Chapter 1

INTRODUCTION

1.1 Impulsive Differential Equations

Differential equations arise in many areas of science and technology, specifically whenever a deterministic relation involving some continuously varying quantities and their rates of change in space and/or time are known or postulated. This is illustrated in classical mechanics where the motion of a body is described by its position and velocity as the time varies. Newton’s laws allow one (given the position, velocity, acceleration and various forces acting on the body) to express these variables dynamically as a differential equation for the unknown position of the body as a function of time. An initial value or initial boundary value problem for an evolution partial differential equation can usually be written as an abstract differential equation

\[ x'(t) = f(t, x(t)) \] (1.1.1)

in a suitable function space, the function \( f \) describing the action of the equation on the space variables with boundary conditions in the definition of the space or of the domain of \( f \). The similarity of (1.1.1) with a true ordinary differential equation is only formal (\( f \) may not be everywhere defined, bounded or continuous) but gives heuristic insight into the problem, suggests ways to extend results from ordinary to partial differential equations and stresses unification leading to discovery of common threads and economy of thought. The “abstract” approach is the best in all situations, many
of the techniques are oblivious to the type of equation and are at least formally similar to classical procedures for systems of ordinary differential equations. These ordinary differential equations are sometimes inadequate as models of certain physical processes and therefore in such cases one or more generalizations of (1.1.1) is necessary.

Various evolutionary processes from fields as diverse as population dynamics, orbital transfer of satellites, sampled-data systems and engineering are characterized by the fact that they undergo abrupt changes is often negligible in comparison with that of the entire evolution. The duration of these changes is often negligible in comparison with that of the entire evolution process and thus the abrupt changes can be well-approximated in terms of instantaneous changes of state, i.e., impulses. These processes tend to be more suitably modeled by impulsive differential equations, which allow for discontinuities in the evolution of the state. Impulsive differential equations are usually defined by a pair of equations, an ordinary differential equation (1.1.1) to be satisfied during the continuous portion of evolution and a difference equation defining the discrete impulsive actions. The impulses occur when some spatio-temporal relation is satisfied. Impulsive differential equations and delay differential equations will be described more fully later.

Impulsive differential equations are usually defined as an ordinary differential equation coupled with a difference equation, although other formulations do exist. The difference equation is usually given by

\[ \Delta x(t) = I(t, x(t)) = x(t^+) - x(t^-), \]  

(1.1.2)

where \( x(t^+) = \lim_{s \to t^+} x(s) \) and \( x(t^-) = \lim_{s \to t^-} x(s) \) denote the right-hand and left-hand limits, respectively. Equation (1.1.2) is to be satisfied when some spatio-temporal relation \( h(t, x(t)) = 0 \) is satisfied. In general then, we are led to an impulsive differential equation having the form

\[ x'(t) = f(t, x(t)), \quad h(t, x(t)) \neq 0, \]  

(1.1.3)

\[ \Delta x(t) = I(t, x(t)), \quad h(t, x(t)) = 0. \]  

(1.1.4)

Thus for as long as we have \( h(t, x(t)) \neq 0 \), then the evolution of the state is governed by the ordinary differential equation (1.1.3). At such time that \( h(t, x(t)) = 0 \), then the
state undergoes an impulse and instantly changes by some amount $I(t, x(t))$ according to (1.1.4). This causes a jump discontinuity in the solution. Following this impulsive action, and assuming $h(t, x(t))$ is nonzero for some time thereafter, then the solution will continue to evolve according to (1.1.3) until it again undergoes an impulse.

The arbitrary nature of the relation $h(t, x(t)) = 0$ makes the study of (1.1.3)-(1.1.4) is extremely difficult. It is therefore common to focus on a particular type of relation. The set of points $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ for which $h(t, x) = 0$ will be assumed to consist of a sequence of hyper surfaces of the form $t = \tau_k(x)$ where $\tau_k \in C(\mathbb{R}^n, \mathbb{R}_+)$ for $k = 0, 1, \ldots$ and $0 = \tau_0(x) < \tau_1(x) < \tau_2(x) < \ldots$ and $\lim_{k \to \infty} \tau_k(x) = \infty$ for each $x \in \mathbb{R}^n$. The system is then written as

$$
\begin{align*}
  x'(t) &= f(t, x(t)), \quad t \neq \tau_k(x(t)), \\
  \Delta x(t) &= I(t, x(t)), \quad t = \tau_k(x(t)).
\end{align*}
$$

(1.1.5) (1.1.6)

When the functions $\tau_k$ depend on the state, then system (1.1.5)-(1.1.6) is said to have impulses at variable times. This is reflected in the fact that different solutions will tend to undergo impulses at different times. If the functions $\tau_k$ are all constant, then (1.1.5)-(1.1.6) is said to be a system having impulses at fixed times. In this case all solutions undergo the impulsive action (1.1.6) at the same times. In this thesis, we shall consider the system having impulses at fixed moments only.

The theory of impulsive differential equations is much richer than the corresponding theory of differential equations without impulse effects. For example, initial value problems of such equations may not, in general, possess any solutions at all even when the corresponding differential equation is smooth enough; fundamental properties such as continuous dependence relative to initial data may be violated, and qualitative properties like stability may need a suitable new interpretation. Moreover, a simple impulsive differential equation may exhibit several new phenomenon such as rhythmical beating, merging of solutions, and noncontinuability of solutions. The properties of the impulsive and the classical continuous models differ greatly, so deep investigations of the impulsive models are required. Therefore it is beneficial to study the theory of impulsive differential equations as a well deserved discipline, due to the increasing applications of impulsive differential equations in various fields.
1.2 History of Fractional Calculus

The fractional calculus is a generalization of the traditional calculus that leads to similar concepts and tools as standard differential calculus, but with a much wider applicability. Fractional calculus can be considered as a branch of mathematical physics which deals with integro-differential equations, having a long history and it has been intensively developed over the past three decades and plays a very important role in diverse fields such as mechanics, viscoelasticity, electrochemistry, biophysics and notable control theory and so on. The success of the fractional methodology is unquestionable with a lot of applications in nonlinear dynamics and complex system dynamics and image processing. This, in turn, led to the sustained interest in studying the theory of fractional differential equations. It is known that the integer derivative of a function is only related to its nearby points, while the fractional derivative has relationship with all of the function history information.

Fractional differential equation is a generalization of ordinary differential equations and integration to arbitrary non-integer orders. In fact, fractional differential equations is considered as model alternative to nonlinear differential equations [2, 4]. The theory of fractional differential equation has been extensively studied by many authors [13, 19, 37, 46, 57, 58, 104, 112, 168]. Many partial fractional differential equations and integro-differential equations can be expressed as fractional differential equation and integro-differential equations in suitable Banach spaces [138]. For basic facts about fractional integral and fractional derivatives one can refer the monographs of Kilbas et al. [95], Miller and Ross [119] and Podulbny [133].

Many authors have investigated the existence result for fractional evolution equations. Moreover, there are different types of mild solutions that have been proved. For example, the first one was constructed interms of a probability density function and Laplace transforms given by El. Borai [57, 58] and latter it was then developed by Y. Zhou et al. [168], and the author Bazhiekova [25] introduced an α-resolvent family under the Riemann-Liouville fractional derivative and some constraints and in [76], the authors proved by well developed theory of resolvent operators for integral equations and recently JinRong Wang et al. [154] initiated the new concept of $PC$-mild
solutions for impulsive Cauchy problems.

However, existence of solutions of fractional differential, integro-differential and controllability of fractional differential equations combined with the impulsive conditions have not yet been fully investigated by using the techniques which is introduced by [25, 57, 76, 154]. So, by using the new concept of mild solutions, we are interested to study the existence results for fractional differential and integro-differential equations with impulsive conditions which is the main theme of this thesis.

1.2.1 Basic Results

The generalization of the concepts of the derivatives and integrals to non-integer order has been subjected to several approaches and so various alternative definitions of fractional derivatives appeared. Here we recall only Riemann-Liouville fractional integral, Riemann-Liouville fractional derivative and Caputo fractional derivative in our discussion.

**Definition 1.2.1** [95, p.no 69](Riemann-Liouville fractional integral). The Riemann-Liouville fractional integral operator of order \( \alpha > 0 \) of a function \( f \in L^1(\mathbb{R}^+) \) is defined as

\[
I_{0+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,
\]

where \( \Gamma(\cdot) \) is the Euler gamma function. For integer \( \alpha > 0, \ t > 0 \), equation (1.2.1) is known as Cauchy’s integral formula.

**Definition 1.2.2** [95, p.no 70](Riemann-Liouville fractional derivative). The Riemann-Liouville fractional derivative of order \( \alpha > 0, \ n-1 < \alpha < n, \ n \in \mathbb{N} \), is defined as

\[
(\text{R-L}) D_{0+}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds,
\]

where the function \( f(t) \) has absolutely continuous derivative up to order \( (n-1) \).

**Definition 1.2.3** [95, p.no 90](Caputo fractional derivative). The Caputo derivative of order \( \alpha \) for a function \( f : [0, \infty) \to \mathbb{R} \) can be written as

\[
D^\alpha f(t) = D^\alpha \left( f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0, \quad n-1 < \alpha < n.
\]
Remark 1.2.1.

(i) If \( f(t) \in C^n[0, \infty) \), then

\[
C^{\alpha} D^a f(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t f^{(n)}(s) (t - s)^{\alpha+1-n} \, ds \\
= \Gamma^{n-\alpha} f^{(n)}(t), \quad t > 0, \quad n - 1 < \alpha < n.
\] (1.2.4)

(ii) The Caputo derivative of a constant is equal to zero.

(iii) If \( f \) is an abstract function with values in \( X \), then integrals which appear in Definition 1.2.1 and 1.2.2 are taken in Bochner’s sense, where \( f' (s) = D f (s) = \frac{df}{ds} \) and \( f \) is an abstract with values in \( X \).

We note that Caputo fractional derivative first computes an ordinary derivative followed by a fractional integral to achieve the desired order of fractional derivative whereas the Riemann-Liouville fractional derivative is computed in the reverse order.

We have the following relationship between Riemann-Liouville and Caputo fractional derivatives [145]:

\[
C^{\alpha} D_0^a f(t) = D_0^a f(t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k - \alpha + 1)} f^{(k+1)}(0^+).
\] (1.2.5)

We observe that, for zero initial conditions, the two derivatives are the same. Thus, because of this condition, we may switch between the two derivatives according to our necessity. Under natural conditions on the function \( f(t) \), for \( \alpha \rightarrow n \), the Caputo derivative becomes a conventional \( n \)th derivative of the function \( f(t) \).

Indeed, let us assume that \( 0 \leq n - 1 < \alpha < n \) and that the function \( f(t) \) has \( n + 1 \) continuous bounded derivatives in \([0, a]\) for every \( a > 0 \). Then

\[
\lim_{\alpha \rightarrow n} C^a_\alpha f(t) = \lim_{\alpha \rightarrow n} \left( \frac{f^{(n)}(0) t^{n-\alpha}}{\Gamma(n - \alpha + 1)} + \frac{1}{\Gamma(n - \alpha + 1)} \int_0^t (t - s)^{n-\alpha} f^{(n+1)}(s) ds \right),
\]

\[= f^{(n)}(0) + \int_0^t f^{(n+1)}(s) ds,
\]

\[= f^{(n)}(t), \quad n = 1, 2, \ldots
\]

The main advantage of Caputo’s approach is that the initial conditions for fractional differential equations with Caputo derivatives take the same form as for integer-order
differential equations, that is, contain the limit values of integer-order derivatives of unknown functions at the lower terminal \( t = 0 \).

The difference between the Riemann–Liouville and the Caputo derivative is that the Caputo derivative of a constant \( k \) is 0, that is, \( \mathcal{C}D^\alpha_t k = 0 \), whereas, the Riemann–Liouville fractional derivative of a constant \( k \) need not be equal to 0, that is,

\[
0D^\alpha_t k = \frac{kt^{-\alpha}}{\Gamma(1-\alpha)}.
\]

1.2.2 Special Functions

(a) Gamma function

One of the basic functions of fractional calculus is Euler’s Gamma function \( \Gamma(z) \), which generalizes the factorial \( n \) and allows \( n \) to take non-integer and even complex values.

**Definition 1.2.4.** [133, p.no 1]. The Gamma function \( \Gamma(z) \) is defined by the integral

\[
\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt,
\]

which converges in the right half of the complex plane \( \Re(z) > 0 \), \( z \in \mathbb{C} \). It has the property that \( \Gamma(z + 1) = z\Gamma(z) \). From this, we note that, for \( n \in \mathbb{N} \), \( \Gamma(n + 1) = n! \).

(b) Beta function

It is more convenient to use the \( \beta \) function instead of a certain combination of values of the gamma function.

**Definition 1.2.5.** [133, p.no.1]. The \( \beta \) function is defined by

\[
\beta(z, w) = \int_0^1 \tau^{z-1}(1 - \tau)^{w-1}d\tau, \quad \Re(z) > 0, \quad \Re(w) > 0, \quad z, w \in \mathbb{C}. \tag{1.2.7}
\]

The relationship between gamma function and beta function is given by

\[
\beta(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z + w)} \tag{1.2.8}
\]

from which it follows that \( \beta(z, w) = \beta(w, z) \).

(c) Mittag-Leffler Function

Besides the gamma function, Euler has brought to light an additional important function called the exponential function \( e^z \) which plays a very important role in the
theory of integer-order differential equations. For differential equations of non-integer order, we define Mittag-Leffler function by replacing the factorial by gamma function which reduces to exponential function when the order is an integer.

**Definition 1.2.6.** [133, p.no 16]. The function $E_\alpha(z)$ defined by

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \quad (z \in C, \ Re(\alpha) > 0) \quad (1.2.9)$$

was introduced by Mittag-Leffler and is, therefore, known as the Mittag-Leffler function.

The Mittag-Leffler function $E_{\alpha,\beta}(z)$, generalizing the one in (1.2.9), is defined by

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad (z, \beta \in C, \ Re(\alpha) > 0) \quad (1.2.10)$$

This function is sometimes called a Mittag-Leffler type function.

In particular, when $\alpha = 1$ and $\alpha = 2$, we have from (1.2.9)

$$E_1(z) = e^z, \quad E_2(z) = \cosh(\sqrt{z}).$$

When $\beta = 1$, $E_{\alpha,1}(z)$ coincides with the Mittag-Leffler function (1.2.9), namely

$$E_{1,1}(z) = E_{\alpha}(z) \quad (z \in C, \ Re(\alpha) > 0).$$

The particular cases of (1.2.10) are given by

$$E_{1,2}(z) = \frac{e^{z-1}}{z}, \quad E_{2,2}(z) = \frac{\sinh(\sqrt{z})}{\sqrt{(z)}}.$$

Functions of Mittag-Leffler type play a special role in fractional calculus. They often arise in the solution of specific problems when the methods of fractional calculus are applied. This is so because the fundamental solution of a fractional linear differential equation with constant coefficients is expressed in terms of functions of Mittag-Leffler type.

### 1.3 Motivation

The objective of this thesis is to study the existence of solutions of various types of impulsive differential, impulsive fractional and fractional integro-differential equa-
tions. We shall motivate our study by giving the occurrence of these types of equations in different fields of science and engineering.

1.3.1 Optimal Control Problem [131]

(i) In an optimal control problem given by a system

\[ x' = f(t, x, u) \]

representing certain physical process, the central problem is to select the function \( u(t) \) from a given set of controls so that the solution \( x(t) \) of the system has a preassigned behaviour on a given time interval \([t_0, T]\) so as to minimize some cost functional. Suppose that the control function \( u(t) \) has to be selected from the set of functions of bounded variation defined on \([t_0, T]\), then the solution \( x(t) \) of the control system may possess discontinuities. Hence the given control problem has to be represented by a differential equation involving impulses.

(ii) The profit of a roadside inn on some prescribed interval of time \( \tau < t < T \) is a function of the number of strangers who pass by on the road each day and of the number of times the inn is painted during the period. The ability to attract new customers into the inn depends on its appearance which is supposed to be indexed by a number \( x_1 \). During time intervals between paint jobs, \( x_1 \) decays according to the law:

\[ x'_1 = -kx_1, \quad k \in \mathbb{R}^+ \]

The total profit in the planning period \( \tau < t < T \) is supposed to be

\[ W(t) = A \int_{\tau}^{T} x_1(t)dt - \sum_{i=1}^{N(T)} C_i, \]

where \( N(T) \) is the number of times the inn is repainted, \( C_i, i = 1, 2, \ldots, T, \) are the cost of each paint job and \( A > 0 \) is a constant. The owner of the inn wishes to maximize his total profit or equivalently to minimize \(-W(T)\).
1.3.2 Extensible Beam Models [160]

(i) Consider the study of dynamical buckling of the hinged extensible beam which is either stretched or compressed by axial force. A mathematical model for this problem is the hyperbolic equation

\[ \frac{\partial^2 z}{\partial t^2} + \frac{\partial^4 z}{\partial x^4} + \left( \alpha + \beta \int_0^L \left| \frac{\partial z(\eta, s)}{\partial \eta} \right|^2 d\eta \right) \frac{\partial^2 z}{\partial x^2} + g \left( \frac{\partial z}{\partial t} \right) = 0, \quad (1.3.1) \]

where \( \alpha, \beta, L > 0 \), \( z(x,t) \) is the deflection of the point \( x \) of the beam at the time \( t \), \( g \) is a non-decreasing numerical function and \( L \) is the length of the beam. The nonlinear friction force \( g(\frac{\partial z}{\partial t}) \) is the dissipative term. When \( g = 0 \), this equation reduces to the equation introduced in [160] as a model for the transverse motion of an extensible beam whose ends are held a fixed distance apart.

Equation (1.3.1) has its analogue in \( \mathbb{R}^n \) and can be included in the general mathematical model

\[ z'' + A^2 z + M(\|A^\frac{1}{2}z\|_H^2)Az + g(z') = 0, \quad (1.3.2) \]

where \( A \) is a linear operator in a Hilbert space \( H \) and \( M \) and \( g \) are real functions. Equations (1.3.1) and (1.3.2) takes the abstract form as

\[
\begin{align*}
  z'' &= A z(t) + B z'(t) + f(t, z(t)), \\
  z(0) &= z_0, \quad z'(0) = z_1,
\end{align*}
\]

where \( A \) and \( B \) are linear operators.

(ii) The physical origin of the problem lies in the theory of vibrations of an extensible beam of length \( L \) whose ends are held a fixed distance apart, hinged or clamped and is either stretched or compressed by an axial force, taking into account the fact that, during vibration, the elements of a beam perform not only a translatory motion but also rotate [132]. A mathematical model for this problem is an initial boundary value problem for the nonlinear hyperbolic equation

\[ \frac{\partial^2 z}{\partial t^2} - \lambda \frac{\partial^4 z}{\partial t^2 \partial x^4} + \frac{\partial^4 z}{\partial x^4} - \left( \alpha + \beta \int_0^L \left| \frac{\partial z(\eta, s)}{\partial \eta} \right|^2 d\eta \right) \frac{\partial^2 z}{\partial x^2} = f \left( \frac{\partial z}{\partial t} \right), \]

where \( z(x,t) \) is the deflection of point \( x \) at time \( t \), \( \alpha \) is a real constant proportional to the axial force acting on the beam when it is constrained to lie along the \( x \) axis.
and $\lambda$ is a non-negative constant. The nonlinearity of the equation is due to the extensibility of the beam. The above equation can be written in abstract form with general nonlinear term as

$$(I - \lambda A)z'' + A^2 z - [\alpha + \beta M(\|A^\frac{1}{2} z\|^2)]Az = f(t, z(t), z'(t)).$$

If we take $K = (I - \lambda A)$ and $[\alpha + \beta M(\|A^\frac{1}{2} z\|^2)]A - A^2 = A$ with appropriate assumptions on $\alpha$, $\beta$, $M$, then it becomes

$$K z''(t) = Az(t) + f(t, z(t), z'(t)).$$

### 1.3.3 Population Growth Model [91]

#### (i) Volterra’s Model

This is the model for population of a species within a closed system characterized by a nonlinear fractional integro-differential equation [145] in the following form

$$\frac{d^\alpha u(t)}{dt^\alpha} = au(t) - bu^2(t) - cu(t) \int_0^t u(x)dx, \ 0 < \alpha < 1, \quad (1.3.3)$$

subject to the initial condition $u(0) = u_0$, where $u(t)$ is the scaled population of identical individuals, $t$ denotes the time, $\alpha$ is a constant describing the order of the time-fractional derivative, $a > 0$ is the birth rate coefficient, $b > 0$ is the crowding coefficient and $c > 0$ is the toxicity coefficient which denotes the essential behaviour of the population evolution before its level falls to zero in the long run and $\beta = \frac{c}{ab}$ is a non-dimensional parameter. The last term of equation (1.3.3) is a functional integral representing the total metabolism or total amount of toxins accumulated from time zero. The individual death rate is proportional to this integral and so the population death rate due to toxicity must include a factor $\alpha$. Numerical experiments for fractional models on population dynamics are examined in [110] and some of the applications of nonlinear fractional differential equations with their approximations are found in [84].

#### (ii) Lokta-Volterra Model

Applications of the theory of differential equations in mathematical ecology have developed rapidly. Various mathematical models have been proposed in the study
of population dynamics. One of the famous models for dynamics of population is the Lotka-Volterra model \([7, 91, 108]\). Birth of many species are not continuous, for example, in some wild animal populations; the birth is an annual birth pulse. To describe a system more accurately, we should use an impulsive fractional differential equation in the following form

\[
\begin{align*}
\frac{\partial^\alpha y_i(t, x)}{\partial t^\alpha} - \frac{\partial^\alpha y_i(t, x)}{\partial x^2} &= y_i(t, x) \left[ -d_i(t) - \sum_{j=1}^{n} a_{ij}(t)y_j(t, x) \right], \quad t \neq t_k, \\
\Delta y_i(t, x)|_{t=t_k} &= y(t_k^+, x) - y(t_k^-, x) = b_{ik}y_i(t_k, x), \quad i = 1, 2, \ldots, n, \ k = 1, 2, \ldots \\
y(0, x) &= y_0(x), \\
y(t, x)|_{\partial \Omega} &= 0,
\end{align*}
\]

where \(0 \leq \alpha \leq 1\) and the variables \(y_i(t, x)\) denote the population densities of the \(i\)th species \(r_i = b_i(t) - d_i(t)\) is the intrinsic growth rate with \(b_i(t)\) and \(d_i(t)\) being the respective birth and death rates; \(a_{ii}(t)\) are the rates of intra-specific competition; \(a_{ij}(t)\) \((i \neq j)\) are the rates of integer-specific competition; \(b_{ik}y_i(t_k, x)\) represents the population \(y_i(t, x)\) at \(t_k\) annual birth pulse. In this model, we assume that \(b_i(t), \ d_i(t), \ a_{ij}(t)\) are all assumed to be continuous positive functions with a common period \(\omega\) and \(\Omega\) is a fixed bounded domain in \(\mathbb{R}^n\) with smooth boundary \(\partial \Omega\). Further more, they are subjected to short-term external influence at fixed moments of time \(t_k\), where \(\{t_k\}, \ k = 1, 2, \ldots\) is a sequence of real numbers \(0 = t_0 < t_1 < t_2 < \ldots < t_k < \ldots\) with \(\lim_{k \to \infty} t_k = +\infty\).

### 1.3.4 Viscoelastic Model \([111]\)

Viscoelasticity seems to be the field of extensive applications of fractional differential and integral operators. The use of fractional derivatives for mathematical modelling of viscoelastic materials is quite natural. We will consider an example from the linear theory of viscoelasticity which involves fractional derivatives. For more examples see \([117, 133, 143]\). The simplest model of viscoelasticity combines Hooke’s elastic solid with Newton’s viscous fluid and it consists of an elastic element (spring) and a viscous element (damper/dashpot) which are joined in a parallel way. These elements form the primary element of a viscoelastic solid. In the more complicated models like
Kelvin, this primary element is joined with other elastic elements, one after another.

The stress $\sigma$ is proportional to the zeroeth-order derivative of strain $\epsilon$ for solids and to the first derivative of strain $\epsilon$ for fluids; it is natural to superpose, that, for "intermediate" materials, stress may be proportional to the stress derivative of "intermediate" (non-integer order):

$$\sigma(t) = \eta_0 D^\alpha_\epsilon(t) \quad (0 < \alpha < 1),$$

where $\eta$ and $\alpha$ are material dependent constants. This relation is got by replacing the linear dashpot by a fractional dashpot.

In the same way, the fractional-order Voigt and Maxwell models are given by equations

$$\frac{d^\alpha \epsilon(t)}{dt^\alpha} = \frac{1}{E} \frac{d^\alpha \sigma(t)}{dt^\alpha} + \frac{\sigma}{\eta},$$

$$\sigma(t) = E\epsilon(t) + \eta \frac{d^\alpha \epsilon(t)}{dt^\alpha}.$$

The three-parameter fractional-order generalized Voigt and Maxwell models are given respectively by

$$\sigma(t) = b_0 \epsilon(t) + b_1 D^\alpha \epsilon(t),$$

$$\sigma(t) = b_0 \epsilon(t) - a_1 D^\alpha \sigma(t).$$

The five-parameter fractional derivative model which is frequently used to describe the behaviour of polymeric materials is given by the stress-strain law

$$\sigma(t) + \beta \left( \frac{d}{dt} \right)^\mu \sigma(t) = \gamma \left( \epsilon(t) + \alpha \left( \frac{d}{dt} \right)^\gamma \epsilon(t) \right), \quad t \in \mathbb{R},$$

where $\alpha, \beta \geq 0$, $\gamma > 0$, $0 \leq \nu \leq \gamma \leq 1$.

Similarly, the most general linear model of viscoelasticity can be obtained by replacing the integer-order derivatives with fractional derivatives:

$$\sum_{k=0}^{n} a_k \frac{d^{\alpha k} \sigma(t)}{dt^{\alpha k}} = \sum_{k=0}^{n} b_k \frac{d^{\beta k} \epsilon(t)}{dt^{\beta k}},$$

and it is possible that the best results may be archived if $n = m$ and $\alpha k = \beta k (k = 0, 1, 2, ...)$.
1.4 Differential Inclusions

There are many problems in applied mathematics that lead us to the study of dynamical systems having velocities non uniquely determined by the state of the system but depending loosely upon it. In these cases the classical equation \( x' = F(t, x) \), describing the dynamics of the system is replaced by a differential inclusion of the form \( x' \in F(t, x) \), where \( F \) is a set valued map that associates to the state \( x \) of the system, the set of feasible velocities.

The motivation for the differential inclusions came from control theory. The dynamics of a particle is described by

\[
x' = f(t, x, u(t)) \tag{1.4.1}
\]

Here the functions \( u(\cdot) \) are controls (realizable by means of mechanical or electronic device) taking values in a bounded subset \( U \) of \( \mathbb{R}^m \). One wants to choose a \( u(\cdot) \) such that something becomes optimal for the corresponding trajectories \( x(\cdot) \) of equation (1.4.1). For example, control the flight of a rocket such that it reaches its goal with minimal amount of fuel or find \( u(\cdot) \) such that a particle \( x_0 \) outside \( B \subset \mathbb{R}^n \) comes to \( B \) in shortest time (along the corresponding trajectory determined by the equation (1.4.1). In many examples of this type the optimal \( u(\cdot) \) has a lot of jumps, so the has to work theoretically with measurable functions \( u(\cdot) \). One way (not the only one) to study the existence problem is to consider the differential inclusions \( x' \in F(t, x) \) with \( F(t, x) = f(t, x, U) = \{ f(t, x, u) : u \in U \} \); if we can solve this one then we have \( x'(t) = f(t, x(t), u(t)) \) and the only question is whether \( u(\cdot) \) is measurable and this formulation has the advantage that the control variables not appear explicitly. For further reading refer the monograph [26] and the papers of [9, 11, 12, 16, 50, 84].

1.5 Existence

When a system described by an ordinary differential equation is subjected to perturbations, the perturbed system is again an ordinary differential equation in which the perturbation function is assumed to be continuous or integrable, and as such, the
state of the system changes continuously with respect to time. However, in many physical problems (optimal control theory in particular), one cannot expect perturbations to be well behaved. Biological systems such as heart beats, blood flows, pulse frequency modulated systems and models for biological neural nets exhibit an impulsive behavior. Therefore, perturbations of impulsive type are more realistic. This gives rise to Measure Differential Equations. The derivative involved in these equations are the distributional derivatives. The fact that their solutions are discontinuous (they are functions of bounded variation), renders most of the classical methods ineffective, thereby making their study interesting.

Impulsive differential equations received scant attention until the 1960’s and it was not until the 1980’s that interest in them began to catch on among mathematicians. Among the earliest articles on impulsive differential equations was the seminal paper by Milman and Myshkis [119]. They considered differential equations with impulses occurring when certain spatio–temporal relations are satisfied. By making use of classical results of ordinary differential equations, they obtained the first result on stability of solutions of impulsive differential equations.

Based on this early work, later research into impulsive differential equations culminated in the publishing of several monographs by Samoilenko and Perestyuk [141], Lakshmikantham, Bainov and Simeonov [102], Bainov and Covachev [17] and Bainov and Dishliev [18] along with the more recent publications during the 1990’s.

These authors considered an impulsive differential equation to be an ordinary differential equation coupled with a difference equation to be satisfied at certain fixed or variable impulse times. The resulting solutions are thereby piecewise continuous with discontinuities occurring at these impulse times. This approach enabled them to apply many well-established results for ordinary differential equations to these systems in order to develop the qualitative theory of the impulsive differential equations. The same approach has been taken in this document with respect to the study of impulsive delay differential equations. Various types of differential and impulsive differential equations have been studied by several researchers see for instance [8–11, 34–37, 39–42, 44, 59, 61, 73, 74, 78–82, 87, 93, 113] using the above mentioned methods in the monographs [18, 26, 102, 131, 141, 164].
1.6 Methods

1.6.1 Semigroup Theory

The theory of semigroups of bounded linear operators is closely related to the solution of differential and integro-differential equations in Banach spaces [49, 65]. This theory has developed quite rapidly since the discovery of the generation theorem by Hille and Yosida in 1948. By now, it is an extensively studied mathematical subject with substantial applications to many fields of analysis. Pazy [130] discussed the existence and uniqueness of evolution equations using semigroup theory and fixed point theorems. By using the abstract approach, it is possible to extend the analysis developed in finite dimensional linear systems to infinite dimensional linear systems. In recent years, the method of semigroups has been applied to study the controllability problems for a large class of nonlinear differential and integro-differential evolution systems in Banach spaces. Many problems in the fields of ordinary and partial differential equations can be recast as integral equations. Several existence and uniqueness results can be derived from the corresponding results of integral equations. Such results can be obtained by applying the fixed point theorems. For instance, several researchers have extended the technique to infinite dimensional spaces and studied the existence and controllability problems for nonlinear evolution equations in abstract spaces.

1.6.2 Fixed Point Method

The fixed point method is the most effective one to study the existence and controllability of nonlinear integro-differential systems. Fixed point method is used to prove the existence theorems for integro-differential equations and study the controllability problem for integro-differential systems. Due to its importance, several researchers have studied the problems represented by evolution equations by using different kinds of fixed point theorems [67, 116]. In this method, the problem is transformed to a fixed point problem for an appropriate nonlinear operator in a function space. Moreover, by using the fixed point theorems, one can obtain existence and controllability results.
in the Banach spaces of continuous functions. We mainly employ the Leray–Schauder of the Kakutani maps, Martelli’s fixed point theorem, Leray-Schauder’s Alternative fixed point theorem, Banach contraction principle, Schafer’s fixed point theorem and Krasnoselskii’s fixed point theorem are used to investigate the existence results for various types of differential systems with impulsive conditions and delay.

1.6.3 Strongly Continuous Cosine Families

The theory of strongly continuous cosine families of bounded linear operators has applications in many branches of analysis and in particular to find the solution of initial and boundary value problems for second order partial differential equations. Strongly continuous cosine families of bounded linear operators are related to abstract linear second order differential equations in the same manner as the strongly continuous semigroups of bounded linear operators are related to abstract linear first order differential equations.

Roughly speaking, every second order differential equation of the form \( x'' = Ax \) which is well-posed in a certain sense give rise to a strongly continuous cosine family of bounded linear operators with infinitesimal generator \( A \) and conversely, every strongly continuous cosine family of bounded linear operators with infinitesimal generator \( A \) may be associated with a well-posed second order differential equation \( x'' = Ax \). The most fundamental and extensive work on cosine families is that of Travis and Webb [146, 147].

1.6.4 Measure of Noncompactness

Measure of noncompactness plays very important role in nonlinear analysis. They are often applied to the theories of differential and integral equations as well as the operator theory and geometry of Banach spaces. In the last few years there has appeared a lot of papers concerning the notion of the so called measure of noncompactness [24, 67, 139].

First Kuratowski introduced measure of noncompactness of a set \( X \) in the following
way

\[ \alpha(X) = \inf \{ d > 0 : X \text{ is represented as a finite sum of sets having diameter smaller than } d \} \]

and another important and very convenient measure is the so called Hausdorff measure of noncompactness (or ball measure). It is defined by the formula

\[ \beta(X) = \inf \{ r > 0 : X \text{ can be covered by a finite sum of balls of radii smaller than } r \} . \]

It is worthwhile mentioning that there exist a lot of definitions of measure of noncompactness [24, 139]. Some axiomatic definitions of measure of noncompactness are very general and not useful in applications. On the other hand, there are axiomatic definitions which are handy and convenient in applications.

A special role in the theory of measure of noncompactness is played by Kuratowski measure of noncompactness and Hausdorff (or ball) measure of noncompactness. Especially the Hausdorff measure is frequently used in many branches of nonlinear analysis and its applications. It is caused by the fact that it is defined in a natural way and has several very useful properties. Roughly speaking, the measure of noncompactness is in almost on the family of all bounded and nonempty subsets of a given metric space such that it is equal to zero on the whole family of relatively compact sets.

1.6.5 Laplace Transform Method

The Laplace transform will prove to be an indispensable tool, especially in our study of fractional differential equations. We briefly inaugurate our discussion of this powerful method in the present section.

(i) Basic Results on the Laplace Transform [133, p.no 108].

Let us recall some basic facts about the Laplace transform. The function \( F(s) \) of the complex variable is defined by

\[ F(s) = L\{f(t); s\} = \int_0^\infty e^{-st} f(t)dt \quad (1.4.2) \]
is called the Laplace transform of the function $f(t)$, which is called the original. For the existence of the integral (1.4.2), the function $f(t)$ must be of exponential order $\alpha$, which means that there exist positive constants $M$ and $T$ such that $e^{-\alpha t}|f(t)| \leq M$ for all $t > T$. In other words, the function $f(t)$ must not grow faster than a certain exponential function when $t \to \infty$.

We will denote the Laplace transforms by uppercase letters and the originals by lowercase letters.

The original $f(t)$ can be restored from the Laplace transform $F(s)$ with the help of the inverse Laplace transform

$$f(t) = L^{-1}\{F(s); t\} = \int_{c-i\infty}^{c+i\infty} e^{st}F(s)ds, \; c = \mathbb{R}e(s) > c_0,$$

where $c_0$ lied in the right half plane of the absolute convergence of the Laplace integral (1.4.2). The direct evaluation of the inverse Laplace transform using the formula (1.4.3) is often complicated. The Laplace transform of the convolution

$$f(t) * g(t) = \int_0^t f(t - \tau)g(\tau)d\tau = \int_0^t f(\tau)g(t - \tau)d\tau$$

(1.4.4)

of the two functions $f(t)$ and $g(t)$, which are equal to zero for $t < 0$, is equal to the product of the Laplace transform of those function:

$$L\{f(t) * g(t); s\} = F(s)G(s),$$

under the assumption that both $F(s)$ and $G(s)$ exist. We will use the property (1.4.4) for the evaluation of the Laplace transform of the Riemann-Liouville fractional integral. Another useful property which we need is the formula for the Laplace transform of the derivative of an integer order $n$ of the function $f(t)$:

$$L\{f^n(t); s\} = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1}f^{(k)}(0) = s^n F(s) - \sum_{k=0}^{n-1} s^{k} f^{(n-k-1)}(0),$$

(1.4.6)

which can be obtained from the definition (1.4.2) by integrating by parts under the assumption that the corresponding integrals exist.

(ii) **Laplace Transform of the Caputo Derivative**[133, p.no 106].

To establish the Laplace transform formula for the Caputo fractional derivative let us write the Caputo derivative

$$C^\alpha_a D^\alpha_t f(t) = \frac{1}{\Gamma(\alpha - n)} \int_a^t f^{(n)}(\tau) (t - \tau)^{\alpha - 1 - n}d\tau, \; (n - 1 < \alpha < n)$$

(1.4.7)
of the form
\[ C_0^D_t^p f(t) = D_t^{-(n-p)} g(t), \ g(t) = f^{(n)}(t), \ (n - 1 < p \leq n). \] (1.4.8)

The Laplace transform of the Riemann-Liouville fractional integral gives
\[ L\{C_0^D_t^p f(t); s\} = s^{-(n-p)} G(s), \] (1.4.9)
where according to (1.4.2),
\[ G(s) = s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0). \] (1.4.10)

Introducing (1.4.10) into (1.4.9), we arrive at the Laplace transform formula for the Caputo fractional derivative:
\[ L\{C_0^D_t^p f(t)\} = s^p F(s) - \sum_{k=0}^{n-1} s^{p-k-1} f^{(k)}(0), \ (n - 1 < p \leq n). \] (1.4.11)

Since this formula for the Laplace transform of the Caputo derivative involves the values of the function \( f(t) \) and its derivatives at the lower terminal \( t = 0 \), for which a certain physical interpretation exists (for example, \( f(0) \) is the initial position, \( f'(0) \) is the initial velocity, \( f''(0) \) is the initial acceleration), we can expect that it can be useful for solving applied problems leading to linear fractional differential equations with constant coefficients with accompanying initial conditions in traditional form.

This thesis mainly presents with the fixed point approach for proving existence results for various kinds of impulsive abstract functional differential, integro-differential equations and fractional differential equations with impulsive and nonlocal conditions in Banach spaces.

In Chapter II of this thesis, we prove the existence results for non-densely defined impulsive neutral functional differential inclusions with state-dependent delay by using Leray-Schauder of the alternative for Kakutani maps.

In Chapter III of this thesis, we study the existence results for a second order impulsive neutral functional integrodifferential inclusions in Banach spaces with infinite delay by using the fixed point theorem for condensing maps due to Martelli combined with theories of a strongly continuous cosine family of bounded linear operators.
Chapter IV is divided into two sections. In first section, we prove the existence and uniqueness results for impulsive fractional integro-differential equations in Banach space by using the Banach contraction principle and Leray–Schauder’s alternative fixed point theorem. In second section, we investigate the existence results for abstract mixed type impulsive fractional semilinear evolution equations by using the different types of fixed point theorems.

In Chapter V, we study the existence results for impulsive fractional semilinear functional integro-differential equations in Banach spaces. The results are obtained by using the Banach fixed point theorem and Krasnoselskii’s fixed point theorem.

Chapter VI of this thesis, in first section, we prove the existence of mild solutions for fractional semilinear integro-differential systems with infinite delay in $\alpha$–norm in Banach spaces by using Banach contraction principle and Schauder’s fixed point theorem and the next section, we establish the existence and controllability results for fractional integro-differential evolution systems involving the Caputo derivative in Banach space. The results are obtained by using the fractional calculus, properties of characteristic solution operators, Mönch fixed point theorem via measures of non compactness. Since we do not assume the characteristic solution operators are compact.

1.7 Contributions of the author

In the light of the above, the author has obtained some significant results on the following topics:

1. Existence results for non-densely defined impulsive neutral functional differential inclusions with state-dependent delay.


5. Existence results for fractional semilinear integro-differential evolution systems with infinite delay in Banach spaces.

The rest of the thesis presents the various results established of the above topics.