Chapter 1

Introduction

Let $F$ be a nonarchimedean local field of characteristic not two and $K$ a separable quadratic extension. Then if $K = F(x_0)$ with $x_0$ an element of $K^*$ whose trace to $F$ is 0 we have an embedding of $K^*$ into $\text{GL}(2, F)$ given by

$$a + b x_0 \mapsto \begin{bmatrix} a & bx_0^2 \\ b & a \end{bmatrix}.$$  

It was Tunnell [Tu] who first gave an answer to the following question: if $\pi$ is an irreducible, admissible representation of $\text{GL}(2, F)$ what are the characters $\chi$ of $K^*$ that occur in the restriction of $\pi$ to $K^*$? It is immediate that if a character $\chi$ occurs in such a restriction then $\chi|_{F^*}$ must be the central character of $\pi$. Assuming that the residue characteristic is odd he showed that the multiplicity of such a character in $\pi|_{K^*}$ must be $\frac{\epsilon(\Pi \otimes \chi^{-1}, \psi_0) + 1}{2}$, where $\Pi$ is a base change lift of $\pi$ to $K$ and $\psi_0$ is nontrivial additive character of $K$ whose restriction to $F$ is trivial. Tunnell also showed that under the given circumstances the value of the epsilon factor must be $\pm 1$. Hence the multiplicity is either 0 or 1. Tunnell did this, case by case, by computing ‘characters’; H. Saito [S] gave a residue characteristic free proof of the same result. D. Prasad [P1] considered a slightly more general problem. He looked at rep-
resentations $\pi$ of $\GL(2,F)$ whose restriction to $\GL(2,F)^+$ (here $\GL(2,F)^+$ denotes the subgroup of index 2 in $\GL(2,F)$ consisting of those matrices whose determinant is in $N_{K/F}(K^*)$ where $N_{K/F}$ is the usual norm map from $K$ to $F$) breaks up as a sum of two irreducible representations $\pi_+ + \pi_-$ (this excluded, for instance, the exceptional representations whose restriction to $\GL(2,F)^+$ remain irreducible). If $\pi$ is supercuspidal then $\pi = r_\theta$, the Weil representation of $\GL(2,F)$ attached to a character $\theta$ of $K^*$ whose restriction to $F^*$ does not factor through the norm map from $K$ to $F$. Using a lemma of Langlands [L], Prasad showed that $\chi$ occurs in $r_{\theta,+}$ if and only if $\epsilon(\theta \chi^{-1}, \psi_0) = \epsilon(\bar{\theta} \chi^{-1}, \psi_0) = 1$ and in $r_{\theta,-}$ if and only if both the epsilon factors are $-1$ ($\bar{\theta}$ is the Galois conjugate of $\theta$). His proof was based on the following identity which is in fact equivalent to his extension theorem:

$$
\epsilon(\omega, \psi) \frac{\omega\left(\frac{x-x_0}{x_0-x_0}\right)}{|\frac{(x-x_0)^2}{x^2}|_{F^*}} = \sum_{\chi \in S} \chi(x)
$$

(1.1)

where $S$ is the set of characters $\chi$ of $K^*$ whose restriction to $F^*$ is $\omega_{K/F}$ and such that $\epsilon(\chi, \psi_0) = 1$. Prasad’s [P1] proof of the identity (1.1) was valid only when the residue characteristic is odd. The residue characteristic even case was open till Prasad [P2] provided a proof. Using a theorem of Waldspurger [W], his own local identity in the odd residue characteristic case, and a local-global technique he was able to conclude that the identity is also true for even residue characteristic (ch $F=0$). Concurrently, Saito [P2] gave a purely local proof of the above identity which was residue characteristic free. He defined an involutive intertwining operator $T : \pi(1_{F^*}, \omega_{K/F}) \longrightarrow \pi(1_{F^*}, \omega_{K/F})$ which commuted with the $\GL(2,F)^+$ action. The eigenspaces corresponding to the eigenvalues 1 and $-1$ turned out to be the spaces for $\pi_+$ and $\pi_-$ and Saito defined explicit eigenfunctions in each space and proved that for such eigenfunctions $f$ we have $Tf = \epsilon(\chi, \psi_0)f$. His proof involves
integral operators and their convergence was the only difficulty in the proof.

We give a local proof of the extension theorem here which depends on the definition of the epsilon factor and nothing else. The proof is transparent and, we hope, better explains the difference between the odd and even residue characteristic cases. It spans the third and fourth chapters of this thesis. We show that in considering \( \sum_{\chi \in S} \chi(x) \) enough cancelations take place to yield the left hand side of the identity. In fact, the right hand side after cancelations reduces to a sum over the characters of a single conductor. Note that the main identity (1.1) does not make sense if \( \text{ch} F = 2 \) for then \( x_0 \in F \).

In chapter 3 we complete the proof of the identity (1.1). We explain how this identity could be used together with a lemma of Langlands ([L], 7.19 the lemma with an embarrassing proof) to prove the extension of Tunnell’s theorem. We state the extension theorem precisely:

**Theorem 1.0.1.** Let \( r_\theta \) be an irreducible admissible representation of \( GL(2, F) \) associated to a regular character \( \theta \) of \( K^* \). Fix embeddings of \( K^* \) in \( GL(2, F) \) and in \( D^{*+}_F \) (there are two conjugacy classes of such embeddings in general), and choose a nontrivial additive character \( \psi \) of \( F \), and an element \( x_0 \) of \( K^* \) with \( \text{tr}(x_0) = 0 \). Then the representation \( r_\theta \) of \( GL(2, F) \) decomposes as \( r_\theta = r_{\theta^+} \oplus r_{\theta^-} \) when restricted to \( GL(2, F) \) and the representation \( r'_\theta \) of \( D^*_F \) decomposes as \( r'_\theta = r'_{\theta^+} \oplus r'_{\theta^-} \) when restricted to \( D^{*+}_F \), such that for a character \( \chi \) of \( K^* \) with \( (\chi^{-1}\theta)|_{F^*} = \omega_{K/F} \), \( \chi \) appears in \( r_{\theta^+} \) if and only if \( \epsilon(\theta^{-1}\chi, \psi_0) = \epsilon(\theta^{-1}\chi^{-1}, \psi_0) = 1 \), \( \chi \) appears in \( r_{\theta^-} \) if and only if \( \epsilon(\theta^{-1}\chi, \psi_0) = \epsilon(\theta^{-1}\chi^{-1}, \psi_0) = -1 \), \( \chi \) appears in \( r'_{\theta^+} \) if and only if \( \epsilon(\theta^{-1}\chi, \psi_0) = 1 \) and \( \epsilon(\theta^{-1}\chi^{-1}, \psi_0) = -1 \), and \( \chi \) appears in \( r'_{\theta^-} \) if and only if \( \epsilon(\theta^{-1}\chi, \psi_0) = -1 \) and \( \epsilon(\theta^{-1}\chi^{-1}, \psi_0) = 1 \).

In the second part of this thesis we use the above theorem to look at the following problem:
1.1 Organization of the thesis

Let \( r_{\theta} \) be as above. We have, by definition of central character, \( \theta|_{F^*} = \omega_{r_{\theta}} \omega \) where \( \omega_{r_{\theta}} \) is the central character of \( r_{\theta} \) and \( \omega = \omega_{K/F} \). As we have already noted, since \( \theta|_{F^*} \neq \omega_{r_{\theta}} \) the character \( \theta \) cannot occur in \( r_{\theta}|_{K^*} \). A necessary condition for a character \( \lambda \) of \( K^* \) to occur in \( r_{\theta}|_{K^*} \) is that its restriction to \( F^* \) should be equal to the central character \( \omega_{r_{\theta}} \). The question we would like to ask at this point is whether \( \theta \) twisted by some character \( \lambda \) of \( K^* \) can occur in \( r_{\theta} \) where \( \lambda|_{F^*} = \omega \) since it satisfies the said necessary condition. That is, whether there exist some \( \lambda \) such that \( \lambda \theta \) occurs in \( r_{\theta}|_{K^*} \). We will prove some results which give an affirmative answer to this question. In fact, we will try to count at each conductor level precisely how many characters are there occurring in \( r_{\theta,+} \) and in \( r_{\theta,-} \).

1.1 Organization of the thesis

We explain the basic terminologies that we use in this work in the second chapter. We also state some known theorems and facts that we frequently use. Our basic references on these will be [Se], [Ne], [Bu], [Ta], [J-L], and [L]. Those terms we have not defined anywhere but used somewhere in this thesis could be found in one of these references. In the third chapter, we lay down the foundation required for proving identity (1.1). Most of the results proved in chapter three will be used in the final chapter also. After proving these basic results, we will turn in chapter four towards proving the identity and then the extension theorem. We in fact imitate [P1] in proving the extension theorem after proving the main identity using our own techniques. We can do this this because the lemma of Langlands used in proving the extension theorem in [P1] is residue characteristic free and so true in all residue characteristics.
1.2 Main results

In chapter five, which is the last chapter of this work, we count the twists of $\theta$ that occur in the restriction of the Weil representation $r_\theta$ attached to $\theta$, a regular character. We make use of the the above cited theorem for this counting process. We do use an interesting result given at the end of chapter 4 heavily for our computations. The last chapter is divided into two parts, when the extension $K$ is unramified over $F$, and when it is ramified. Unramified case is quite simple where as the ramified situation consists of some what lengthy, but straight forward computations.

We collect our main results of this thesis for quick reference in the next section to conclude this introductory chapter.

1.2 Main results

The main theorem in the first part of this thesis (that is, chapters 3 and 4) is the following. This is in fact equivalent to the extension of Tunnell’s theorem. All most all the notations used here are explained in chapter 2.

**Theorem 1.2.1.** Let $K$ be a separable quadratic extension of a local field $F$ of characteristic not two. Let $\psi$ be a nontrivial additive character of $F$, and $x_0 \in K^*$ such that $\text{tr}(x_0) = 0$. Define an additive character $\psi_0$ of $K$ by $\psi_0(x) = \psi(\text{tr}[-xx_0/2])$. Then

$$\epsilon(\omega, \chi) \frac{\omega(\frac{x-x_0}{x_0})}{\frac{(x-x_0)^2}{2x}} \left| \frac{x}{x_0} \right|_{F^*} \sum_{\chi \in S} \chi(x)$$

$x \in K^* - F^*$ where as is usual, the summation on the right is by partial sums over all characters of $K^*$ of conductor $\leq n$.

The following lemma is simple, but plays a pivotal role in our computations. Note that the summation is over $\chi$’s of a fixed conductor, not over all
1.2 Main results

\(\chi\)'s such that \(\chi|_{F^*} = \omega\).

**Lemma 1.2.2.**

\[
\sum_{\chi \in S(2f) \cup S'(2f)} \chi(y) = 0 \text{ if } y \notin F^* U_{K'}^{2f-1}.
\]

**Theorem 1.2.3.** \(|S(l)| = |S'(l)|\) for each feasible \(l\), that is when \(l = 2d - 1\) or \(l = 2f\) with \(f \geq d\).

The above theorem states that with a given conductor \(l\), out of all \(\chi \in \hat{K}^*\) with restriction to \(F^*\) equal to \(\omega\), half will have epsilon factor +1 and the remaining will have epsilon factor -1.

**Theorem 1.2.4.** Let \(x = 1 + \pi^{-1} \pi_K x'\) where \(x' \in U_F\), then

\[
\sum_{\chi \in S(2r+2m)} \chi(x) = \begin{cases} 
-q^{-1} & \text{if } m = 0 \\
0 & \text{if } m = 1, 2, \ldots \text{ and } m \neq d - 1 \\
\epsilon(\omega, \psi) \frac{\omega(\frac{\pi - r}{\pi - q})}{|x|^2}_{F^*} & \text{if } m = d - 1 
\end{cases}
\]

and

\[
\sum_{\chi' \in S'(2r+2m)} \chi'(x) = \begin{cases} 
-q^{-1} & \text{if } m = 0 \\
0 & \text{if } m = 1, 2, \ldots \text{ and } m \neq d - 1 \\
-\epsilon(\omega, \psi) \frac{\omega(\frac{\pi - r}{\pi - q})}{|x|^2}_{F^*} & \text{if } m = d - 1 
\end{cases}
\]

The above is in fact stronger than theorem (1.2.1). In chapter 5, we have the following results:

If \(K/F\) ramified, then

**Lemma 1.2.5.** If \(\theta = (-1)^{\nu_K}\) then no \(\tilde{\omega} \theta\) can occur in \(r_\theta\) where \(\tilde{\omega} \in S_{2d-1}\).

**Theorem 1.2.6.** Let \(0 \neq a(\frac{\theta}{\tilde{\theta}}) < a(\tilde{\theta})\). Then among all \(\tilde{\omega} \in S(2d - 1)\) half and only half will be such that \(\tilde{\omega} \theta\) occur in \(r^+_\theta\) and among all \(\tilde{\omega} \in S'(2d - 1)\) half and only half will be such that \(\tilde{\omega} \theta\) occur in \(r^-_\theta\).
Theorem 1.2.7. Let $\lambda \in S(2f + 2d), f \geq 0$ \ $a(\frac{\theta}{q}) \leq a(\lambda) - 2d = 2f$. Then all the elements in $\{\lambda \theta : \lambda \in S(2f + 2d)\}$ will occur in $r^+$, Similarly if $\lambda' \in S'(2f + 2d)$, then all the elements in $\{\lambda' \theta : \lambda' \in S'(2f + 2d)\}$ will occur in $r^-$. Therefore the number of $\lambda \theta$ where $\lambda \in S_{2f+2d}$ occurring in $r_\theta$ is $|S_{2f+2d}|$.

Corollary 1.2.8. If $\frac{\theta}{q} = (-1)^{\kappa}$, then all $\lambda \in S(2f + 2d)$ will be such that $\lambda \theta$ will occur in $r^+$. Similarly all $\lambda' \in S'(2f + 2d)$ will be such that $\lambda' \theta$ will occur in $r^-$. 

Theorem 1.2.9. Let $\lambda \in S(2f + 2d), 2f + 2 < a(\frac{\theta}{q}) < a(\lambda)$. Then among all $\lambda \theta$ where $\lambda \in S(2f + 2d)$ exactly half will occur in $r^+$. Similarly, let $\lambda' \in S'(2f + 2d), 2f + 2 < a(\frac{\theta}{q}) < a(\lambda')$. Then among all $\lambda' \theta$ where $\lambda' \in S'(2f + 2d)$ exactly half will occur in $r^-$. Therefore the number of $\lambda \theta$ where $\lambda \in S_{2f+2d}$ occurring in $r_\theta$ is $|S_{2f+2d}|/2$.

Theorem 1.2.10. Let $\lambda \in S(2f + 2d), a(\frac{\theta}{q}) = 2f + 2 < a(\lambda)$. Then number of $\lambda \theta$ appearing in $r^+$ where $\lambda \in S(2f + 2d)$ is $\frac{q-2}{2}q^{f+d-1}$. Similarly, let $\lambda' \in S'(2f + 2d), a(\frac{\theta}{q}) = 2f + 2 < a(\lambda')$. Then number of $\lambda' \theta$ appearing in $r^-$ where $\lambda' \in S'(2f + 2d)$ is $\frac{q-2}{2}q^{f+d-1}$. The number of $\lambda \theta$ where $\lambda \in S_{2f+2d}$ occurring in $r_\theta$ is therefore $(q-2)q^{f+d-1}$.

Theorem 1.2.11. Let $a(\lambda) = 2f + 2d < a(\frac{\theta}{q}) = 2m < a(\lambda) + 2d$. Then the number of $\lambda \theta$ with $\lambda \in S_{2f+2d}$ appearing in $r_\theta$ is $|S(2f + 2d)| = |S_{2f+2d}|/2$.

When $a(\lambda)$ is too small compared to $a(\frac{\theta}{q})$ the occurrence of $\lambda \theta$ in $r^+$ or $r^-$ depends only on $\theta$.

Theorem 1.2.12. Suppose $\lambda \in S(2m), m \geq d$ and $a(\frac{\theta}{q}) = 2n \geq a(\lambda) + 2d$. Then either all the elements in $\{\lambda \theta : \lambda \in S(2m)\}$ will occur in $r^+$ or all the elements in $\{\lambda' \theta : \lambda' \in S'(2m)\}$ will occur in $r^-$ not both. Therefore the number of $\lambda \theta$ where $\lambda \in S_{2m}$ occurring in $r_\theta$ is $|S_{2m}|/2$. 


1.2 Main results

The only one theorem in which we didn’t give an exact count, but only a lower bound is the following:

**Theorem 1.2.13.** If $a(\frac{\theta}{F}) = a(\lambda) = 2f + 2d$, $\lambda|_{F^*} = \omega$ then the number of $\lambda \theta$ appearing in $r_\theta$ is greater than or equal to $q^{f+d-1}$.

And we conclude the thesis in the fifth chapter by describing the possibilities of occurrence of twists when $K/F$ is unramified.