the effect of anisotropy of the environment as well as the hysteresis phenomena. The function \( e(t) \) represents an external force acting on the mass and the multifunction \( G \) is defined as

\[ G(s) = \begin{cases} [-\mu_0, \mu_0], & \text{if } s = 0, \\ \mu(s) \text{ sign}(s), & \text{if } s \neq 0, \end{cases} \]

where \( \mu_0 \) and \( \mu(\cdot) \) are the coefficients of static and sliding friction respectively.
1.3. INTEGRODIFFERENTIAL EQUATIONS

Integrodifferential equations or evolution equations serve as an abstract formulation of many partial integrodifferential equations which arise in problems connected with heat-flow in materials with memory, viscolasticity and many other physical phenomena. So it becomes important to study the existence of solutions of such equations in infinite dimensional spaces.

(a) Chemical Model

The modelling of a particular type of reaction-diffusion system, with the unknown functions $u(x,t)$ and $\gamma(t)$ representing the concentrations of two different chemicals, may be written as

\[ u_t(x,t) = u_{xx}(x,t), \quad 0 < x < 1, \quad t > 0 \]
\[ u(x,0) = 1, \quad 0 < x < 1, \]
\[ u_x(0,t) = 0, \quad t > 0 \]
\[ u_x(1,t) = \left( \frac{Em}{1+L} \right) (L(\gamma(t) - (1 - \gamma(t))u(1,t)), \quad t > 0 \]
\[ m\gamma(t) + \int_0^1 u(x,t)dx = 1, \quad t > 0 \]

where $E, L, m$ are positive constants.

By using Laplace transforms, the above initial-boundary value problem can be transformed into the following nonlinear integrodifferential equation

\[ \dot{\gamma}(t) = -C\{L\gamma(t) - (1 - \gamma(t))(1 - m\int_0^t k(t-s)\gamma(s)ds), \quad t > 0 \]

with $\gamma(0) = 0$,

where $k(t) = \frac{1}{\sqrt{\phi t}} \left( 1 + 2 \sum_{i=1}^{\infty} \exp(-\frac{n^2}{t}) \right), \quad t > 0$

and $C = \frac{E}{1+L}$. 

7
Furthermore the value of the dependent variable at \( x = 1 \) can be obtained from
\[
u(1, t) = 1 - m \int_0^t k(t - s) \gamma(s) ds > 0.
\]
This permits an efficient calculation of \( u(1, t) \) through the evaluation of \( u_x(1, t) \), which is often the quantity required by electro chemists, for example, as it represents the flux to the surface at \( x = 1 \).

(b) Abstract Equations

(i) Consider the classical heat equation for materials with memory,
\[
q(t, x) = -E u_x(t, x) - \int_0^t b(t - s) u_x(s, x) ds
\]
\[
u_t(t, x) = -\frac{\partial}{\partial x} q(t, x) + f(t, x)
\]
\[
u(x, 0) = \nu_0(x).
\]
The first equation gives the heat flux and the second is the balance equation. The above equations take the form:
\[
u_t(t, x) = \frac{\partial^2}{\partial x^2} [u(t, x) + \int_0^t b(t - s) u(s, x) ds] + f(t, x),
\]
which can be written as an abstract semilinear differential equation
\[
u'(t) = A[u(t) + \int_0^t F(t - s) u(s) ds] + f(t, u(t)), \quad 0 \leq t \leq T
\]
in a Banach space with \( A \) as the generator of a strongly continuous semigroup and \( F(t) \), a bounded operator for \( t \in [0, T] \). If the nonlocal condition \( u(0) + g(u) = u_0 \) is introduced to the above equation, it will have better effect than the classical condition.

(ii) Consider the following partial differential equations
\[
u_t(x, t) = a(x, t) v_{xx}(x, t) + f(t, v(x, t - r), v_x(x, t - r)), \quad 0 \leq x \leq \pi, \quad t \geq 0
\]
\[
u_t(x, t) = a(x, t), \quad t \geq 0
\]
\[
u_t(x, t) = \phi(x, t), \quad 0 \leq x \leq \pi, \quad -r \leq t \leq 0
\]
where \( v_t - a(x,t)v_{xx} \) is uniformly parabolic differentiable, with \( a(x,t) \) continuous on \( 0 \leq x \leq \pi, \ 0 \leq t \leq T \), and \( f \) is a linear or nonlinear scalar valued function. The above partial functional differential equation can be written as an abstract ordinary functional differential equations in a Banach space as

\[
\dot{x}(t) = Ax(t) + f(t, x_t), \quad t > 0
\]

\[
x_0 = \phi
\]

where \(-A\) is the infinitesimal generator of an analytic semigroup of a linear operator.

(iii) In one dimensional heat flow in a material with memory, one can take the following model

\[
w_t(x,t) = w_{xx}(x,t) + \int_0^t f(s, w(x, s - r))ds, \quad 0 < x < 1, \ t > 0,
\]

\[
w(0,t) = w(1,t) = 0, \quad t > 0,
\]

\[
w(x,t) = \phi(x,t), \quad -r \leq t \leq 0.
\]

The above equations can be written in the abstract form as

\[
x'(t) = Ax(t) + \int_0^t f(s, x_s)ds,
\]

\[
x_0 = \phi,
\]

where \( A \) is a linear closed densely defined operator in a Banach space \( X \).

Consider the following integrodifferential equation

\[
\frac{\partial^2 z(x,t)}{\partial t^2} + c(t) \frac{\partial z(x,t)}{\partial t} = M \left( \int_{-\infty}^{+\infty} \left| \frac{\partial z(x,s)}{\partial s} \right|^2 ds \right) \frac{\partial^2 z(x,t)}{\partial x^2} + z(t, x)
\]

\[
= h(t, x, u(t, x)), \quad 0 \leq t < \infty, \ x \in \mathbb{R},
\]

\[
z(0, x) = z_0(x), \quad \frac{\partial z(x,0)}{\partial t} = z_1(x), \quad x \in \mathbb{R}.
\]

Equations of this type occur in the study of the nonlinear behavior of elastic strings. The basic physical assumptions are that the longitudinal strain of the string is very small and that the tension \( F \) is uniform along the string but may vary with time to accommodate changes in the arc length of the string. The nonlinearity arises from
the assumption that $F$ depends on the arc length $S$ of the string at time $t \geq 0$ by the relation $F = F_0 + C[(S - L)/L]$ where $F_0$ is the minimum tension, $L$ is the minimum length and $C$ is a physical constant.

(c) Quasilinear Integrodifferential Equations

Quasilinear integrodifferential equations which occur in the study of the nonlinear behavior of elastic strings can be written as

$$
\frac{\partial^2 z(x,t)}{\partial t^2} + c(t) \frac{\partial z(x,t)}{\partial t} - M \left( \int_{-\infty}^{+\infty} | \frac{\partial z(x,s)}{\partial s} |^2 ds \right) \frac{\partial^2 z(x,t)}{\partial x^2} + z(t,x)
$$

$$
= h(t,x,z(t,x)), \quad 0 \leq t < \infty, \quad x \in R,
$$

$$
z(0,x) = z_0(x), \quad \frac{\partial z(x,0)}{\partial t} = z_1(x), \quad x \in R.
$$

The derivation of this equation for finite a string is given in [85] and the basic physical assumptions are that the longitudinal strain of the string is very small and that the tension $F$ is uniform along the string but may vary with time to accommodate changes in the arc length of the string. The nonlinearity arises from the assumption that $F$ depends on the arc length $S$ of the string at time $t \geq 0$ by the relation $F = F_0 + C[(S - L)/L]$ where $F_0$ is the minimum tension, $L$ is the minimum length and $C$ is a physical constant.

There are other types of integrodifferential equations, similar to the above, which occur in the study of dynamical buckling of the hinged extensible beam which is either stretched or compressed by axial force. An equation of the following form

$$
u_t(t,x) + \Psi(u(t,x))_x = \int_0^t b(t-s) \Psi(u(s,x))_x ds + f(t,x), \quad t \in [0,b], \quad x \in R,
$$

$$
u(0,x) = \phi(x), \quad x \in R,
$$

occurs in a nonlinear conservation law with memory.

The above type of equation can be formulated abstractly as

$$
\dot{x}(t) + A(t,x(t))x(t) = Bu(t) + f(t,x(t),\int_0^t g(t,s,x(s))ds), \quad t \in [0,b],
$$

$$
x(0) = x_0,
$$

where $-A$ is the infinitesimal generator of an analytic semigroup of linear operators and $f$, $g$ are given nonlinear operators.
1.4. OTHER MODELS

(a) Neutral Functional Differential Equations

Consider the model for the heat conduction in materials with memory which can be represented as a partial functional neutral integrodifferential equation of the form

$$\frac{\partial}{\partial t} \left[ z(t, \xi) + \int_{-\infty}^{t} \int_{0}^{\eta} b(s - t, \eta, \xi) z(s, \eta) d\eta ds \right]$$

$$= \frac{\partial^2}{\partial \xi^2} z(t, \xi) + a_0(\xi) z(t, \xi) + \int_{-\infty}^{t} a(s - t) z(s, \xi) ds + a_1(t, \xi),$$

$$z(t, 0) = z(t, \pi) = 0, \quad t \geq 0,$$

$$z(\theta, \xi) = \phi(\theta, \xi), \quad \theta \leq 0, \quad 0 \leq \xi \leq \pi,$$

where the functions $a_0, a_1, b$ and $\phi$ satisfy appropriate conditions for the existence of solutions.

The above problem arises from control systems described by abstract retarded functional differential equations with a feedback control governed by a proportional integrodifferential law and it can be written as

$$\frac{d}{dt} [x(t) + h(t, x_t)] = Ax(t) + f(t, x_t), \quad t \geq \sigma,$$

$$x_{\sigma} = \phi,$$

where the initial function $\phi$ takes values in some approximated phase space and $h, f$ are nonlinear functions.

(b) Second order Integrodifferential Equations

There are other types of integrodifferential equations similar to the above and, for example, the one occurring in the study of dynamical buckling of the hinged extensible beam which is either stretched or compressed by axial force. A mathematical model for this problem is the hyperbolic equation

$$\frac{\partial^2 z}{\partial t^2} + \frac{\partial^4 z}{\partial x^4} - \left( \alpha + \beta J_{0}^{L} \left| \frac{\partial z(x, s)}{\partial s} \right|^2 ds \right) \frac{\partial^2 z}{\partial x^2} + g \left( \frac{\partial z}{\partial t} \right) = 0,$$

in which $\alpha, \beta, L > 0, z(x, t)$ is the deflection of the point $x$ of the beam at the time $t, g$ is a nondecreasing numerical function, and $L$ is the length of the beam. The nonlinear
friction force $g(\frac{\partial x}{\partial t})$ is the dissipative term. When $g = 0$, this equation reduces to the equation as a model for the transverse motion of an extensible beam whose ends are held a fixed distance apart. These equations take the abstract form as

$$z'' + A^2 z + M(\|A^{1/2} z\|_H)Az + g(z') = 0$$

where $A$ is a linear operator in a Hilbert space $H$ and $M$ and $g$ are real functions.

(c) Nonlocal Cauchy problem

Consider the following partial differential equation:

$$u_{xt}(x, t) = F(x, t, u(x, t), u_x(x, t), u_t(x, t)), (x, t) \in Q$$

$$u(x, 0) + \sum_{i=1}^{p} h_i(x, t_i)u(x, t_i) = \phi(x), \quad x \in [0, a],$$

$$u(0, t) = \psi(t), \quad t \in [0, a],$$

$$\psi(0) + \sum_{i=1}^{p} \psi(t_i) = \phi(0),$$

where $Q = [0, a] \times [0, a]$, $t_i, i = 1, ..., p$, are finite numbers such that $0 < t_1 < t_2 < ... < t_p \leq a$ and $F, \phi, \psi, h_i, i = 1, ..., p$ are given functions with appropriate assumptions. In the theory of elasticity, the sum $u(x, 0) + \sum_{i=1}^{p} h_i(x, t_i)u(x, t_i)$ is more precise to measurement of a state of vibrating system than the only one measurement $u(x, 0)$ of the state of the vibrating system. The sum may be interpreted as the sum of the $p+1$ measurements of positions of a vibrating elastic string and the functions $h_i(x, t_i)$ can be interpreted as the properties of the medium in which the string vibrates.

1.5. METHODS

Many problems in the fields of ordinary and partial differential equations can be recast as integral equations. Several existence and uniqueness results can be derived from the corresponding results of integral equations. Such results can be obtained by applying the fixed point theorems. Fixed point method is the most powerful method in proving existence theorems for integrodifferential equations. Due to their importance, several researchers have studied the problems represented by evolution equations by using different kinds of fixed point theorems. Schauder’s fixed point
Theorem will be helpful in asserting the existence of solutions of integrodifferential equations. The Banach fixed point theorem is an important source of existence and uniqueness theorem in different branches of analysis.

Theory of semigroups of bounded linear operators developed quite rapidly since the discovery of the generation theorem by Hille and Yosida in 1943. By now, it is an extensive mathematical subject with substantial applications to many fields of analysis. The theory of semigroups of bounded linear operators is closely related to the solution of differential and integrodifferential equations in Banach spaces. In recent years, the theory of semigroups of bounded linear operators has been extended to a larger class of differential equations in Banach spaces. Using the method of semigroups, existence and uniqueness of mild, strong and classical solutions of semilinear evolution equations have been discussed by Pazy [90]. Byszewski [35] extended this method to nonlocal Cauchy problems in Banach spaces. The present study deals mainly with the fixed point approach of proving existence theorems for integrodifferential equations and integrodifferential inclusions with nonlocal conditions in Banach spaces.

The theory of strongly continuous cosine families of bounded linear operators has applications in many branches of analysis and in particular, to find the solution of initial and boundary value problems for second order partial differential equations. It is closely related to the solution of second order ordinary differential equations in Banach spaces. In recent years, the theory of strongly continuous cosine families of bounded linear operators has been extensively applied to study the existence problems in second order differential equations [78, 79, 82]. The most fundamental and extensive work on cosine families is that of Travis and Webb [97-99]. In this thesis, we shall investigate the existence of solutions of some second order integrodifferential inclusions in Banach spaces.

1.6. CONTRIBUTIONS OF THE AUTHOR

In the light of the above, the author has obtained some significant results on the following topics:


The rest of the thesis contains a detailed account of the above topics.

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