VI. EXISTENCE OF SOLUTIONS OF SECOND ORDER INTEGRODIFFERENTIAL INCLUSIONS

6.1. INTRODUCTION

The existence of mild, strong and classical solutions for second order differential and integrodifferential equations in abstract spaces have been studied by several authors [79,82,85,99]. Benchohra and Ntouyas [29] discussed the existence results for second order functional differential inclusions in Banach spaces. The existence of mild solutions of initial value problem of the first order evolution inclusions and semilinear integrodifferential inclusions with nonlocal conditions has been established by Benchohra and Ntouyas [28]. But no work has been reported regarding the existence of solutions of second order evolution inclusions. In this chapter, we study the existence of solutions of second order mixed integrodifferential inclusions and evolution integrodifferential inclusions by using a fixed point theorem due to Martelli [75]. The results generalize those of [29].

6.2. MIXED INTEGRODIFFERENTIAL INCLUSIONS

6.2.1 Preliminaries

Consider the second order mixed integrodifferential inclusion of the form
\[
\frac{d^2u}{dt^2} - Au \in G \left( t, u, \int_0^t k(t,s,u)ds, \int_0^T B(t,s,u)ds \right) \quad t \in J = [0,T],
\]
\[u(0) = u_0, u'(0) = v_0,
\]
where \(G : J \times X \times X \times X \to 2^X\) is a bounded, closed, convex multivalued map, \(u_0 \in X, v_0 \in X\), \(A\) is the infinitesimal generator of a strongly continuous cosine family \(\{C(t) : t \in R\}\) in a real Banach space \(X\) and \(\{S(t) : t \in R\}\) is a strongly continuous sine family associated with the cosine family \(\{C(t) : t \in R\}\).
In this section, we introduce the notations, definitions and preliminary results related to this chapter. Let $C(J, X)$ be a Banach space of continuous functions from $J$ into $X$ with the norm

$$
\|y\|_\infty = \sup\{|y(t)| : t \in J\}.
$$

Let us denote the Banach space of bounded linear operators from $X$ into $X$ by $B(X)$. A measurable function $u : J \to X$ is Bochner integrable if and only if $|u|$ is Lebesgue integrable. Let $L^1(J, X)$ denote the Banach space of continuous functions $u : J \to X$ which are Bochner integrable with the norm

$$
\|u\|_{L^1} = \int_0^T \|u(t)\| dt \text{ for all } y \in L^1(J, X).
$$

Let $BCC(X)$ denote the set of all nonempty bounded, closed and convex subsets of $X$. A multivalued map $G : J \to BCC(X)$ is said to be measurable, if for each $x \in X$, the distance between $x$ and $G(x)$ is a measurable function on $J$.

An upper semi-continuous map $G : X \to 2^X$ is said to be condensing if for each subset $B \subseteq X$ with $\alpha(B) \neq 0$, we have $\alpha(G(B)) < \alpha(B)$, where $\alpha$ denotes the Kuratowski measure of noncompactness.

We say that a family $\{C(t) : t \in R\}$ of operators in $B(X)$ is a strongly continuous cosine family if

(i) $C(0) = I$ (I is the identity operator in $X$),

(ii) $C(t + s) + C(t - s) = 2C(t)C(s)$ for all $t, s \in R$, and

(iii) the map $t \to C(t)u$ is strongly continuous for each $u \in X$.

The strongly continuous sine family $\{S(t) : t \in R\}$ associated with the given strongly continuous cosine family $\{C(t) : t \in R\}$ is defined by

$$
S(t)u = \int_0^t C(s)uds, \ u \in X, t \in R.
$$

The infinitesimal generator $A : X \to X$ of a cosine family $\{C(t) : t \in R\}$ is defined by

$$
Au = \left. \frac{d^2C(t)}{dt^2} \right|_{t=0} u.
$$
For more details on strongly continuous cosine and sine families, we refer the papers of Travis and Webb [97,98].

Assume the following conditions:

(H1) $G : J \times X \times X \times X \rightarrow BCC(X)$ is measurable, with respect to $t$, for each $u, v \in X$, upper semicontinuous with respect to $u, v$ for each $t \in J$ and, for each $u, v \in C(J, X)$, the set

$$S_{G,u} = \left\{ g \in L^1(J; X) : g(t) \in G(t, u, \int_0^t k(t, s, u)ds, \int_0^T B(t, s, u)ds) \text{ for a.e. } t \in J \right\}$$

is non empty.

(H2) There exist functions $a(t), b(t) \in C(J; R_+)$ such that

$$\left| \int_0^t k(t, s, u)ds \right| \leq a(t)\|u\| \text{ and } \left| \int_0^T B(t, s, u)ds \right| \leq b(t)\|u\| \text{ for a.e. } t \in J, u \in X.$$  

(H3) There exists a function $\alpha_0(t) \in L^1(J, R_+)$ such that

$$\|G(t, u, v, w)\| \leq \alpha_0(t)\Omega_0(\|u\| + \|v\| + \|w\|)$$

for a.e. $t \in J, u \in X$, where $\Omega_0 : R_+ \rightarrow (0, \infty)$ is a continuous, increasing function satisfying $\Omega_0(a(t)u + b(t)v + c(t)w) \leq a(t)\Omega_0(u) + b(t)\Omega_0(v) + c(t)\Omega_0(w)$ and

$$M_1 T \int_0^T \alpha_0(s)\beta(s)ds < \int_0^\infty \frac{du}{\Omega_0(u)},$$

where $\beta(s) = 1 + a(s) + b(s)$, $c_1 = M_1\|u_0\| + M_1 T\|v_0\|$ and $M_1 = \sup\{\|C(t)\|; t \in J\}$.

(H4) For each bounded set $B \subset C(J, X)$ and $u \in B$, the set

$$\left\{ C(t)u_0 + S(t)v_0 + \int_0^t S(t-s)G(s, u, \int_0^s k(s, \tau, u(\tau))d\tau, \int_0^T B(s, \tau, u(\tau))d\tau)ds : g \in S_{G,u} \right\}$$

is relatively compact.

**Definition 6.1.** A continuous solution $u(t)$ of the integral inclusion

$$u(t) \in C(t)u_0 + S(t)v_0 + \int_0^t S(t-s)G(s, u, \int_0^s k(s, \tau, u(\tau))d\tau, \int_0^T B(s, \tau, u(\tau))d\tau)ds$$

is called a mild solution of (6.1) on $J$. 

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Lemma 6.1.[75] Let $X$ be a Banach space and $\mathcal{N}: X \to BCC(X)$ be a condensing map. If the set
$$\zeta = \{y \in X : \lambda y \in \mathcal{N}(y) \text{for some } \lambda > 1\}$$
is bounded, then $\mathcal{N}$ has a fixed point.

6.2.2. Main Result

Theorem 6.1. If the assumptions (H1)-(H4) are satisfied, then the initial value problem (6.1) has at least one mild solution on $J$.

Proof. Consider the multivalued map $\mathcal{N} : C(J, X) \to 2^{C(J, X)}$ defined by
$$\mathcal{N}u = \left\{ h \in C(J, X) : h(t) = C(t)u_0 + S(t)v_0 + \int_0^t S(t - s)g(s)ds, \ g \in S_{G,u} \right\},$$
where
$$S_{G,u} = \{g \in L^1(J, X) : g(t) \in G(t, u, \int_0^t k(t, s, u(s))ds, \int_0^T B(t, s, u(s))ds) \text{ for a.e } t \in J\}.$$

Let $h_1, h_2 \in \mathcal{N}u$. Then there exist $g_1, g_2 \in S_{G,u}$ such that, for each $t \in J$,
$$h_1(t) = C(t)u_0 + S(t)v_0 + \int_0^t S(t - s)g_1(s)ds,$$
and
$$h_2(t) = C(t)u_0 + S(t)v_0 + \int_0^t S(t - s)g_2(s)ds.$$
Let $0 \leq \alpha_1 \leq 1$. Then, for each $t \in J$, we have
$$(\alpha_1 h_1 + (1 - \alpha_1)h_2)(t) = C(t)u_0 + S(t)v_0 + \int_0^t S(t - s)(\alpha_1 g_1(s) + (1 - \alpha_1)g_2(s))ds.$$
Since $S_{G,u}$ is convex,
$$\alpha_1 h_1 + (1 - \alpha_1)h_2 \in \mathcal{N}u.$$
Hence $\mathcal{N}u$ is convex for each $u \in C(J, X)$. Let $h \in \mathcal{N}u$. Then there exists $g \in S_{G,u}$ such that, for each $t \in J$,
$$h(t) = C(t)u_0 + S(t)v_0 + \int_0^t S(t - s)g(s)ds$$

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and we have

\[
\|h(t)\| \leq \|C(t)\|\|u_0\| + \|S(t)\|\|v_0\| + \int_0^t \|S(t-s)g(s)\|ds \\
\leq M_1\|u_0\| + M_1T\|v_0\| + M_1T \int_0^t \alpha_0(s)\beta(s)\Omega(||u||)ds \\
\leq M_1\|u_0\| + M_1T\|v_0\| + M_1T \beta\|\sup_{u\in B} \Omega(\|u\|) \left(\int_0^t \alpha_0(s)ds\right).
\]

Thus \(\mathcal{N}\) is bounded on bounded subsets of \(C(J, X)\). Let \(t_1, t_2 \in J, t_1 < t_2\). For each \(u \in B\) and \(h \in \mathcal{N}u\),

\[
\|h(t_2) - h(t_1)\| \leq \|C(t_2)u_0 - C(t_1)u_0\| + \|S(t_2)v_0 - S(t_1)v_0\| \\
+ \|\int_{t_1}^{t_2} (S(t_2-s) - S(t_1-s))g(s)ds\| + \|\int_{t_1}^{t_2} S(t_1-s)g(s)ds\| \\
\leq \|C(t_2)u_0 - C(t_1)u_0\| + \|S(t_2)v_0 - S(t_1)v_0\| \\
+ M_1(t_2 - t_1) \int_0^T \|g(s)\|ds + M_1T \int_{t_1}^{t_2} \|g(s)\|ds.
\]

As \(t_2 \to t_1\) the right-hand side of the above inequality tends to zero. Hence, by Ascoli-Arzela theorem, we conclude that \(\mathcal{N} : C(J, X) \to 2^{C(J, X)}\) is a completely continuous map. Finally we shall prove that \(\mathcal{N}\) is upper semicontinuous. Let \(u_n \to u_*\), \(h_n \in \mathcal{N}(u_n)\) and \(h_n \to h_0\). We shall prove that \(h_0 \in \mathcal{N}(u_*)\). \(h_n \in \mathcal{N}(u_n)\) means that there exists \(g_n \in S_{G,u_n}\) such that

\[
h_n(t) = C(t)u_0 + S(t)v_0 + \int_0^t S(t-s)g_n(s)ds, \quad t \in J.
\]

We must prove that there exists \(g_0 \in S_{G,u}\) such that

\[
h_0(t) = C(t)u_0 + S(t)v_0 + \int_0^t S(t-s)g_0(s)ds, \quad t \in J.
\]

Now we consider the linear continuous operator

\[
\Gamma : L^1(J, X) \to C(J, X)
\]

defined by \(\Gamma(g)(t) = \int_0^t S(t-s)g(s)ds\). Obviously we have

\[
\|(h_n - C(t)u_0 - S(t)v_0) - (h_0 - C(t)u_0) - S(t)v_0)\|_\infty \to 0 \text{ as } n \to \infty.
\]

From Lemma 4.1, it follows that \(\Gamma \circ S_{G,u}\) is upper semicontinuous. Since \(u_n \to u_*\), and from Lemma 4.1, we have

\[
h_0(t) - C(t)u_0 - S(t)v_0 = \int_0^t S(t-s)g_0(s)ds \text{ for some } g_0 \in S_{G,u_*}.
\]
Finally we shall prove that the set \( \zeta = \{ u \in C(J, X) : \lambda u \in \mathcal{N}(u), \text{ for some } \lambda > 1 \} \) is bounded. Let \( u \in \zeta \). Then \( \lambda u \in \mathcal{N}(u) \) for some \( \lambda > 1 \). Thus there exists \( g \in S_{G,u} \) such that

\[
 u(t) = \lambda^{-1} C(t)u_0 + \lambda^{-1} S(t)v_0 + \lambda^{-1} \int_0^t S(t-s)g(s)ds,
\]

for each \( t \in J \) and we have

\[
 \|u(t)\| \leq M_1\|u_0\| + M_1T\|v_0\| + M_1T\int_0^t \alpha_0(s)\beta(s)\Omega_0(\|u\|)ds.
\]

Let

\[
 v_1(t) = M_1\|u_0\| + M_1T\|v_0\| + M_1T\int_0^t \alpha_0(s)\beta(s)\Omega_0(\|u\|)ds.
\]

Then we have \( v(0) = M_1\|u_0\| + M_1T\|v_0\| = c_1 \) and \( \|u(t)\| \leq v_1(t), t \in J \). Using the increasing character of \( \Omega_0 \), we have

\[
 \int_{v(0)}^{v_1(t)} \frac{du}{\Omega_0(u)} \leq M_1T \int_0^T \alpha_0(s)\beta(s)ds < \int_{c_1}^{\infty} \frac{du}{\Omega_0(u)}.
\]

This proves that there exists a constant \( M_2 \) such that \( v_1(t) \leq M_2, t \in J \), and hence \( \|u(t)\| \leq M_2 \), where \( M_2 \) depends on \( T \) and on the functions \( \alpha, a, b, \Omega_0 \). Hence \( \zeta \) is bounded. Thus, by Lemma 6.1, \( \mathcal{N} \) has a fixed point which is a mild solution of (6.1).

### 6.3. EVOLUTION INTEGRODIFFERENTIAL INCLUSIONS

#### 6.3.1. Preliminaries

Consider the following second order evolution integrodifferential inclusion of the form

\[
 \frac{d^2u}{dt^2} - A(t)u \in G(t, u, \int_0^t K(t, s, u)ds, \int_0^T H(t, s, u)ds) \quad t \in J = [0, T] \tag{6.2}
\]

\[
 u(0) = u_0, u'(0) = v_0,
\]

where \( G : J \times X \times X \times X \to 2^X \) is a bounded, closed, convex multivalued map, \( K, H : D \times X \to X \), where \( D = \{(t, s) \in J \times J ; t \geq s \} \), \( u_0 \in X, v_0 \in X \), \( T \) is a real constant, \( A(t) : X \to X \) denotes a closed densely defined operator and \( X \), a real Banach space with the norm \( \| \cdot \| \).
Definition 6.2. A family $S$ of bounded linear operators $S(t, s) : X \to X$, $t, s \in [0, T]$ is called a fundamental solution of the second order equation $\frac{d^2 u}{dt^2} = A(t)u(t)$ if

$[Z_1]$ For each $x \in X$, the mapping $[0, T] \times [0, T] \ni (t, s) \mapsto S(t, s)x \in X$ is of class $C^1$ and $(Z_{11})$ for each $t \in [0, T], S(t, t) = 0$,

$(Z_{12})$ for all $t, s \in [0, T]$ and for each $x \in X$,

$$\left. \frac{\partial}{\partial t} S(t, s) \right|_{t=s} x = x, \quad \left. \frac{\partial}{\partial s} S(t, s) \right|_{t=s} x = -x.$$ 

$[Z_2]$ For all $t, s \in [0, T]$, if $x \in D(A)$, then $S(t, s)x \in D(A)$ and the mapping $[0, T] \times [0, T] \ni (t, s) \mapsto S(t, s)x \in X$ is of class $C^2$ and $(Z_{21}) \frac{\partial^2}{\partial t^2} S(t, s)x = A(t)S(t, s)x,$

$(Z_{22}) \frac{\partial^2}{\partial s^2} S(t, s)x = S(t, s)A(s)x,$

$(Z_{23}) \left. \frac{\partial}{\partial s} \frac{\partial}{\partial t} S(t, s) \right|_{t=s} x = 0.$

$[Z_3]$ For all $t, s \in [0, T]$, if $x \in D(A)$, then $\left. \frac{\partial}{\partial s} S(t, s)x \right|_{s=0} D(A)$ and there exist

$$\frac{\partial^2}{\partial t^2} \frac{\partial}{\partial s} S(t, s)x, \quad \frac{\partial^2}{\partial s^2} \frac{\partial}{\partial t} S(t, s)x$$

and

$$(Z_{31}) \frac{\partial^2}{\partial t^2} \frac{\partial}{\partial s} S(t, s)x = A(t) \frac{\partial}{\partial s} S(t, s)x,$$

$$\left. (Z_{32}) \frac{\partial^2}{\partial s^2} \frac{\partial}{\partial t} S(t, s)x = \frac{\partial}{\partial t} S(t, s)A(s)x, \right.$$ 

and the mapping $[0, T] \times [0, T] \ni (t, s) \mapsto A(t) \frac{\partial}{\partial s} S(t, s)x$ is continuous.

For convenience let us denote $U(t, 0) = \left. -\frac{\partial}{\partial s} S(t, s) \right|_{s=0}$. We assume the following conditions:
(A1) \(G : J \times X \times X \times X \to BCC(X)\) is measurable, with respect to \(t\), for each \(u \in X\), upper semicontinuous with respect to \(u, v\) for each \(t \in J\) and, for each \(u \in C(J, X)\), the set
\[ S_{G,u} = \{ g \in L^1(J; X) : g(t) \in G(t, u, \int_0^T K(t, s, u)ds, \int_0^T H(t, s, u)ds) \text{ for a.e } t \in J \} \]
is nonempty.

(A2) There exist functions \(a(t), b(t) \in C(J; R_+)\) such that
\[ \left| \int_0^T K(t, s, u)ds \right| \leq a(t)\|u\| \text{ and } \left| \int_0^T H(t, s, u)ds \right| \leq b(t)\|u\| \text{ for a.e } t \in J, u \in X. \]

(A3) There exists a function \(\beta(t) \in L^1(J; R_+)\) such that
\[ \|G(t, u, v, w)\| \leq \beta(t)\Omega(\|u\| + \|v\| + \|w\|) \]
for a.e \(t \in J, u \in X\), where \(\Omega : R_+ \to (0, \infty)\) is a continuous, increasing function satisfying \(\Omega(a_1(t)u + a_2(t)v + a_3(t)w) \leq a_1(t)\Omega(u) + a_2(t)\Omega(v) + a_3(t)\Omega(w)\) for every \(a_1(t), a_2(t), a_3(t) \in C(J; R_+)\) and
\[ M \int_0^T \beta(s)\gamma(s)ds < \int_c^\infty \frac{du}{\Omega(u)}, \]
where \(\gamma(s) = 1 + a(s) + b(s)\), \(c = M^*\|u_0\| + M\|v_0\|\) and
\[ M = \sup\{\|S(t, s)\|, t, s \in J\}, \quad M^* = \sup\{\|\frac{\partial}{\partial s}S(t, s)\|\} \]

(A4) For each bounded set \(B \subset C(J, X), u \in B\), the set
\[ \{U(t, 0)u_0 + S(t, 0)v_0 + \int_0^t S(t, s)g(s)ds : g \in S_{G,u}\}, \]
is relatively compact.

**Definition 6.3.** A continuous solution \(u(t)\) of the integral inclusion
\[ u(t) \in U(t, 0)u_0 + S(t, 0)v_0 + \int_0^t S(t, s)G \left( s, u, \int_s^t K(s, \tau, u(\tau))d\tau, \int_0^T H(s, \tau, u(\tau))d\tau \right) ds \]
is called a mild solution of (6.1) on \(J\).
6.3.2. Main Result

Theorem 6.2. If the assumptions (A1)-(A4) are satisfied, then the initial value problem (6.2) has at least one mild solution on J.

Proof. Consider the multivalued map $\mathcal{N}: C(J, X) \to 2^{C(J, X)}$ defined by

$$\mathcal{N}u = \left\{ h \in C(J, X) : h(t) = U(t, 0)u_0 + S(t, 0)v_0 + \int_0^t S(t, s)g(s)ds : g \in S_{G, u} \right\},$$

where

$$S_{G, u} = \{ g \in L^1(J, X) : g(t) \in G(t, u, \int_0^t K(t, s, u(s))ds, \int_0^T H(t, s, u(s))ds) \text{ for a.e } t \in J \}$$

Let $h_1, h_2 \in \mathcal{N}u$. Then there exist $g_1, g_2 \in S_{G, u}$ such that, for each $t \in J,$

$$h_1(t) = U(t, 0)u_0 + S(t, 0)v_0 + \int_0^t S(t, s)g_1(s)ds,$$

$$h_2(t) = U(t, 0)u_0 + S(t, 0)v_0 + \int_0^t S(t, s)g_2(s)ds.$$

Let $0 < \alpha_1 < 1$ and, for each $t \in J,$ we have

$$(\alpha_1 h_1 + (1 - \alpha_1) h_2)(t) = U(t, 0)u_0 + S(t, 0)v_0 + \int_0^t S(t, s)(\alpha_1 g_1(s) + (1 - \alpha_1) g_2(s))ds.$$

Since $S_{G, u}$ is convex,

$$\alpha_1 h_1 + (1 - \alpha_1) h_2 \in \mathcal{N}u.$$

Hence $\mathcal{N}u$ is convex for each $u \in C(J, X)$. Let $h \in \mathcal{N}u$. Then there exist $g \in S_{G, u}$ such that, for each $t \in J,$

$$h(t) = U(t, 0)u_0 + S(t, 0)v_0 + \int_0^t S(t, s)g(s)ds.$$

and we have

$$\|h(t)\| \leq \|U(t, 0)\|\|u_0\| + \|S(t, 0)\|\|v_0\| + \int_0^t \|S(t, s)g(s)\|ds$$

$$\leq M^*\|u_0\| + M\|v_0\| + M\int_0^t \beta(s)\gamma(s)\Omega(\|u\|)ds$$

$$\leq M^*\|u_0\| + M\|v_0\| + M\|\gamma\|\sup_{u \in B} \Omega(\|u\|) \left( \int_0^t \beta(s)ds \right).$$
Thus $\mathcal{N}$ is bounded on bounded subsets of $C(J, X)$. Let $t_1, t_2 \in J, t_1 < t_2$. For each $u \in B$ and $h \in \mathcal{N}u$,

$$
\|h(t_2) - h(t_1)\| \leq \| - \partial_s [S(t_2, s) - S(t_1, s)]_{s=0} u_0 \| + \|[S(t_1, 0) - S(t_2, 0)]v_0\|
+ \| \int_{t_1}^{t_2} (S(t_2, s) - S(t_1, s))g(s)ds \| + \| \int_{t_1}^{t_2} S(t_1, s)g(s)ds \|
\leq \|[U(t_2, 0) - U(t_1, 0)]u_0\| + \|[S(t_1, 0) - S(t_2, 0)]v_0\|
+ M(t_2 - t_1) \int_{t_1}^{t_2} \|g(s)\|ds + M \int_{t_1}^{t_2} \|g(s)\|ds.
$$

As $t_2 \to t_1$, the right-hand side of the above inequality tends to zero. Hence, by Ascoli-Arzela theorem, we conclude that $\mathcal{N} : C(J, X) \to 2^{C(J, X)}$ is a completely continuous map. Next we shall prove that $\mathcal{N}$ is upper semicontinuous. Let $u_n \to u_*$, $h_n \in \mathcal{N}(u_n)$ and $h_n \to h_0$. We shall prove that $h_0 \in \mathcal{N}(u_*)$. $h_n \in \mathcal{N}(u_n)$ means that there exists $g_n \in S_{G, u_n}$ such that

$$
h_n(t) = U(t, 0)u_0 + S(t, 0)v_0 + \int_{t_0}^{t} S(t, s)g_n(s)ds, \quad t \in J.
$$

We must prove that there exists $g_0 \in S_{G, u}$ such that

$$
h_0(t) = U(t, 0)u_0 + S(t, 0)v_0 + \int_{t_0}^{t} S(t, s)g_0(s)ds, \quad t \in J.
$$

Now we consider the linear continuous operator

$$
\Gamma : L^1(J, X) \to C(J, X)
$$

defined by $\Gamma(g)(t) = \int_{t_0}^{t} S(t, s)g(s)ds$. Obviously we have

$$
\|(h_n - U(t, 0)u_0 - S(t, 0)v_0) - (h_0 - U(t, 0)u_0 - S(t, 0)v_0)\|_{\infty} \to 0 \text{ as } n \to \infty.
$$

From Lemma 4.1, it follows that $\Gamma \circ S_{G, u}$ is a closed graph operator. Since $u_n \to u_*$ and, from Lemma 4.1, we have

$$
h_0(t) - U(t, 0)u_0 - S(t, 0)v_0 = \int_{t_0}^{t} S(t, s)g_0(s)ds \text{ for some } g_0 \in S_{G, u_*}.
$$

Finally we shall prove that the set $\zeta = \{u \in C(J, X) : \lambda u \in \mathcal{N}(u)\}$, for some $\lambda > 1$, is bounded. Let $u \in \zeta$. Then $\lambda u \in \mathcal{N}(u)$ for some $\lambda > 1$. Thus there exists $g \in S_{G, u}$ such that

$$
u(t) = \lambda^{-1}U(t, 0)u_0 + \lambda^{-1}S(t, 0)v_0 + \lambda^{-1} \int_{t_0}^{t} S(t, s)g(s)ds,
$$

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for each \( t \in J \) and we have
\[
\|u(t)\| \leq M^*\|u_0\| + M\|w_0\| + M \int_0^t \beta(s)\gamma(s)\Omega(\|v\|)ds.
\]
Let
\[
v(t) = M^*\|u_0\| + M\|w_0\| + M \int_0^t \beta(s)\gamma(s)\Omega(\|u\|)ds.
\]
Then we have \( v(0) = M^*\|u_0\| + M\|w_0\| = c \) and \( \|u(t)\| \leq v(t), \) \( t \in J. \) Using the increasing character of \( \Omega, \) we have
\[
\int_{v(0)}^{v(t)} \frac{du}{\Omega(u)} \leq MT \int_0^T \beta(s)\gamma(s)ds < \int_c^\infty \frac{du}{\Omega(u)}.
\]
This proves that there exists a constant \( M_1 \) such that \( v(t) \leq M_1, \) \( t \in J, \) and hence \( \|u(t)\| \leq M_1, \) where \( M_1 \) depends on \( T \) and on the functions \( \beta, a, b, \Omega. \) Hence \( \zeta \) is bounded. Thus, by Lemma 6.1, \( \mathcal{N} \) has a fixed point which is a mild solution of (6.2).

### 6.3.3 Nonlocal Initial conditions

Consider the second order mixed integro-differential inclusion (6.1) with the nonlocal initial condition of the form
\[
u(0) + f(u) = u_0, \quad u'(0) = v_0.
\]
Further we assume the following:

**(A5)** \( f : C(J, X) \to X \) is continuous and there exists a constant \( L > 0 \) such that \( \|f(u)\| \leq L, \) for each \( u \in X. \)

**(A6)** There exists a function \( \beta(t) \in L^1(J; R_+) \) such that
\[
\|G(t, u, v, w)\| \leq \beta(t)\Omega(\|u\| + \|v\| + \|w\|)
\]
for a.e \( t \in J, \) \( u \in X, \) where \( \Omega : R_+ \to (0, \infty) \) is a continuous, increasing function satisfying \( \Omega(a_1(t)u + a_2(t)v + a_3(t)w) \leq a_1(t)\Omega(u) + a_2(t)\Omega(v) + a_3(t)\Omega(w) \) for every \( a_1(t), a_2(t), a_3(t) \in C(J; R_+) \) and
\[
M \int_0^T \beta(s)\gamma(s)ds < \int_{c_2}^\infty \frac{du}{\Omega(u)},
\]
where \( c_2 = M^*\|u_0\| + M^*L\|u_0\| + M\|v_0\|. \)
(A7) For each bounded set $B \subset C(J, X), u \in B$, the set

$$\left\{ U(t, 0)u_0 - U(t, 0)f(u) + S(t, 0)v_0 + \int_0^t S(t, s)g(s)ds : g \in G_u \right\}$$

is relatively compact.

**Definition 6.4.** A continuous solution $u(t)$ of the integral inclusion

$$u(t) \in U(t, 0)u_0 - U(t, 0)f(u) + S(t, 0)v_0 + \int_0^t S(t, s)G(s, u, \int_0^s K(s, r, u(r))dr, \int_0^T H(s, r, u(r))dr)ds$$

is called a mild solution of (6.2)-(6.3) on $J$.

**Theorem 6.3.** If the assumptions (A1),(A2), (A5)-(A7) are satisfied, then the non-local initial value problem (6.2)-(6.3) has at least one mild solution on $J$.

The proof of the above theorem is similar to that of Theorem 6.2 and hence is omitted.