5.1. INTRODUCTION

The existence of mild, strong and classical solutions for differential and integrodifferential equations in abstract spaces have been studied by several authors [57-59,68]. The existence of resolvent operators for an integrodifferential equations in Banach spaces has been established by Grimmer [54]. Based on [54], Balachandran and Sakthivel [20] studied the existence theorem for nonlinear integrodifferential equations in Banach spaces. Many authors extensively investigated the existence of solutions of differential inclusions in Banach spaces [2,5,6,11,12,25,28,30,40,41,70,87,89]. Avgerinos and Papageorgiou [8], Papageorgiou [88,89] and Benchohra [25] discussed the existence of solutions for differential inclusions on unbounded intervals. In this chapter, we study the existence of mild solutions for a nonlinear mixed integrodifferential inclusion using resolvent operators. Further we establish the existence of solutions of an initial value problem of integrodifferential inclusions using the fixed point theorem due to Ma [74].

5.2. MIXED INTEGRODIFFERENTIAL INCLUSIONS

5.2.1 Preliminaries

Consider the initial value problem of the nonlinear mixed integrodifferential inclusions of the form

\[
\frac{du}{dt} - Au \in G \left( t, u, \int_0^t k(t, s, u)ds, \int_0^t B(t, s, u)ds \right) \quad t \in I = [0, \infty),
\]

\[u(0) = u_0,\]

where \( G : I \times X \times X \times X \to 2^X \) is a bounded, closed, convex multivalued map, \( k, B : \Delta \times X \to X \), with \( \Delta = \{(t,s) \in I \times I; t \geq s\} \), \( u_0 \in X \), \( T \) is a real constant, \( A \) is the infinitesimal generator of a strongly continuous semigroup \( T(t), t \geq 0 \) and \( X \), a real Banach space with the norm \( \| \cdot \| \).
We assume the following conditions:

(\textit{H1}) $G : I \times X \times X \times X \to BCC(X)$ is measurable, with respect to $t$, for each $u \in X$, upper semi continuous with respect to $u,v$ for each $t \in I$ and, for each $u,v \in C(I,X)$, the set

$$S_{G,u} = \{g \in L^1(I;X) : g(t) \in G(t,u, \int_0^t k(t,s,u)ds, \int_0^T B(t,s,u)ds) \text{ for a.e } t \in I \}$$

is nonempty.

(\textit{H2}) There exist functions $a(t), b(t) \in C(I;R_+)$ such that

$$\left| \int_0^t k(t,s,u)ds \right| \leq a(t)||u|| \text{ and } \left| \int_0^T B(t,s,u)ds \right| \leq b(t)||u|| \text{ for a.e } t \in I.$$

(\textit{H3}) There exists a function $\alpha(t) \in L^1(I;R_+)$ such that

$$||G(t,u,v,w)|| \leq \alpha(t)\Omega(||u|| + ||v|| + ||w||)$$

for a.e $t \in I$, $u \in X$, where $\Omega : R_+ \to (0, \infty)$ is a continuous, increasing function satisfying $\Omega(a_1(t)u + a_2(t)v + a_3(t)w) \leq a_1(t)\Omega(u) + a_2(t)\Omega(v) + a_3(t)\Omega(w)$ for every $a_1(t), a_2(t), a_3(t) \in C(I, R_+)$ and

$$M \int_0^m \alpha(s)\beta(s)ds < \int_{c_2}^{\infty} \frac{du}{\Omega(u)}$$

for each $m \in N$, where $\beta(s) = 1 + a(s) + b(s)$, $c_2 = M||u_0||$ and $M = \sup\{||T(t)||; t \in I\}$.

(\textit{H4}) For each neighbourhood $U_p$ of $0, u \in U_p$ and $t \in I$, the set

$$\left\{T(t)u_0 + \int_0^t T(t-s)g(s)ds, \; g \in S_{G,u} \right\}$$

is relatively compact.

\textbf{Definition 5.1.} A continuous solution $u(t)$ of the integral inclusion

$$u(t) \in T(t)u_0 + \int_0^t T(t-s)G\left(s,u, \int_0^s k(s,\tau,u(\tau))d\tau, \int_0^T B(s,\tau,u(\tau))d\tau \right)ds$$

is called a mild solution of (5.1) on $I$. 54
5.2.1 Main Results

**Theorem 5.1** If the assumptions (H1)-(H4) are satisfied, then the initial value problem (5.1) has at least one mild solution on $I$.

**Proof.** A solution to (5.1) is a fixed point for the multivalued map $N : C(I, X) \rightarrow 2^{C(I, X)}$ defined by

$$
N(u) = \left\{ h \in C(I, X) : h(t) = T(t)u_0 + \int_0^t T(t - s)g(s)ds, \; g \in S_{G,u} \right\},
$$

where

$$S_{G,u} = \{ g \in L^1(I, X) : g(t) \in G(t, u, \int_0^t k(t, s, u(s))ds, \int_0^T B(t, s, u(s))ds) \text{ for a.e } t \in I \}.$$

First we shall prove $N(u)$ is convex for each $u \in C(I, X)$.

Let $h_1, h_2 \in N(u)$. Then there exist $g_1, g_2 \in S_{G,u}$ such that

$$h_i(t) = T(t)u_0 + \int_0^t T(t - s)g_i(s)ds, \; i = 1, 2, t \in I,$$

Let $0 \leq k_1 \leq 1$. Then, for each $t \in I$, we have

$$(k_1h_1 + (1 - k_1)h_2)t = T(t)u_0 + \int_0^t T(t - s)(k_1g_1(s) + (1 - k_1)g_2(s))ds.$$

Since $S_{G,u}$ is convex, $k_1h_1 + (1 - k_1)h_2 \in N(u)$. Hence $N(u)$ is convex for each $u \in C(I, X)$.

Let $U_p = \{ u \in C(I, X) : \|u\| \leq p \}$ be a neighbourhood of 0 in $C(I, X)$ and $u \in U_p$. Then for each $h \in N(u)$, there exists $g \in S_{G,u}$ such that, for $t \in I$, we have

$$\|h(t)\| \leq \|T(t)\|\|u_0\| + \int_0^t \|T(t - s)\|\|g(s)\|ds$$

$$\leq M\|u_0\| + M\int_0^t a(s)\Omega(\|u\|) + a(s)\|u\| + b(s)\|u\|)ds$$

$$\leq M\|u_0\| + M\int_0^t a(s)\Omega(\|u\|) + a(s)\Omega(\|u\|) + b(s)\Omega(\|u\|)ds$$

$$\leq M\|u_0\| + M\|a\|_{L^1(I_m)}\|b\|_{L^1(I_m)} \sup_{u \in U_p} \Omega(\|u\|).$$

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Hence $N(U_p)$ is bounded in $C(I, X)$ for each $p \in N$.

Next we shall prove $N(U_p)$ is an equicontinuous set in $C(I, X)$ for each $p \in N$. Let $t_1, t_2 \in I_n$, $t_1 < t_2$. Then, for all $h \in N(u), u \in U_p$ and we have

$$
\|h(t_1) - h(t_2)\| \leq \|(T(t_2) - T(t_1))u_0\| + \int_0^{t_1} \|T(t_2 - s) - T(t_1 - s)\|g(u)ds \\|
+ \int_{t_1}^{t_2} T(t_1 - s)g(u)ds \\|
\leq \|(T(t_2) - T(t_1))u_0\| + \int_0^{t_1} \|T(t_2 - s) - T(t_1 - s)\|g(u)ds \\|
+ M(t_2 - t_1) \int_0^m \|g(s)\| ds.
$$

Hence, by the Ascoli-Arzela theorem, we conclude that $N : C(I, X) \to 2^{C(I, X)}$ is a completely continuous multivalued map. Next we shall prove that $N$ is upper semicontinuous.

Let $u_n \to u_*$, $h_n \in N(u_n)$ and $h_n \to h_0$. We shall prove that $h_0 \in N(u_*).h_n \in N(u_n)$ means that there exists $g_n \in S_{G,u_n}$ such that

$$
h_n(t) = T(t)u_0 + \int_0^t T(t - s)g_n(s)ds, \quad t \in I.
$$

We must prove that there exists $g_0 \in S_{G,u}$ such that

$$
h_0(t) = T(t)u_0 + \int_0^t T(t - s)g_0(s)ds, \quad t \in I.
$$

(5.2)
The idea is then to use the fact that $h_n \to h_0$ and $h_n - T(t)u_0 \in \Gamma(S_{G,u})$ where

$$
(\Gamma g)(t) = \int_0^t T(t - s)g(s)ds, \quad t \in I.
$$

So we consider the functions $u_n, h_n - T(t)u_0, g_n$ defined on the interval $[k, k + 1]$ for any $k \in N \cup \{0\}$. Then, using Lemma 4.1, in this case we are able to say that (5.2) is true on the compact interval $[k, k + 1]$, that is,

$$
[h_0(t)|_{[k,k+1]} = T(t)u_0 + \int_0^t T(t - s)g_0^k(s)ds
$$

for a suitable $L^1$-selection $g_0^k$ of $G(t,u, \int_0^t k(t,s,u)ds, \int_0^T B(t,s,u)ds)$ on the interval $[k,k+1]$. Let $g_0(t) = g_0^k(t)$ for $t \in [k,k+1]$. We obtain then that $g_0$ is an $L^1$-selection.
and (5.2) will be satisfied. Clearly we have \(\| (h_n - T(t)u_0) - (h_0 - T(t)u_0) \|_\infty \to 0\) as \(n \to \infty\). Consider, for all \(k \in \mathbb{N} \cup \{0\}\), the mapping

\[
S^k_{\mathcal{G}} : C([k, k+1], X) \to L^1([k, k+1], X),
\]

\[
y \to S^k_{\mathcal{G}, y} = \{ g \in L^1([k, k+1], X) : g(t) \in G(t, u, \int_0^t k(t, s, u)ds, \int_0^T B(t, s, u)ds) \text{ for a.e } t \in [k, k+1]\}.
\]

Now we consider the linear continuous operators

\[
\Gamma_k : L^1([k, k+1], X) \to C([k, k+1], X),
\]

\[
g \to \Gamma_k(g)(t) = \int_0^t T(t-s)g(s)ds.
\]

From Lemma 4.1, it follows that \(\Gamma_k \circ S^k_{\mathcal{G}}\) is a closed graph operator for all \(k \in \mathbb{N} \cup \{0\}\). Moreover, we have

\[
(h_n(t) - T(t)u_0)|_{[k, k+1]} \in \Gamma_k(S^k_{\mathcal{G}, u_0})
\]

and \(u_n \to u_*\). From Lemma 4.1, we have \((h_0(t) - T(t)u_0)|_{[k, k+1]} \in \Gamma_k(S^k_{\mathcal{G}, u_0})\), and

\[
(h_0(t) - T(t)u_0)|_{[k, k+1]} = \int_0^t T(t-s)g^k_0(s)ds \text{ for some } g^k_0 \in S^k_{\mathcal{G}, u_*}.
\]

Hence the function \(g_0\) defined on \(I\) by \(g_0(t) = g^k_0(t)\) for \(t \in [k, k+1]\) is in \(S^k_{\mathcal{G}, u_*}\). Therefore \(\mathcal{N}(U_p)\) is relatively compact, for each \(p \in N\), and \(\mathcal{N}\) is upper semi continuous with convex closed values. Finally we prove the set \(\zeta = \{ u \in C(I, X) ; \lambda u \in \mathcal{N}u \}\) for some \(\lambda > 1\), is bounded.

Let \(\lambda u = \mathcal{N}u\) for some \(\lambda > 1\). Then there exists \(g \in S_{\mathcal{G}, u}\) such that

\[
u(t) = \lambda^{-1}T(t)u_0 + \lambda^{-1}\int_0^t T(t-s)g(s)ds, t \in I,
\]

and \(\|u(t)\| \leq M\|u_0\| + M\int_0^t \alpha(s)\beta(s)\Omega(\|u\|)ds\).

Let

\[
u(t) = M\|u_0\| + M\int_0^t \alpha(s)\beta(s)\Omega(\|u\|)ds.
\]

Then we have \(v(0) = M\|u_0\| = c_2\) and \(\|u(t)\| \leq v(t), t \in I_m\). Using the increasing character of \(\Omega\), we get

\[
v'(t) \leq M\alpha(t)\beta(t)\Omega(v(t)), t \in I_m.
\]
This proves, for each $t \in I_m$, that
\[ \int_{v(0)}^{v(t)} \frac{du}{\Omega(u)} \leq M \int_0^m \alpha(s)\beta(s)ds < \int_0^\infty \frac{du}{\Omega(u)}. \]
This inequality implies that there exists a constant $M_0$ such that $v(t) \leq M_0, t \in I_m$, and hence $\|u\|_\infty \leq M_0$ where $M_0$ depends on $m$ and on the functions $\alpha, \alpha, \beta, \Omega$. Hence $\zeta$ is bounded. Thus, by Lemma 4.2, $\mathcal{N}$ has a fixed point which is a mild solution of (5.1).

**5.2.2 Nonlocal Initial conditions**

Consider a first order mixed integro-differential inclusion (5.1) with the nonlocal initial condition of the form
\[ u(0) + f(u) = u_0. \] (5.3)

In addition to the previous assumptions, we assume the following:

(H5) Let $f : C(I, X) \rightarrow X$ be a continuous function and there exist a constant $L > 0$ such that $\|f(u)\| \leq L$ for each $u \in X$.

(H6) There exists a function $\alpha(t) \in L^1(I; R_+)$ such that
\[ \|G(t, u, v, w)\| \leq \alpha(t)\Omega(\|u\| + \|v\| + \|w\|) \]
for a.e $t \in I, u \in X$, where $\Omega : R_+ \rightarrow (0, \infty)$ is continuous, increasing function satisfying $\Omega(a_1(t)u + a_2(t)v + a_3(t)w) \leq a_1(t)\Omega(u) + a_2(t)\Omega(v) + a_3(t)\Omega(w)$ for every $a_1(t), a_2(t), a_3(t) \in C(I, R_+)$ and
\[ M \int_0^m \alpha(s)\beta(s)ds < \int_0^\infty \frac{du}{\Omega(u)} \]
where $c_3 = M\|u_0\| + LM\|u_0\|.

(H7) For each neighbourhood $U_p$ of 0, $u \in U_p$ and $t \in I$, the set
\[ \{T(t)u_0 - T(t)f(u) + \int_0^t T(t-s)g(s)ds, \ g \in S_{G,p}\} \]
is relatively compact.
Definition 5.2. A continuous solution $u(t)$ of the integral inclusion

$$
\dot{u}(t) \in T(t)u_0 - T(t)f(u) + \int_0^t T(t-s)G \left( s, u, \int_0^s k(s, \tau, u(\tau))d\tau, \int_0^T B(s, \tau, u(\tau))d\tau \right) ds
$$

is called a mild solution of (5.1)-(5.3) on $I$.

Theorem 5.2. If the assumptions (H1),(H2), (H5)-(H7) are satisfied, then the non-local initial value problem (5.1)-(5.3) has at least one mild solution on $I$.

The proof of Theorem 5.2 is similar to that of Theorem 5.1 and hence is omitted.

5.3. NONLINEAR MIXED INTEGRODIFFERENTIAL INCLUSIONS

5.3.1. Preliminaries

Consider the nonlinear mixed integrodifferential inclusion of the form

$$
\frac{du}{dt} - A[u(t)] + \int_0^t F(t-s)u(s) ds \in G(t, u, \int_0^t k(t, s, u(s)) ds, \int_0^T h(t, s, u(s)) ds), \quad t \in I = [0, \infty), \quad (5.4)
$$

$$
u(0) = u_0,
$$

where $G : I \times X \times X \times X \rightarrow 2^X$ is a bounded, closed, convex multivalued map, $k, h : \Delta \times X \rightarrow X$, are given functions, $u_0 \in X$, $F(t) : Y \rightarrow Y$, $AF(.)u(.) \in L^1(I, X)$, $F(t) \in B(X), t \in I$ and for $u \in X, F'(t)u$ is continuous in $t \in I$, where $B(X)$ is the space of all bounded linear operators on $X$ and $Y$ is the Banach space formed from $D(A)$, the domain of $A$ endowed with the graph norm. $T$ is a real constant, $A$ is the infinitesimal generator of a strongly continuous semigroup in a Banach space $X$.

Definition 5.3. A continuous solution $u(t)$ of the integral inclusion

$$
u(t) \in R(t)u_0 + \int_0^t R(t-s)G \left( s, u, \int_0^s k(s, \tau, u(\tau))d\tau, \int_0^T h(s, \tau, u(\tau))d\tau \right) ds
$$

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is called a mild solution of (5.3) on $I$, where $R(t)$ is a resolvent operator of (5.3) with $G = 0$ and $R(t) \in B(X)$, for $t \in I$, satisfying the following conditions (see [54]):

(a) $R(0) = I$ (the identity operator on $X$),

(b) for all $x \in X, R(t)x$ is continuous for $t \in I$,

(c) $R(t) \in B(Y), t \in I$. For $y \in Y, R(t)y \in C^1(I, X) \cap C(I, Y)$ and

$$\frac{d}{dt} R(t)y = A \left[ R(t)y + \int_0^t F(t-s)R(s)yds \right]$$

$$= R(t)Ay + \int_0^t R(t-s)AF(s)yds, \ t \in I.$$

We assume the following conditions:

(A1) $G : I \times X \times X \times X \to BCC(X)$ is measurable with respect to $t$, for each $u \in X$, upper semi continuous with respect to $u$ for each $t \in I$ and for each $u \in C(I, X)$, the set

$$S_{G,u} = \{ g \in L^1(I; X) : g(t) \in G(t, u(t), \int_0^t k(t, s, u(s))ds, \int_0^t h(t, s, u(s))ds) \}$$

for a.e $t \in I$ is non empty.

(A2) There exist functions $a(t), b(t) \in C(I, \mathbb{R}^+)$ such that

$$\left| \int_0^t k(t, s, u)ds \right| \leq a(t)||u|| \text{ and } \left| \int_0^t h(t, s, u)ds \right| \leq b(t)||u|| \text{ for a.e } t, s \in I, u \in X.$$

(A3) The resolvent operator $R(t)$ is compact such that $\max_{t \geq 0} ||R(t)|| \leq M$, where $M > 0$.

(A4) There exists a function $\alpha(t) \in L^1(I, \mathbb{R}^+)$ such that

$$||G(t, u, v, w)|| \leq \alpha(t)\Omega(||u|| + ||v|| + ||w||)$$

for a.e $t \in I, u \in X$, where $\Omega : R_+ \to (0, \infty)$ is a continuous, increasing function satisfying $\Omega(a_1(t)x + a_2(t)y + a_3(t)z) \leq a_1(t)\Omega(x) + a_2(t)\Omega(y) + a_3(t)\Omega(z)$ for every $a_1(t), a_2(t), a_3(t) \in C(I, \mathbb{R}^+)$ and

$$M \int_0^m \alpha(s)\beta(s)ds < \int_c^\infty \frac{du}{\Omega(u)}$$

for each $m \in N$, where $\beta(s) = 1 + a(s) + b(s), \ c = M||u_0||$.  

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(A5) For each neighbourhood $U_p$ of $0, u \in U_p$ and $t \in I$, the set
$$\{R(t)u_0 + \int_0^t R(t-s)g(s)ds, \ g \in S_{G,u}\}$$

is relatively compact.

5.3.2 Main Results

**Theorem 5.3.** If the assumptions (A1)-(A5) are satisfied, then the initial value problem (5.4) has at least one mild solution on $I$.

**Proof.** A solution to (5.4) is a fixed point for the multivalued map
$$\mathcal{N}: C(I, X) \rightarrow 2^{C(I, X)}$$
defined by
$$\mathcal{N}(u) = \{y \in C(I, X) : y(t) = R(t)u_0 + \int_0^t R(t-s)g(s)ds, \ g \in S_{G,u}\},$$
where
$$S_{G,u} = \left\{ g \in L^1(I, X) : g(t) \in G(t, u, \int_0^t k(t, s, u(s))ds, \int_0^T h(t, s, u(s))ds) \text{ for a.e } t \in I \right\}.$$

First we shall prove that $\mathcal{N}(u)$ is convex for each $u \in C(I, X)$. Let $y_1, y_2 \in \mathcal{N}(u)$. Then there exist $g_1, g_2 \in S_{G,u}$ such that
$$y_i(t) = R(t)u_0 + \int_0^t R(t-s)g_i(s)ds, i = 1, 2, t \in I.$$

Let $0 \leq k_1 \leq 1$. Then, for each $t \in I$, we have
$$(k_1y_1 + (1-k_1)y_2)t = R(t)u_0 + \int_0^t R(t-s)(k_1g_1(s) + (1-k_1)g_2(s))ds.$$ 

Since $S_{G,u}$ is convex, $k_1y_1 + (1-k_1)y_2 \in \mathcal{N}(u)$. Hence $\mathcal{N}(u)$ is convex for each $u \in C(I, X)$.

Let $U_p = \{ u \in C(I, X) : \|u\| \leq p \}$ be a neighbourhood of $0$ in $C(I, X)$ and $u \in U_p$, then, for each $y \in \mathcal{N}(u)$, there exists $g \in S_{G,u}$ such that for $t \in I$ and we have
\[
\|y(t)\| \leq \|R(t)\|\|u_0\| + \int_0^t \|R(t-s)\|\|g(s)\|\,ds \\
\leq M\|u_0\| + M\int_0^t \alpha(s)\Omega(\|u\| + a(s)\|u\| + b(s)\|u\|)\,ds \\
\leq M\|u_0\| + M\int_0^t \alpha(s)\beta(s)\Omega(\|u\|)\,ds \\
\leq M\|u_0\| + M\|\alpha\|_{L^1(I_m)}\|\beta\| \sup_{u \in U_p} \Omega(\|u\|). 
\]

Hence \(\mathcal{N}(U_p)\) is bounded in \(C(I, X)\) for each \(p \in N\).

Next we shall prove that \(\mathcal{N}(U_p)\) is an equicontinuous set in \(C(I, X)\), for each \(p \in N\). Let \(t_1, t_2 \in I_m, t_1 < t_2\). Then, for all \(h \in \mathcal{N}(u), u \in U_p\), we have

\[
\|y(t_1) - y(t_2)\| \leq \|(R(t_2) - R(t_1))u_0\| + \int_0^{t_2} \|(R(t_2 - s) - R(t_1 - s))g(u)\,ds\| \\
+ \|\int_{t_1}^{t_2} R(t_1 - s)g(u)\,ds\| \\
\leq \|(R(t_2) - R(t_1))u_0\| + \int_0^{t_2} \|(R(t_2 - s) - R(t_1 - s))g(u)\,ds\| \\
+ M(t_2 - t_1)\int_0^m \|g(s)\|\,ds. 
\]

Hence, by the Ascoli-Arzela theorem, we conclude that \(\mathcal{N}\) is a completely continuous multivalued map.

Now we shall prove that \(\mathcal{N}\) is upper semi continuous. Let \(u_n \to u_*, y_n \in \mathcal{N}(u_n)\) and \(y_n \to y_0\). We shall prove that \(y_0 \in \mathcal{N}(u_*)\). \(y_n \in \mathcal{N}(u_n)\) means that there exists \(g_n \in S_{G,u_n}\) such that

\[
y_n(t) = R(t)u_0 + \int_0^t R(t-s)g_n(s)\,ds, \quad t \in I.
\]

We must prove that there exists \(g_0 \in S_{G,u}\) such that

\[
y_0(t) = R(t)u_0 + \int_0^t R(t-s)g_0(s)\,ds, \quad t \in I. \tag{5.5}
\]

The idea is then to use the fact that \(y_n \to y_0\) and \(y_n - R(t)u_0 \in \Gamma(S_{G,u})\) where

\[
(\Gamma g)(t) = \int_0^t R(t-s)g(s)\,ds, \quad t \in I.
\]
So we consider the functions \( u_n, y_n - R(t)u_0, g_n \) defined on the interval \([k, k + 1]\) for any \( k \in \mathbb{N} \cup \{0\} \). Then, using Lemma 4.1, in this case we are able to say that (5.5) is true on the compact interval \([k, k + 1]\), that is,

\[
[y_0(t)]_{[k,k+1]} = R(t)u_0 + \int_0^t R(t-s)g_k^*(s)\,ds
\]

for a suitable \( L^1 \)-selection \( g_k^* \) of \( G \left( t, u, \int_0^t k(t,s,u)\,ds, \int_0^T h(t,s,u)\,ds \right) \) on the interval \([k, k + 1]\). Let \( g_0(t) = g_k^*(t) \) for \( t \in [k, k + 1] \). We obtain then that \( g_0 \) is an \( L^1 \)-selection and (5.2) will be satisfied. Clearly we have \( \|(y_n - R(t)u_0) - (y_0 - R(t)u_0)\|_\infty \to 0 \) as \( n \to \infty \). Consider, for all \( k \in \mathbb{N} \cup \{0\} \), the mapping

\[
S_{G,u}^k : C([k, k + 1], X) \to L^1([k, k + 1], X),
\]

\[
y \to S_{G,u}^k = \{ g \in L^1([k, k + 1], X) : g(t) \in G \left( t, u, \int_0^t k(t,s,u)\,ds, \int_0^T h(t,s,u)\,ds \right) \text{ for a.e } t \in [k, k + 1] \}.
\]

Now we consider the linear continuous operators

\[
\Gamma_k : L^1([k, k + 1], X) \to C([k, k + 1], X),
\]

\[
g \to \Gamma_k(g)(t) = \int_0^t R(t-s)g(s)\,ds.
\]

From Lemma 4.1, it follows that \( \Gamma_k \circ S_{G,u}^k \) is upper semi continuous for all \( k \in \mathbb{N} \cup \{0\} \). Moreover, we have

\[
(y_n(t) - R(t)u_0)_{|[k,k+1]} \in \Gamma_k(S_{G,u}^k)
\]

and \( u_n \to u_* \). From Lemma 4.1, we have \( (y_0(t) - R(t)u_0)_{|[k,k+1]} \in \Gamma_k(S_{G,u_*}^k) \), and

\[
(y_0(t) - R(t)u_0)_{|[k,k+1]} = \int_0^t R(t-s)g_0^*\,ds \text{ for some } g_0^* \in S_{G,u_*}^k.
\]

Hence the function \( g_0 \), defined on \( I \), by \( g_0(t) = g_0^*(t) \) for \( t \in [k, k + 1] \) is in \( S_{G,u_*} \). Therefore \( \mathcal{N}(U_p) \) is relatively compact for each \( p \in \mathbb{N} \) and \( \mathcal{N} \) is upper semi continuous with convex closed values. Finally we prove that the set \( \zeta = \{ u \in C(I, X) : \lambda u \in \mathcal{N}u \} \), for some \( \lambda > 1 \), is bounded.

Let \( \lambda u = \mathcal{N}u \) for some \( \lambda > 1 \). Then there exists \( g \in S_{G,u} \) such that

\[
u(t) = \lambda^{-1} R(t)u_0 + \lambda^{-1} \int_0^t R(t-s)g(s)\,ds, \quad t \in I,
\]

and \( \|u(t)\| \leq M\|u_0\| + M \int_0^t \alpha(s)\beta(s)\Omega(\|u\|)\,ds \).
Let
\[ v(t) = M\|u_0\| + M \int_0^t \alpha(s)\beta(s)\Omega(||u||)ds. \]

Then we have \( v(0) = M\|u_0\| = c \) and \( \|u(t)\| \leq v(t), t \in I_m \). Using the increasing character of \( \Omega \), we get
\[ v'(t) \leq M\alpha(t)\beta(t)\Omega(v(t)), \quad t \in I_m. \]

This proves, for each \( t \in I_m \), that
\[ \int_{v(0)}^{v(t)} \frac{du}{\Omega(u)} \leq M \int_c^m \alpha(s)\beta(s)ds < \int_c^\infty \frac{du}{\Omega(u)}. \]

This inequality implies that there exists a constant \( M_0 \) such that \( v(t) \leq M_0, t \in I_m \), and hence \( \|u\|_\infty \leq M_0 \) where \( M_0 \) depends on \( m \) and on the functions \( \alpha, \beta, \Omega \). Hence \( \zeta \) is bounded. Thus, by Lemma 4.2, \( N \) has a fixed point which is a mild solution of (5.4).

### 5.3.3 Nonlocal Initial condition

Consider the nonlinear mixed integrodifferential inclusion (5.4) with the nonlocal initial condition
\[ u(0) + f(u) = u_0. \]  
(5.6)

In addition to the previous assumptions, we assume the following:

(A6) Let \( f : C(I, X) \to X \) be a continuous function and there exist a constant \( L > 0 \) such that \( \|f(u)\| \leq L \) for each \( u \in X \).

(A7) There exists a function \( \alpha(t) \in L^1(I; R_+) \) such that
\[ \|G(t,u,v,w)\| \leq \alpha(t)\Omega(||u|| + ||v|| + ||w||) \]
for a.e \( t \in I, u \in X \), where \( \Omega : R_+ \to (0, \infty) \) is a continuous, increasing function and
\[ M \int_0^m \alpha(s)\beta(s)ds < \int_{c_1}^\infty \frac{du}{\Omega(u)} \]
where \( c_1 = M\|u_0\| + LM\|u_0\| \).
For each neighbourhood $U_p$ of 0, $u \in U_p$ and $t \in I$, the set

$$\{ R(t)u_0 - R(t)f(u) + \int_0^t R(t - s)g(s)ds, \ g \in S_{G,u} \}$$

is relatively compact.

**Definition 5.4.** A continuous solution $u(t)$ of the integral inclusion

$$u(t) \in R(t)u_0 - R(t)f(u) + \int_0^t R(t - s)G \left( s, u, \int_0^s h(s, \tau, u(\tau))d\tau, \int_0^T h(s, \tau, u(\tau))d\tau \right) ds$$

is called a mild solution of $(5.4)-(5.6)$ on $I$.

**Theorem 5.4.** If the assumptions (A1)-(A3) and (A5)-(A7) are satisfied, then the nonlocal initial value problem $(5.4)-(5.6)$ has at least one mild solution on $I$.

The proof of Theorem 5.4 is similar to that of Theorem 5.3 and hence is omitted.

\*\*\*\*\*