4. SOME PROPERTIES OF COMPACT FUZZY METRIC SPACES

4.1 Introduction

In this chapter a totally F-bounded fuzzy metric space is introduced. In Section 4.2 the properties of compatible fuzzy metric are discussed. The main result of this chapter, a necessary and sufficient condition for a complete and totally F-bounded fuzzy metric space to be a compact fuzzy metric space, is proved in Section 4.3 (Theorem 4.3.2). In Section 4.4 some equivalent cases involving totally F-bounded fuzzy metric space, precompact fuzzy metric space and compatible fuzzy metric are dealt with. Part of this chapter is published in [55].

4.2 Some Properties of Compatible Fuzzy Metric

A classical result in the theory of metrizable topological spaces is the celebrated Kelly metrization lemma [31] which is stated as follows.

**Lemma 4.2.1:** A $T_1$ topological space $(X, \tau)$ is metrizable if and only if it admits a compatible uniformity with a countable base.
Lemma 4.2.2 [14]: Let \((X, M, \ast)\) be a fuzzy metric space. Then \(\tau_M\) is a Hausdorff topology and for each \(x \in X\), \(\{B(x, 1/n, 1/n) : n \in \mathbb{N}\}\) is a neighborhood base at \(x\) for the topology \(\tau_M\).

Theorem 4.2.1[20]: Let \((X, M, \ast)\) be a fuzzy metric space. Then \((X, \tau_M)\) is a metrizable topological space.

Proof: For each \(n \in \mathbb{N}\) define

\[ U_n = \{(x, y) \in X \times X : M(x, y, 1/n) > 1 - (1/n)\} \]

Then \(\{ U_n : n \in \mathbb{N}\}\) is a base for a uniformity \(U\) on \(X\) whose induced topology coincides with \(\tau_M\): Note that for each \(n \in \mathbb{N}\),

\[ \{(x, y) : x \in X \subseteq U_n, U_{n+1} \subseteq U_n \text{ and } U_n = U_n^{-1}\}. \]

On the other hand, for each \(n \in \mathbb{N}\), there is, by the continuity of \(\ast\), an \(m \in \mathbb{N}\) such that \(m > 2n\) and \((1 - (1/m)) \ast (1 - (1/m)) > 1 - (1/n)\).

Then, \(U_m \circ U_m \subseteq U_n\): Indeed, let \((x, y) \in U_m\) and \((y, z) \in U_m\). Since \(M(x, y.)\) is nondecreasing [19],

\[ M(x, z, 1/n) \geq M(x, z, 2/m). \]

So \(M(x, z, 1/n) \geq M(x, y, 1/m) \ast M(y, z, 1/m) \geq 1 - (1/m) \ast (1 - (1/m)) > 1 - (1/n)\).

Therefore \((x, z) \in U_n\). Thus \(\{U_n : n \in \mathbb{N}\}\) is a base for a uniformity \(U\) on \(X\).
Since for each \( x \in X \) and each \( n \in \mathbb{N} \),
\[
U_n(x) = \{ y \in X : M(x, y, 1/n) > 1 - (1/n) = B(x, 1/n, 1/n) \},
\] and from Lemma 4.2.2, the topology induced by \( U \) coincides with \( \tau_M \). By Lemma 4.2.1, \((X, \tau_M)\) is a metrizable topological space.

**Corollary 4.2.1:** A topological space is metrizable if and only if it admits a compatible fuzzy metric.

**Theorem 4.2.2 [20]:** Let \((X, M, *)\) be a complete fuzzy metric space. Then \((X, \tau_M)\) is completely metrizable.

**Proof:** It follows from the Theorem 4.2.1 that \( \{ U_n : n \in \mathbb{N} \} \) is a base for a uniformity \( U \) on \( X \) compatible with \( \tau_M \), where \( U_n = \{ (x, y) \in X \times X : M(x, y, 1/n) > 1 - (1/n) \} \) for every \( n \in \mathbb{N} \). Then there exists a metric \( d \) on \( X \) whose induced uniformity coincides with \( U \). Then the metric \( d \) is complete.

Let \((x_n)_{n \in \mathbb{N}}\) be a Cauchy sequence in \((X, d)\). To prove \((x_n)_{n \in \mathbb{N}}\) is a Cauchy sequence in \((X, M, *)\), fix \( r, t \) with \( 0 < r < 1 \) and \( t > 0 \). Choose a \( k \in \mathbb{N} \) such that \( 1/k \leq \min\{t, r\} \). Then, there is \( n_0 \in \mathbb{N} \) such that \((x_n, x_m) \in U_k \) for every \( n, m \geq n_0 \). Consequently, for each \( n, m \geq n_0 \),
\[
M(x_n, x_m, t) \geq M(x_n, x_m, 1/k) > 1 - (1/k) \geq 1 - r.
\]
(1-s) * (1-s) > 1-r. On the other hand, there is \( n_0 \geq k(n_0) \) such that for each
\( n, m \geq n_1, M(x_n, x_m, t/2) > 1-s \). Therefore, for each \( n \geq n_1 \), we have

\[
M(x, x_n, t) \geq M(x, x_{k(n)}, t/2) \cdot M(x_{k(n)}, x_n, t/2)
\]

\[
\geq (1-s) * (1-s) > 1-r.
\]

Therefore the Cauchy sequence \( (x_n)_{n \in \mathbb{N}} \) converges to \( x \).

Now we will define the totally F-bounded fuzzy metric space as follows.

**Definition 4.3.1 [55]:** A fuzzy metric space \((X, M, \ast)\) is said to be totally F-bounded if there exists a finite covering of open balls \( B(x, r, t) \) for every \( r, 0 < r < 1 \).

From the definitions of totally F-bounded spaces and precompact spaces one can easily verify that precompactness implies totally F-boundedness.
Therefore \((x_n)_{n \in \mathbb{N}}\) is a Cauchy sequence in the complete fuzzy metric space \((X, \tau_M, *)\), so it is convergent with respect to \(\tau_M\). Hence, \(d\) is complete metric on \(X\). Therefore \((X, \tau_M)\) is completely metrizable.

**Corollary 4.2.2:** A topological space is completely metrizable if and only if it admits a compatible complete fuzzy metric.

Since every completely metrizable space is a Baire Space [10], from Theorem 4.2.2 we have the following corollary

**Corollary 4.2.3:** Every complete fuzzy metric space is a Baire space.

### 4.3 Totally F-Bounded Spaces

**Lemma 4.3.1** [20]: Let \((X, M, *)\) be a fuzzy metric space. If a Cauchy sequence clusters to a point \(x \in X\), then the sequence converges to \(x\).

**Proof:** Let \((x_n)_{n \in \mathbb{N}}\) be a Cauchy sequence in \((X, M, *)\) having a cluster point \(x \in X\). Then there is a subsequence \((x_{k(n)})_{n \in \mathbb{N}}\) of \((x_n)_{n \in \mathbb{N}}\) that converges to \(x\) with respect to \(\tau_m\). Thus given \(r\), with \(0 < r < 1\), and \(t > 0\) there is an \(n_0 \in \mathbb{N}\) such that for each \(n \geq n_0\), \(M(x, x_{k(n)}, t/2) > 1 - s\), where \(s > 0\) satisfies
In the following, we prove an important theorem, which is useful in the proof of the main theorem of this chapter.

**Theorem 4.3.1:** A fuzzy metric space is totally F-bounded if and only if every sequence in F-bounded set has a Cauchy subsequence.

**Proof:** Let $(X, M, *)$ be a totally F-bounded fuzzy metric space. Consider the sequence $\{x_n\}$ in $X$. Since $X$ is totally F-bounded we can find F-bounded sets $A_1, A_2, A_3, \ldots, A_n$ such that $X = \bigcup_{x \in A_i} B(x, r, t)$ for $0 < r < 1$.

Consider the F-bounded set $A_1$ containing the subsequence $x_{n_1}$ of $x_n$.

If $x_1 \in A_1$ then the subsequence $x_{n_1} \in B(x_1, 1, 1)$. We can find another F-bounded set $B_2$ in $A_1$ containing the point $x_2$ and the subsequence $x_{n_2}$ of $x_{n_1}$ such that $x_{n_2} \in B(x_2, 1/2, 1/2)$. Similarly we can find F-bounded set $B_m$ containing the point $x_m$ and the subsequence $x_{n_m}$ such that $x_{n_m} \in B(x_m, 1/m, 1/m)$. Now consider the subsequences $x_k$ and $x_s$ of $(x_{n_m})$. Given $r$, with $0 < r < 1$, and $t > 0$, there is an $n_0 \in \mathbb{N}$.
such that \[ 1 - \frac{1}{n_0} \geq 1 - \frac{1}{n_0} > 1 - r \text{ and } \left( \frac{2}{n_0} \right) < t. \] Then for every \( k, s \geq n_0 \) we have

\[ M(x_k, x_s, t) \geq M(x_h, x_s, 2/n_0) \]

\[ \geq M(v_k, a_{n_0}, 1/n_0) \cdot M(a_{n_0}, x, 1/n_0) \]

\[ \geq \left( 1 - \frac{1}{n_0} \right) \left( 1 - \frac{1}{n_0} \right) > 1 - r. \]

Hence \( x_{nm} \) is a Cauchy sequence in \( (X, M, *) \).

Conversely let every sequence has a Cauchy subsequence. Consider that \( (X, M, *) \) is not totally \( F \)-bounded. Therefore no open covering for \( X \) is existing.

Therefore we have \( X \neq \bigcup_{x \in A_i} B(x, r, t) \) where \( A_i \)'s are \( F \)-bounded sets.

For \( x_1 \in A_1 \), we can find \( x_2 \in A_2 = X \setminus A_1, x_3 \in X \setminus (A_1 \cup A_2), \ldots \) such that

\[ x_{n+1} \in \bigcup_{k=1}^n B(x, r, t). \]

Therefore the sequence \( (x_n)_{n \in \mathbb{N}} \) has no Cauchy subsequence which contradict our assumption. Therefore \( (X, M, *) \) is totally \( F \)-bounded.

The following lemma is proved for precompact spaces in [20] as follows.
Lemma 4.3.2: A fuzzy metric space is precompact if and only if every sequence has a Cauchy subsequence.

Note: In Theorem 4.3.1 only F-bounded sets are considered for totally F-bounded sets and in Lemma 4.3.2 precompact deals with all sets.

Now we prove the main theorem of this chapter.

Theorem 4.3.2: A fuzzy metric space is compact if and only if it is complete and totally F-bounded.

Proof: Let \((X, M, \ast)\) be a compact fuzzy metric space. Every Cauchy sequence \((x_n)_{n \in \mathbb{N}}\) in \((X, M, \ast)\) has a cluster point \(y \in X\). By Lemma 4.3.1, the sequence \((x_n)_{n \in \mathbb{N}}\) converges to a point \(y\). Therefore \((X, M, \ast)\) is complete. \((X, M, \ast)\) compact implies every open covering has a finite subcovering, therefore covering of \((X, M, \ast)\) by all open balls \(B(x, r, t)\) must contain a finite subcovering. Therefore \((X, M, \ast)\) is totally F-bounded.

Conversely, let \((x_n)_{n \in \mathbb{N}}\) be a sequence in \(X\). By Theorem 4.3.1 and the completeness of \((X, M, \ast)\), it follows that \((x_n)_{n \in \mathbb{N}}\) has a cluster point. Since \((X, \tau)\) is metrizable and every sequentially compact metrizable space is compact, we have \((X, M, \ast)\) is compact.
By using the Definition (1.17) the following necessary and sufficient condition for a fuzzy metric space to be complete is given like this in [15].

**Theorem 4.3.3:** A necessary and sufficient condition that a fuzzy metric space \((X, M, *)\) be complete is that every nested sequence of nonempty closed sets \(\{F_n\}_{n=1}^{\infty}\) with fuzzy diameter zero have nonempty intersection.

**Proof:** First suppose that the given condition is satisfied. Let \(\{x_n\}\) be a Cauchy sequence in \(X\). Take \(A_n = \{x_n, x_{n+1}, x_{n+2}, \ldots\}\) and \(F_n = \overline{A_n}\), then \(\{F_n\}\) has fuzzy diameter zero. For, given \(s, t > 0, 0 < s < 1\), we can find a \(r \in (0, 1)\), such that \((1-r)^* (1-r)^* (1-r) > (1-s)\). Since \(\{x_n\}\) is a Cauchy sequence, for \(r, t > 0, 0 < r < 1\), there exits \(n_0 \in \mathbb{N}\) such that \(M(x_m, x_n, t/3) > 1-r\) for all \(m, n \geq n_0\). Therefore \(M(x, y, t/3) > 1-r\) for all \(x, y \in A_{n_0}\).

Let \(x, y \in F_{n_0}\). Then there exists sequence \(\{x_n'\}\) and \(\{y_n'\}\) in \(A_{n_0}\) such that \(x_n'\) converges to \(y\). Hence \(x_n' \in B(x, r, t/3)\) and \(y_n' \in B(y, r, t/3)\) for sufficiently large \(n\). Now
\[ M(x, y, t) \geq M(x, x_n^*, t/3) \ast M(y_n^*, y, t/3) \ast M(y, y_n^*, t/3) \]

\[ > (1-r) \ast (1-r) \ast (1-r) > (1-s). \]

Therefore, \( M(x, y, t) > (1-s) \) for all \( x, y \in F_{n_0} \). Thus \( \{ F_n \} \) has fuzzy diameter zero. Hence by hypothesis \( \bigcap_{n=1}^{\infty} F_n \) is nonempty.

Take \( x \in \bigcap_{n=1}^{\infty} F_n \). Then for \( r, t > 0 \), \( 0 < r < 1 \), there exits \( n_1 \) such that

\[ M(x_n, x, t) > (1-r) \text{ for all } n \geq n_1. \]

Therefore, for each \( t > 0 \) \( M(x_n, x, t) \) converges to 1 as \( n \) tends to \( \infty \). Hence \( x_n \) converges to \( x \). Therefore ( \( X, M^* \) ) is a complete fuzzy metric space.

Conversely, suppose that ( \( X, M^* \) ) is a complete fuzzy metric space and \( \{ F_n \}_{n=1}^{\infty} \) is a nested sequence of nonempty closed sets with fuzzy diameter zero. Let \( x_n \in F_n, n = 1, 2, 3, \ldots \). Since \( \{ F_n \} \) has fuzzy diameter zero, for \( r, t > 0, 0 < r < 1 \), there exits \( n_0 \in \mathbb{N} \) such that

\[ M(x, y, t) > 1-r \text{ for all } x, y \in F_{n_0}. \]

Therefore \( M(x_n, x_m, t/3) > 1-r \) for all \( n, m \geq n_0 \). Since \( x_n \in F_n \subset F_{n_0} \) and \( x_m \in F_m \subset F_{n_0} \), \( \{ x_n \} \) is a Cauchy sequence. But ( \( X, M^* \) ) is a complete fuzzy metric space and hence \( x_n \) converges to \( x \) for some \( x \in X \). Now for each fixed \( n \), \( x_k \in F_n \) for \( k \geq n \).
Therefore, \( x \in \overline{F_n} = F_n \) for every \( n \), and hence \( x \in \bigcap_{n=1}^{\infty} F_n \). This completes the proof.

**Theorem 4.3.2:** A metrizable topological space is compact if and only if every compatible fuzzy metric is complete.

**Proof:** Suppose that \((X,\tau)\) is a compact metrizable space. By Theorem 4.3.2, every compatible fuzzy metric is complete.

Conversely, let \( d \) be any fuzzy metric \( M_d \), induced by \( d \). By hypothesis, \( M_d \) is complete. So by [15], \( d \) is complete. Applying the Niemytzki-Tychonoff theorem [10], \((X,\tau)\) is compact.

### 4.4 Some Equivalent Cases in Fuzzy Metric Spaces

**Theorem 4.4.1:** A metrizable topological space is compact if and only if every compatible fuzzy metric is totally \( F \)-bounded.

**Proof:** Suppose that \((X,\tau)\) is a compact metrizable space by Theorem 4.3.2, every compatible fuzzy metric is totally \( F \)-bounded.

Conversely, let \( d \) be any metric on \( X \) compatible with \( \tau \). Consider the fuzzy metric \( M_d \), induced by \( d \). By hypothesis, \( M_d \) is totally