Peristaltic flow of a Prandtl fluid through a porous medium in a channel
PERISTALTIC FLOW OF A PRANDTL FLUID THROUGH A POROUS MEDIUM IN A CHANNEL

2.1 Introduction

Peristaltic transport is a form of fluid transport generated by a progressive wave of area contraction or expansion along a length of a distensible tube containing fluid. Peristaltic transport widely occurs in many biological systems for example, food swallowing through the esophagus, intra-urine fluid motion, circulation of blood in small blood vessels and the flows of many other glandular ducts. Several theoretical and experimental studies have been undertaken to understand peristalsis through abrupt changes in geometry and realistic assumptions. A review of much of the early literature is presented in an article by Jaffrin and Shapiro (1971). All the important literature up to 1978 on peristaltic transport has been documented by Rath (1980).

In the literature, several works pertaining to peristaltic motion have been done for Newtonian fluid. Such approach is true in ureter but it fails to give an adequate understanding of peristalsis in blood vessels, chyme moment in intestine, semen transport in ductus efferentus of male reproductive tract, in transport of spermatozoa and in cervical canal. Peristaltic transport of blood in smaller vessels has been investigated with a variety of treatments of blood characterized by power law, viscoelastic, Casson and micropolar fluid model by quite a good number of researchers (Radhakrishnamacharya, 1982; Bohme and Friedrich, 1983; Srivastava and Srivastava, 1984; Srinivasacharya, 2003). The peristaltic flow of a power-law fluid in an asymmetric channel was investigated by Subba Reddy et al. (2007). Nagendra et al. (2008) have studied the peristaltic flow of a Jeffrey fluid in a tube. Recently, Akbar et al. (2012) have discussed the peristaltic flow of a Prandtl fluid in an asymmetric channel.

Moreover, flow through a porous medium has been of considerable interest in recent years particularly among geophysical fluid dynamicists. Examples of natural porous media are beach sand, stand stone, limestone, rye bread, wood, the human lung, bile duct, gall bladder with stones and in small blood vessels. Flow through a porous medium has been studied by a number of workers employing Darcy’s law Scheidegger (1974).

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The first study of peristaltic flow through a porous medium is presented by Elsehawey et al. (1999). Elsehawey et al. (2000) have studied peristaltic motion of a generalized Newtonian fluid through a porous medium. Peristaltic transport through a porous medium in an inclined planar channel has investigated by Mekheimer (2003) taking the gravity effect on pumping characteristics. Recently, Subba Reddy and Prasnath Reddy (2010) have investigated the effect of variable viscosity on peristaltic flow of a Jeffrey fluid through a porous medium in a planar channel.

In view of these, we studied the peristaltic transport of a Prandtl fluid through a porous medium in a uniform channel under the assumptions of long wavelength and low Reynolds number. Series solutions of axial velocity and pressure gradient are given by using regular perturbation technique when Prandtl number is small. The effects of various pertinent parameters on the pressure gradient and pumping characteristics are discussed in detail through graphs.

2.2 Mathematical Formulation

We consider the peristaltic flow of a Prandtl fluid through a porous medium in a two dimensional channel of width $2a$. The walls of the channel are flexible. The flow is induced by periodic peristaltic wave of length $\lambda$ and amplitude $b$ with constant speed $c$ along the channel walls. The physical model of the symmetric channel is shown in Fig. 2.1.

The equation of the wall is given by

$$Y = \pm H(X,t) = \pm a \pm b \sin \frac{2\pi}{\lambda}(X - ct)$$

where $t$ is the time, $\lambda$ is the wavelength and $(X,Y)$ are the Cartesian co-ordinates in laboratory frame of reference.

We introduce a wave frame of reference $(x, y)$ moving with velocity $c$ in which the motion becomes independent of time when the channel length is an integral multiple of the wavelength and the pressure difference at the ends of the channel is a constant (Shapiro et al., 1969). The transformation from the fixed frame of reference $(X, Y)$ to the wave frame of reference $(x, y)$ is given by

$$x = X - ct, \quad y = Y, \quad u = U - c, \quad v = V \quad \text{and} \quad p(x) = P(X, t).$$

where \((u, v)\) and \((U, V)\) are the velocity components, \(p\) and \(P\) are pressures in the wave and fixed frames of reference, respectively.

\[ (u, v) \text{ and } (U, V) \text{ are the velocity components. } p \text{ and } P \text{ are pressures in the wave and fixed frames of reference, respectively.} \]

\[ \text{Porous medium} \]

\[ \text{Fig. 2.1 The physical model} \]

The Constitutive equation for Prandtl fluid is given by (Patel and Timaol, 2010)

\[ \tau = A \sin^{-1} \left( \frac{1}{C} \left( \frac{\partial u}{\partial y} + \left( \frac{\partial v}{\partial x} \right)^2 \right) \right) \]

\[ \left( \frac{\partial v}{\partial y} \right) + \left( \frac{\partial v}{\partial x} \right)^2 \]

\[ \text{in which } A \text{ and } C \text{ are material constants of Prandtl fluid model.} \]

The equations governing the flow in wave frame of reference are given by

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \] \hspace{1cm} \text{(2.2.4)}

\[ \rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xu}}{\partial x} + \frac{\partial \tau_{uy}}{\partial y} - \frac{\mu(u+1)}{k} \] \hspace{1cm} \text{(2.2.5)}
The boundary conditions are

$$u = -c \quad \text{at} \quad y = H$$

(2.2.7)

$$\frac{\partial u}{\partial y} = 0 \quad \text{at} \quad y = 0$$

(2.2.8)

Introducing the following non-dimensional variables

$$x = \frac{x}{\lambda}, \quad y = \frac{y}{a}, \quad u = \frac{u}{c}, \quad v = \frac{v}{c\mu}, \quad p = \frac{pa^2}{\mu}, \quad \theta = \frac{ct}{\lambda}, \quad h = \frac{H}{a},$$

$$\tau = \frac{ar}{\mu}, \quad \phi = \frac{b}{a}, \quad \delta = \frac{a}{\lambda}, \quad \text{Re} = \frac{\rho \alpha c}{\mu}$$

where $\mu$ is the viscosity of the fluid, in the l'qs. (2.2.4) - (2.2.6), we get (after dropping the bars)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

(2.2.9)

$$\text{Re} \theta \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = - \frac{\partial p}{\partial x} + \frac{\partial^2 \tau_{yn}}{\partial x^2} + \frac{\partial^2 \tau_{yn}}{\partial y^2} - \frac{1}{Da} (u + 1)$$

(2.2.10)

$$\text{Re} \phi \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = - \frac{\partial p}{\partial y} + \delta \frac{\partial \tau_{yn}}{\partial x} + \delta \frac{\partial \tau_{yn}}{\partial y} - \frac{\delta^2 v}{Da}$$

(2.2.11)

where $Da = \frac{k}{a^2}$ is the Darcy number.

Under the assumptions of long wave length ($d < < 1$) and low Reynolds number ($\text{Re} \rightarrow 0$), the Equations (2.2.10) and (2.2.11) become

$$\frac{\partial p}{\partial x} = \frac{\partial^2 \tau_{yn}}{\partial x^2} - \frac{1}{Da} (u + 1)$$

(2.2.12)

$$\frac{\partial p}{\partial y} = 0$$

(2.2.13)

here $\tau_{yn} = \alpha \frac{\partial u}{\partial y} + \beta \left( \frac{\partial u}{\partial y} \right)^3$. 
The corresponding boundary conditions in wave frame of reference are given by

\[ u = -1 \quad \text{at} \quad y = h. \tag{2.2.14} \]

\[ \frac{\partial u}{\partial y} = 0 \quad \text{at} \quad y = 0. \tag{2.2.15} \]

Equations (2.2.12) and (2.2.13) indicate that \( p \) is independent of \( y \). Therefore Eq. (2.2.12) can be rewritten as

\[ \frac{dp}{dx} = \alpha \frac{\partial^2 u}{\partial y^2} + \frac{\beta}{6} \left[ \left( \frac{\partial u}{\partial y} \right)^2 \right] - \frac{1}{D_{u}} (u + 1). \tag{2.2.16} \]

The volume flow rate \( q \) in a wave frame of reference is given by

\[ q = \int_{0}^{h} u dy. \tag{2.2.17} \]

The instantaneous flux \( Q(x,t) \) in the laboratory frame is

\[ Q(x,t) = \int_{0}^{b} u dy = \int_{0}^{h} (u + 1) dy = q + h. \tag{2.2.18} \]

The time averaged flux over one period \( T \left( \frac{\lambda}{c} \right) \) of the peristaltic wave is

\[ \bar{Q} = \frac{1}{T} \int_{0}^{h} Q dt = \int_{0}^{h} (q + h) dx = q + 1. \tag{2.2.19} \]

### 2.3 Solution of the Problem

The Equation (2.2.16) is non-linear and its closed form solution is not possible. Hence, we linearize this equation in terms of \( \beta (<< 1) \). So we expand \( u, p \) and \( q \) as

\[ u = u_{0} + \beta u_{1} + O(\beta^2) \]

\[ p = p_{0} + \beta p_{1} + O(\beta^2) \]

\[ q = q_{0} + \beta q_{1} + O(\beta^2) \tag{2.3.1} \]

Substituting (2.3.1) in the Equation (2.2.16) and in the boundary conditions (2.2.14) and (2.2.15) and equating the coefficients of like powers of \( \beta \) to zero and neglecting the terms of \( \beta^2 \) and higher order, we get the following equations:
2.3.1 System of Order Zero ($\beta^0$)

\[ \frac{\partial^2 u}{\partial y^2} - \frac{1}{Da} u = \frac{dp_o}{dx} + \frac{1}{Da} \]  

(2.3.2)

with the corresponding boundary conditions are

\[ u_o = -1 \quad \text{at} \quad y = h = 1 + \phi \cos 2\pi x. \]  

(2.3.3)

\[ \frac{\partial u_o}{\partial y} = 0 \quad \text{at} \quad y = 0. \]  

(2.3.4)

2.3.2 System of Order One ($\beta$)

\[ \frac{\partial^2 u}{\partial y^2} - \frac{1}{Da} u = \frac{dp}{dx} - \frac{1}{6} \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right) \]  

(2.3.5)

with the corresponding boundary conditions are

\[ u_i = 0 \quad \text{at} \quad y = h = 1 + \phi \cos 2\pi x. \]  

(2.3.6)

\[ \frac{\partial u_i}{\partial y} = 0 \quad \text{at} \quad y = 0. \]  

(2.3.7)

2.3.3 Solution of Order Zero ($\beta^0$)

Solving Eq. (2.3.2) together with the boundary conditions (2.3.3) and (2.3.4), we obtain

\[ u_o = Da \frac{dp_o}{dx} \left[ \frac{\cosh Nh}{\cosh Nh} - 1 \right] - 1 \]  

(2.3.8)

where \( N = \frac{1}{\sqrt{Da}}. \)

The volume flow rate \( q_o \) in the moving coordinate system is given by

\[ q_o = \int u_o dy = Da \frac{dp_o}{dx} \left[ \frac{\sinh Nh}{N \cosh Nh} - h \right] - h \]  

(2.3.9)

From Eq. (2.3.9), we have

\[ \frac{dp_o}{dx} = \frac{(q_o + h) N \cosh Nh}{Da \left( \sinh Nh - Nh \cosh Nh \right)} \]  

(2.3.10)

2.3.4 Solution of Order One ($\beta$)

Solving the Equation (3.5) by using the Equation (3.8) and the boundary conditions (3.6) and (3.7), we obtain
\[ u_i = \frac{D_a}{dx} \left[ \cosh Ny \frac{\cosh Nh}{\cosh Nh} - 1 \right] + \frac{N(A_1,D_a)}{8} \left( \frac{dp}{dx} \right) \left[ \frac{\cosh 3Ny}{2N} - D_a \frac{\cosh 3Nh}{8} - A_i \cosh Ny \right] \]

(2.3.11)

where \( A_i = \frac{N}{\cosh Nh} \) and \( A_i = \left( \frac{h \sinh Nh - D_a \cosh 3Nh}{2aN} \right) \frac{1}{\cosh Nh} \).

and the volume flow rate \( q_i \) is given by

\[ q_i = \int u_i dy = \frac{D_a}{N \cosh Nh} \frac{dp}{dx} \left[ \sinh Nh - Nh \cosh Nh \right] + A_i \left( \frac{dp}{dx} \right) \]

(2.3.12)

where \( A_i = \frac{N(A_1,D_a)}{8} \left[ \cosh Nh \frac{\sinh Nh}{2aN^2} \right] - \frac{D_a \sinh 3Nh}{24N} + \frac{A_i \sinh Nh}{N} \).

From Eq. (2.3.12), we have

\[ \frac{N \cos Nh}{D_a (\sinh Nh - hN \cosh Nh)} \left[ q_i - A_i \left( \frac{dp}{dx} \right) \right] \]

(2.3.13)

Substituting Equations (2.3.10) and (2.3.13) into the second Equation of (2.3.1) and using the relation \( \frac{dp}{dx} = \frac{dp}{dx} - \beta \frac{dp}{dx} \) and neglecting terms greater than \( O(\beta) \), we get

\[ \frac{dp}{dx} = \frac{N \cosh Nh}{D_a (\sinh Nh - Nh \cosh Nh)} \left[ q + h - \beta A_i \left( \frac{(q_i + h) N \cosh Nh}{D_a (\sinh Nh - Nh \cosh Nh)} \right) \right] \]

(2.3.14)

The dimensionless pressure rise per one wavelength in the wave frame is defined as

\[ \Delta p = \int \frac{dp}{dx} dx \]

(2.3.15)

Note that, as \( D_a \rightarrow \infty, \alpha \rightarrow 1 \) and \( \beta \rightarrow 0 \) our results coincides with the results of Shapiro et al. (1969).
2.4. Results and Discussions

Fig. 2.2 illustrates the variation of axial pressure gradient $\frac{dp}{dx}$ with $\beta$ for $\phi = 0.5$, $\alpha = 1.5$ and $Da = 0.1$. It is observed that, the axial pressure gradient $\frac{dp}{dx}$ increases with increasing $\beta$.

The variation of axial pressure gradient $\frac{dp}{dx}$ with $Da$ for $\phi = 0.5$, $\alpha = 1.5$ and $\beta = 0.1$ is shown in Fig. 2.3. It is noted that, the axial pressure gradient $\frac{dp}{dx}$ decreases with an increase in $Da$.

Fig. 2.4 depicts the variation of axial pressure gradient $\frac{dp}{dx}$ with $\alpha$ for $\phi = 0.5$, $\beta = 0.1$ and $Da = 0.1$. It is found that, the axial pressure gradient $\frac{dp}{dx}$ increases an increasing $\alpha$.

The variation of axial pressure gradient $\frac{dp}{dx}$ with $\phi$ for $\beta = 0.1$, $\alpha = 1.5$ and $Da = 0.1$ is depicted in Fig. 2.5. It is observed that, the axial pressure gradient $\frac{dp}{dx}$ increases with increasing $\phi$. 

Fig. 2.6 shows the variation of pressure rise $\Delta p$ with time averaged flux $\bar{Q}$ for different values of $\beta$ with $\phi = 0.5$, $\alpha = 1.5$ and $Da = 0.1$. It is observed that, the time averaged flux $\bar{Q}$ increases with increasing $\beta$ in the pumping region ($\Delta p > 0$), while it decreases with increasing $\beta$ in both the free-pumping ($\Delta p = 0$) and co-pumping ($\Delta p < 0$) regions. Further, it is observed that, the pumping is more for Prandtl fluid than that of Newtonian fluid ($\alpha = 1, \beta = 0$).

The variation of pressure rise $\Delta p$ with time averaged flux $\bar{Q}$ for different values of $Da$ with $\phi = 0.5$, $\alpha = 1.5$ and $\beta = 0.1$ is depicted in Fig. 2.7. It is found that, the time
averaged flux decreases with increasing \( Da \) in the pumping region, while it increases with increasing \( Da \) in both the free-pumping and co-pumping regions.

Fig. 2.8 depicts the variation of pressure rise \( \Delta p \) with time averaged flux \( \bar{Q} \) for different values of \( \alpha \) with \( \phi = 0.5 \), \( \beta = 0.1 \) and \( Da = 0.1 \). It is noted that, the time averaged flux \( \bar{Q} \) increases with increasing \( \alpha \) in both the pumping and free-pumping regions, while it decreases with increasing \( \alpha \) in the co-pumping region.

The variation of pressure rise \( \Delta p \) with time averaged flux \( \bar{Q} \) for different values of \( \phi \) with \( \beta = 0.1 \), \( \alpha = 1.5 \) and \( Da = 0.1 \) is depicted in Fig. 2.9. It is noted that, the time averaged flux \( \bar{Q} \) increases with increasing \( \phi \) in both the pumping and free-pumping regions, while it decreases with increasing \( \phi \) in the co-pumping region.

2.5. Conclusions

In this chapter, we studied the peristaltic transport of a Prandtl fluid through a porous medium in a uniform channel under the assumptions of long wavelength and low Reynolds number. Series solutions of axial velocity and pressure gradient are given by using regular perturbation technique when Prandtl number is small. It is observed that, the axial pressure gradient increases with the increasing \( \beta, \alpha \) and \( \phi \), while it decreases with increasing \( Da \). In the pumping region, time averaged flux \( \bar{Q} \) increases with increasing \( \beta, \alpha \) and \( \phi \), while it decreases with increasing \( Da \). Also, it is observed that, the pumping is more for Prandtl fluid than that of Newtonian fluid.
Fig. 2.2 The variation of axial pressure gradient $\frac{dp}{dx}$ with $\beta$ for $\phi = 0.5, \alpha = 1.5$ and $Du = 0.1$. 

\[ \beta = 0.2, 0.1, 0.0 \]
Fig. 2.3 The variation of axial pressure gradient $\frac{dp}{dx}$ with $Du$ for $\phi = 0.5, \alpha = 1.5$ and $\beta = 0.1$. 
Fig. 2.4 The variation of axial pressure gradient $\frac{dp}{dx}$ with $\alpha$ for $\phi = 0.5$, $\beta = 0.1$ and $Da = 0.1$. 
Fig. 2.5 The variation of axial pressure gradient $\frac{dp}{dx}$ with $\phi$ for $\beta = 0.1$, $\alpha = 1.5$ and $Da = 0.1$. 
Fig. 2.6 The variation of pressure rise $\Delta p$ with time averaged flux $\bar{Q}$ for different values of $\beta$ with $\phi = 0.5$, $\alpha = 1.5$ and $\Delta \omega = 0.1$. 
Fig. 2.7 The variation of pressure rise $\Delta p$ with time averaged flux $\bar{Q}$ for different values of $Da$ with $\phi = 0.5$, $\alpha = 1.5$ and $\beta = 0.1$. 
Fig. 2.8 The variation of pressure rise $\Delta p$ with time averaged flux $\overline{Q}$ for different values of $\alpha$ with $\phi = 0.5$, $\beta = 0.1$ and $Da = 0.1$. 
Fig. 2.9 The variation of pressure rise $\Delta p$ with time averaged flux $\bar{Q}$ for different values of $\phi$ with $\beta = 0.1$, $\alpha = 1.5$ and $Da = 0.1$. 