CHAPTER VI

NON-MARKOVIAN BULK ARRIVAL GENERAL BULK SERVICE QUEUEING SYSTEM WITH MULTIPLE VACATIONS AND REPAIR OF SERVICE STATION ON REQUEST BY A LEAVING BATCH OF CUSTOMERS

There are queueing models in which a leaving customer may be dissatisfied with the low quality of service rendered and he may complain and request that the station be adjusted to its proper level of service. In some queueing models, a leaving customer may request for some additional service. Avi-Itzhak and Naor [3] have analysed a M/G/1 queueing model with repair of service station on request by a leaving customer.

In this chapter we deal with a "M'/G(a,b)/1 queueing system with multiple vacations and repair of service station on request by a leaving batch of customers". The leaving batch of customers may request for repair of service station with probability \( \pi \) and it is assumed that more than one request will never be made by the same batch. After the repair time or service completion without request for repair, if the queue length is less than 'a', the server avails multiple vacations till the queue length reaches 'a'. After a vacation or a service completion without request for repair or a repair completion, if the server finds
atleast 'a' customers waiting for service, he serves according to general bulk service rule with minimum of 'a' customers and maximum of 'b' customers.

A practical situation in which this model occurs is in a boat house of a tourist centre. Customers arrive in bulk to the boat house and the boat is operated with minimum of 'a' customers and maximum of 'b' customers. Though the service is done for a specific duration, some customers may request for additional trip for about fifteen minutes which will be less than the specific duration. If required number of customers is not available in the queue, the operator will start some maintenance work.

For the above queueing model, the probability generating function of the steady state queue size at an arbitrary time is obtained. Expressions for expected length of queue, busy period and idle period are obtained. A cost model is developed. Numerical illustration is presented.

6.1 The Mathematical Model

Let \( \lambda \) be the Poisson arrival rate, \( X \) be the group size random variable of the arrival, \( g_k \) be the probability that \( k \) customers arrive in a batch and \( X(z) \) be the probability generating function of \( X \). Let \( S(.) \), \( V(.) \) and \( G(.) \) be the cumulative distribution functions of service time, vacation time and repair time, respectively.
Let $s(x)$, $v(x)$ and $g(x)$ be the probability density functions of $S$, $V$ and $G$, respectively. $\tilde{S}(0)$, $\tilde{V}(0)$ and $\tilde{G}(0)$ denote the Laplace - Stieltjes transforms of $S$, $V$ and $G$, respectively. The remaining service time of a batch in service at an arbitrary time $t$ is denoted by $S^0(t)$, $V^0(t)$ is the remaining vacation time of the server on vacation and $G^0(t)$ is the remaining repair time at time $t$. $N_q(t)$ and $N_s(t)$ denote the number of customers in the queue and under service, respectively, at time $t$. We define $Y(t)$ such that $Y(t) = 0$, when the server is on vacation, $Y(t) = 1$, when the server is attending to repair, $Y(t) = 2$, when the server is busy with service. We define $Z(t)$ such that $Z(t) = j$ implies that the server is on $j^{th}$ vacation.

Let $P_{ij}(x, t)dt = Pr\{N_s(t) = i, N_q(t) = j, x < S^0(t) \leq x + dt, Y(t) = 2\}$, $a \leq i \leq b$, $j \geq 0$

$Q_{in}(x, t)dt = Pr\{N_q(t) = n, x < V^0(t) \leq x + dt, Y(t) = 0, Z(t) = j\}$, $j \geq 1$, $n \geq 0$

and $G_{n}(x, t)dt = Pr\{N_q(t) = n, x < G^0(t) \leq x + dt, Y(t) = 1\}$, $n \geq 0$

The following equations are obtained for the queueing system under consideration, using supplementary variables technique:

$P_{i0}(x - \Delta t, t + \Delta t) = P_{i0}(x, t)(1 - \lambda \Delta t) + (1 - \pi) \sum_{m=a}^{b} P_{mi}(0, t)s(x)\Delta t$

$+ \sum_{l=1}^{\infty} Q_{li}(0, t)s(x)\Delta t + G_{i}(0, t)s(x)\Delta t, a \leq i \leq b$

$P_{ij}(x - \Delta t, t + \Delta t) = P_{ij}(x, t)(1 - \lambda \Delta t) + \sum_{k=1}^{i} P_{ij-k}(x, t)\lambda g_k \Delta t, \quad a \leq i \leq b-1, j \geq 1$
\[
P_{bj}(x - \Delta t, t + \Delta t) = P_{bj}(x, t)(1 - \lambda \Delta t) + (1 - \pi) \sum_{m=a}^{b} P_{m,b+j}(0, t)s(x)\Delta t \\
+ \sum_{k=1}^{l} P_{b,j-k}(x, t)\lambda g_k \Delta t + \sum_{l=1}^{\infty} Q_{l,b+j}(0, t)s(x)\Delta t \\
+ G_{b+j}(0, t)s(x)\Delta t, \quad j \geq 1
\]

\[
Q_{10}(x - \Delta t, t + \Delta t) = Q_{10}(x, t)(1 - \lambda \Delta t) + (1 - \pi) \sum_{m=a}^{b} P_{m0}(0, t)v(x)\Delta t \\
+ G_{0}(0, t)v(x)\Delta t
\]

\[
Q_{ln}(x - \Delta t, t + \Delta t) = Q_{ln}(x, t)(1 - \lambda \Delta t) + (1 - \pi) \sum_{m=a}^{b} P_{mn}(0, t)v(x)\Delta t \\
+ \sum_{k=1}^{n} Q_{1,n-k}(x, t)\lambda g_k \Delta t + G_{n}(0, t)v(x)\Delta t, \quad n = 1,2,...,a-1
\]

\[
Q_{1n}(x - \Delta t, t + \Delta t) = Q_{1n}(x, t)(1 - \lambda \Delta t) + \sum_{k=1}^{n} Q_{1,n-k}(x, t)\lambda g_k \Delta t, \quad n \geq a
\]

\[
Q_{j0}(x - \Delta t, t + \Delta t) = Q_{j0}(x, t)(1 - \lambda \Delta t) + Q_{j-1,0}(0, t)v(x)\Delta t, \quad j \geq 2
\]

\[
Q_{jn}(x - \Delta t, t + \Delta t) = Q_{jn}(x, t)(1 - \lambda \Delta t) + \sum_{k=1}^{n} Q_{j,n-k}(x, t)\lambda g_k \Delta t \\
+ Q_{j-1,n}(0, t)v(x)\Delta t, \quad j \geq 2, \quad n = 1,2,...,a-1
\]

\[
Q_{jn}(x - \Delta t, t + \Delta t) = Q_{jn}(x, t)(1 - \lambda \Delta t) + \sum_{k=1}^{n} Q_{j,n-k}(x, t)\lambda g_k \Delta t, \quad j \geq 2, \quad n \geq a
\]

\[
G_{0}(x - \Delta t, t + \Delta t) = G_{0}(x, t)(1 - \lambda \Delta t) + \pi \sum_{m=a}^{b} P_{m0}(0, t)g(x)\Delta t
\]

\[
G_{n}(x - \Delta t, t + \Delta t) = G_{n}(x, t)(1 - \lambda \Delta t) + \pi \sum_{m=a}^{b} P_{mn}(0, t)g(x)\Delta t \\
+ \sum_{k=1}^{n} G_{n-k}(x, t)\lambda g_k \Delta t, \quad n \geq 1
\]
From the above equations, the steady state queue size equations are obtained as:

\[
\begin{align*}
-\frac{d}{dx} P_{i0}(x) &= -\lambda P_{i0}(x) + (1 - \pi) \sum_{m=a}^{b} P_{mi}(0)s(x) + \sum_{l=1}^{\infty} Q_{li}(0)s(x) + G_{i}(0)s(x), \quad a \leq i \leq b \\
-\frac{d}{dx} P_{ij}(x) &= -\lambda P_{ij}(x) + \lambda \sum_{k=1}^{i} P_{i,j-k}(x)g_k, \quad a \leq i \leq b-1, \quad j \geq 1 \\
-\frac{d}{dx} P_{bj}(x) &= -\lambda P_{bj}(x) + (1 - \pi) \sum_{m=a}^{b} P_{m,b+j}(0)s(x) + \sum_{k=1}^{j} P_{bj-k}(x)g_k + \sum_{l=1}^{\infty} Q_{l,b+j}(0)s(x) + G_{b+j}(0)s(x), \quad j \geq 1 \\
-\frac{d}{dx} Q_{i0}(x) &= -\lambda Q_{i0}(x) + (1 - \pi) \sum_{m=a}^{b} P_{m0}(0)v(x) + G_{0}(0)v(x) \\
-\frac{d}{dx} Q_{in}(x) &= -\lambda Q_{in}(x) + (1 - \pi) \sum_{m=a}^{b} P_{mn}(0)v(x) + \sum_{k=1}^{n} Q_{1,n-k}(x)g_k + G_{n}(0)v(x), \quad n = 1, 2, \ldots, a-1 \\
-\frac{d}{dx} Q_{in}(x) &= -\lambda Q_{in}(x) + \lambda \sum_{k=1}^{n} Q_{1,n-k}(x)g_k, \quad n \geq a \\
-\frac{d}{dx} Q_{j0}(x) &= -\lambda Q_{j0}(x) + Q_{j-1,0}(0)v(x), \quad j \geq 2 \\
-\frac{d}{dx} Q_{jn}(x) &= -\lambda Q_{jn}(x) + Q_{j-1,n}(0)v(x) + \sum_{k=1}^{n} Q_{j,n-k}(x)g_k, \quad j \geq 2, \quad n = 1, 2, \ldots, a-1 \\
-\frac{d}{dx} Q_{jn}(x) &= -\lambda Q_{jn}(x) + \lambda \sum_{k=1}^{n} Q_{j,n-k}(x)g_k, \quad n \geq a \\
-\frac{d}{dx} G_{0}(x) &= -\lambda G_{0}(x) + \pi \sum_{m=a}^{b} P_{m0}(0)g(x)
\end{align*}
\]
\[- \frac{d}{dx} G_n(x) = - \lambda G_n(x) + \pi \sum_{m=a}^b P_{mn}(0)g(x) + \lambda \sum_{k=1}^n G_{n-k}(x)g_k, \quad n \geq 1 \quad (6.11)\]

Taking Laplace - Stieltjes transform on both sides of the equations (6.1) through (6.11), we have

\[\theta \tilde{P}_{10}(\theta) - P_{10}(0) = \lambda \tilde{P}_{10}(\theta) - (1 - \pi) \sum_{m=a}^b P_{mn}(0)\tilde{S}(\theta) - \tilde{S}(\theta)\sum_{l=1}^{\infty} Q_{li}(0), \quad a \leq i \leq b \quad (6.12)\]

\[\theta \tilde{P}_{ij}(\theta) - P_{ij}(0) = \lambda \tilde{P}_{ij}(\theta) - \lambda \sum_{k=1}^i P_{i;j-k}(\theta)g_k, \quad a \leq i \leq b-1, \quad j \geq 1 \quad (6.13)\]

\[\theta \tilde{P}_{bj}(\theta) - P_{bj}(0) = \lambda \tilde{P}_{bj}(\theta) - (1 - \pi) \sum_{m=a}^b P_{m,b+j}(0)\tilde{S}(\theta)
- \lambda \sum_{k=1}^j \tilde{P}_{b;j-k}(\theta)g_k - \tilde{S}(\theta)\sum_{l=1}^{\infty} Q_{1,b+j}(0)
- \tilde{S}(\theta)G_{b+j}(0), \quad j \geq 1 \quad (6.14)\]

\[\theta \tilde{Q}_{10}(\theta) - Q_{10}(0) = \lambda \tilde{Q}_{10}(\theta) - (1 - \pi) \sum_{m=0}^{b} P_{m0}(0)\tilde{V}(\theta) - G_0(0)\tilde{V}(\theta) \quad (6.15)\]

\[\theta \tilde{Q}_{1n}(\theta) - Q_{1n}(0) = \lambda \tilde{Q}_{1n}(\theta) - (1 - \pi) \sum_{m=a}^b P_{mn}(0)\tilde{V}(\theta)
- \lambda \sum_{k=1}^n \tilde{Q}_{1,n-k}(\theta)g_k - G_n(0)\tilde{V}(\theta), \quad 1 \leq n \leq a-1 \quad (6.16)\]

\[\theta \tilde{Q}_{1n}(\theta) - Q_{1n}(0) = \lambda \tilde{Q}_{1n}(\theta) - \lambda \sum_{k=1}^n \tilde{Q}_{1,n-k}(\theta)g_k, \quad n \geq a \quad (6.17)\]

\[\theta \tilde{Q}_{j0}(\theta) - Q_{j0}(0) = \lambda \tilde{Q}_{j0}(\theta) - Q_{j-1,0}(0)\tilde{V}(\theta), \quad j \geq 2 \quad (6.18)\]

\[\theta \tilde{Q}_{jn}(\theta) - Q_{jn}(0) = \lambda \tilde{Q}_{jn}(\theta) - Q_{j-1,n}(0)\tilde{V}(\theta) - \lambda \sum_{k=1}^n \tilde{Q}_{j,n-k}(\theta)g_k,
+ \lambda \tilde{Q}_{j,n-k}(\theta)g_k, \quad j \geq 2, \quad 1 \leq n \leq a-1 \quad (6.19)\]
\[ \theta \tilde{Q}_{jn}(\theta) - Q_{jn}(0) = \lambda \tilde{Q}_{jn}(\theta) - \lambda \sum_{k=1}^{n} \tilde{Q}_{j,n-k}(\theta)g_k, \quad n \geq a \]  
(6.20)

\[ \theta \tilde{G}_0(\theta) - G_0(0) = \lambda \tilde{G}_0(\theta) - \pi \sum_{m=a}^{b} P_{m0}(0)\tilde{G}(\theta) \]  
(6.21)

\[ \theta \tilde{G}_n(\theta) - G_n(0) = \lambda \tilde{G}_n(\theta) - \pi \sum_{m=a}^{b} P_{mn}(0)\tilde{G}(\theta) - \lambda \sum_{k=1}^{n} \tilde{G}_{n-k}(\theta)g_k, \quad n \geq 1 \]  
(6.22)

### 6.2 Queue Size Distribution

We define the following probability generating functions:

\[ \tilde{P}_i(z, \theta) = \sum_{j=0}^{\infty} \tilde{P}_{ij}(\theta)z^j \quad \text{and} \quad P_i(z, 0) = \sum_{j=0}^{\infty} P_{ij}(0)z^j; \quad a \leq i \leq b \]

\[ \tilde{Q}_j(z, \theta) = \sum_{n=0}^{\infty} \tilde{Q}_{jn}(\theta)z^n \quad \text{and} \quad Q_j(z, 0) = \sum_{n=0}^{\infty} Q_{jn}(0)z^n; \quad j \geq 1 \]

\[ \tilde{G}(z, \theta) = \sum_{n=0}^{\infty} \tilde{G}_n(\theta)z^n \quad \text{and} \quad G(z, 0) = \sum_{n=0}^{\infty} G_n(0)z^n. \]  
(6.23)

Multiplying the equations (6.15) by \( z^0 \), (6.16) by \( z^n(1 \leq n \leq a-1) \), (6.17) by \( z^n(n \geq a) \) and taking the summation from \( n = 0 \) to \( \infty \) and using (6.23), we have

\[ (\theta - \lambda + \lambda X(z))\tilde{Q}_1(z, \theta) = Q_1(z, 0) - \tilde{V}(\theta) \sum_{n=0}^{a-1} \left[ (1 - \pi) \sum_{m=a}^{b} P_{mn}(0)z^n + G_n(0)z^n \right] \]  
(6.24)

Multiplying the equations (6.18) by \( z^0 \), (6.19) by \( z^n(1 \leq n \leq a-1) \), (6.20) by \( z^n(n \geq a) \) and taking the summation from \( n = 0 \) to \( \infty \) and using (6.23), we have

\[ (\theta - \lambda + \lambda X(z))\tilde{Q}_j(z, \theta) = Q_j(z, 0) - \tilde{V}(\theta) \sum_{n=0}^{a-1} Q_{j-1,n}(0)z^n, \quad j \geq 2 \]  
(6.25)
Multiplying the equations (6.12) by \( z^a \), (6.13) by \( z^j (j \geq 1) \) and taking the summation from \( j = 0 \) to \( \infty \) and using (6.23), we have

\[
(\theta - \lambda + \lambda X(z))\tilde{P}_1(z, \theta) = P_1(z, 0) - \tilde{S}(\theta)[(1 - \pi) \sum_{m=a}^{b} P_{mi}(0) + \sum_{i=1}^{\infty} Q_{fi}(0) + G_i(0)], \quad a \leq i \leq b-1 \tag{6.26}
\]

Multiplying the equations (6.12) by \( z^a \), (6.14) by \( z^j (j \geq 1) \) and taking the summation from \( j = 0 \) to \( \infty \) and using (6.23), we have

\[
z^b(\theta - \lambda + \lambda X(z))\tilde{P}_b(z, \theta) = z^b P_b(z, 0) - \tilde{S}(\theta)[(1 - \pi) \sum_{m=a}^{b} (P_m(z, 0) - \sum_{j=0}^{b-1} P_{mj}(0)z^j)
\]

\[
+ \sum_{i=1}^{\infty} (Q_i(z, 0) - \sum_{i=1}^{b-1} Q_{ij}(0)z^j) + (G(z, 0) - \sum_{n=0}^{b-1} G_n(0)z^n)] \tag{6.27}
\]

Multiplying the equations (6.21) by \( z^a \), (6.22) by \( z^n (n \geq 1) \) and taking the summation from \( n = 0 \) to \( \infty \) and using (6.23), we have

\[
(\theta - \lambda + \lambda X(z))\tilde{G}(z, \theta) = G(z, 0) - \pi \sum_{m=a}^{b} P_m(z, 0)\tilde{G}(\theta) \tag{6.28}
\]

Substituting \( \theta = \lambda - \lambda X(z) \) in equations (6.24), (6.25), (6.26), (6.27) and (6.28), we get

\[
Q_1(z, 0) = \tilde{V}(\lambda - \lambda X(z)) \sum_{n=0}^{a-1} \left( (1 - \pi) \sum_{m=a}^{b} P_{mn}(0)z^n + G_n(0)z^n \right) \tag{6.29}
\]

\[
Q_j(z, 0) = \tilde{V}(\lambda - \lambda X(z)) \sum_{n=0}^{a-1} Q_{j-1,n}(0)z^n, \quad j \geq 2 \tag{6.30}
\]
\[ P_i(z, 0) = \tilde{S}(\lambda - \lambda X(z)) \left[ (1 - \pi) \sum_{m=a}^{b} P_{mi}(0) + \sum_{l=1}^{\infty} Q_{li}(0) + G_i(0) \right], \]
a \leq i \leq b-1 \quad (6.31)

\[ G(z, 0) = \pi \sum_{m=a}^{b} P_m(z, 0) \tilde{G}(\lambda - \lambda X(z)) \]

\[ z^{b-1} P_b(z, 0) = \tilde{S}(\lambda - \lambda X(z)) \left[ (1 - \pi) \sum_{m=a}^{b} (P_m(z, 0) - \sum_{j=0}^{b-1} P_{mj}(0)z^j) \right. \]
\[ + \sum_{i=1}^{\infty} (Q_{i}(z, 0) - \sum_{j=0}^{b-1} Q_{ij}(0)z^j) + (G(z, 0) - \sum_{n=0}^{b-1} G_n(0)z^n) \right] \quad (6.33) \]

Let \( p_i = \sum_{m=a}^{b} P_{mi}(0), \ q_i = \sum_{l=1}^{\infty} Q_{li}(0), \ G_i = G_i(0) \) and \( c_i = (1 - \pi)p_i + q_i + G_i \)

Substituting in (6.33) for \( P_i(z, 0), \ a \leq i \leq b-1, \) from (6.31) and for \( G(z, 0) \) from (6.32) and then solving for \( P_b(z, 0), \) we get

\[ P_b(z, 0) = \frac{\tilde{S}(\lambda - \lambda X(Z))f(z)}{z^{b-1}(1 - \pi)\tilde{S}(\lambda - \lambda X(z)) - \pi \tilde{S}(\lambda - \lambda X(z)) \tilde{G}(\lambda - \lambda X(z))} \]

\[ where \ f(z) = \sum_{m=a}^{b-1} \left[ (1 - \pi) + \pi \tilde{G}(\lambda - \lambda X(z)) \right] \tilde{S}(\lambda - \lambda X(z)) - z^m \right] c_m \]
\[ + \sum_{m=0}^{a-1} \left( \tilde{V}(\lambda - \lambda X(z)) - 1 \right) c_m z^m \]

Using (6.31) and (6.34) in (6.32), we have

\[ G(z, 0) = \pi \sum_{m=a}^{b-1} \tilde{S}(\lambda - \lambda X(z)) \tilde{G}(\lambda - \lambda X(z)) c_m \]
\[ + \frac{\tilde{S}(\lambda - \lambda X(z)) \tilde{G}(\lambda - \lambda X(z))}{z^{b-1}(1 - \pi)\tilde{S}(\lambda - \lambda X(z)) - \pi \tilde{S}(\lambda - \lambda X(z)) \tilde{G}(\lambda - \lambda X(z))} \]

(6.35)
Equations (6.24) and (6.29) give us
\[ Q_1(z, 0) = \frac{\left( \tilde{V}(\lambda - \lambda X(z)) - \tilde{V}(\theta) \right) \sum_{n=0}^{a-1} [(1-\pi)p_n + G_n] z^n}{\theta - \lambda + \lambda X(z)} \]  
\[ (6.36) \]

From equations (6.25) and (6.30), we have
\[ Q_j(z, 0) \]
\[ ~ \sim \sim \frac{1}{\theta - \lambda + \lambda X(z)} \] 
\[(6.37)\]

Equations (6.26) and (6.31) give us
\[ P_i(z, 0) = \frac{\left( \tilde{S}(\lambda - \lambda X(z)) - \tilde{S}(\theta) \right) c_i}{\theta - \lambda + \lambda X(z)} \] 
\[ (6.38) \]

From equations (6.27) and (6.34), we get
\[ P_h(z, 0) = \frac{\left( \tilde{S}(\lambda - \lambda X(z)) - \tilde{S}(\theta) \right) f(z)}{\left( z^b - (1-\pi)\tilde{S}(\lambda - \lambda X(z)) - \pi \tilde{S}(\lambda - \lambda X(z)) \tilde{G}(\lambda - \lambda X(z)) - \theta - \lambda + \lambda X(z) \right)} \] 
\[ (6.39) \]

From equations (6.28) and (6.35), we have
\[ (\theta - \lambda + \lambda X(z)) \tilde{G}(z, \theta) = \pi \left( \tilde{G}(\lambda - \lambda X(z)) - \tilde{G}(\theta) \right) \left[ \sum_{m=a}^{b-1} \tilde{S}(\lambda - \lambda X(z)) c_m \right] 
+ \frac{\tilde{S}(\lambda - \lambda X(z)) f(z)}{z^b - (1-\pi)\tilde{S}(\lambda - \lambda X(z)) - \pi \tilde{S}(\lambda - \lambda X(z)) \tilde{G}(\lambda - \lambda X(z))} \] 
\[ (6.40) \]
Let \( P(z) \) be the probability generating function of the queue size at an arbitrary time epoch. Then

\[
P(Z) = \sum_{i=a}^{b-1} \tilde{P}_i(z, 0) + \tilde{P}_b(z, 0) + \sum_{j=1}^{\infty} \tilde{Q}_j(z, 0) + \tilde{G}(z, 0)
\]

which, using equations, (6.36) through (6.40), gives

\[
P(z) = \frac{\left[ \tilde{S}(\lambda - \lambda X(z)) - 1 \right] \sum_{i=a}^{b-1} c_i}{-\lambda + \lambda X(z)}
\]

\[
+ \frac{\left[ \tilde{S}(\lambda - \lambda X(z)) \right] f(z)}{z^b - (1 - \pi) \tilde{S}(\lambda - \lambda X(z)) - \pi \tilde{S}(\lambda - \lambda X(z)) \tilde{G}(\lambda - \lambda X(z))} (-\lambda + \lambda X(z))
\]

\[
+ \frac{\left[ \tilde{V}(\lambda - \lambda X(z)) - 1 \right] \sum_{i=0}^{a-1} c_i z^i}{-\lambda + \lambda X(z)} + \frac{\pi \left[ \tilde{G}(\lambda - \lambda X(z)) - 1 \right] \sum_{m=a}^{b-1} \tilde{S}(\lambda - \lambda X(z)) c_m}{-\lambda + \lambda X(z)}
\]

\[
+ \frac{\pi \left[ \tilde{G}(\lambda - \lambda X(z)) - 1 \right] \tilde{S}(\lambda - \lambda X(z)) f(z)}{z^b - (1 - \pi) \tilde{S}(\lambda - \lambda X(z)) - \pi \tilde{S}(\lambda - \lambda X(z)) \tilde{G}(\lambda - \lambda X(z))} (-\lambda + \lambda X(z))
\]

(6.41)

and on further simplification, we have

\[
P(z) = \left[ ((\tilde{S}(\lambda - \lambda X(z)) - 1) + \pi (\tilde{G}(\lambda - \lambda X(z)) - 1) \tilde{S}(\lambda - \lambda X(z))) \sum_{m=a}^{b-1} (z^b - z^m) c_m \right.
\]

\[
+ \left. \left( \tilde{V}(\lambda - \lambda X(z)) - 1 \right) \sum_{m=0}^{a-1} (z^b - 1) c_m z^m \right]
\]

\[
(-\lambda + \lambda X(z)) h(z)
\]

(6.42)
where \( h(z) = z^b - (1 - \pi)\tilde{S}(\lambda - \lambda X(z)) - \pi\tilde{S}(\lambda - \lambda X(z))\tilde{G}(\lambda - \lambda X(z)) \)

The probability generating function \( P(z) \) has to satisfy \( P(1) = 1 \). In order to satisfy this condition, applying L'Hospital's rule and evaluating \( \lim_{z \to 1} P(z) \) and equating the expression to 1, we have to satisfy

\[
[E(S) + \pi E(G)] \sum_{i=0}^{b-1} (b - i)c_i + bE(V) \sum_{i=0}^{a-1} c_i = b - \lambda E(X)[E(S) + \pi E(G)]
\]

Since \( p_i, q_i, \) and \( G_i \) are the probabilities of \( i \) customers being in the queue, at a customer departure epoch, vacation completion epoch and at a repair completion epoch, respectively, it follows that \( c_n = (1 - \pi) p_n + q_n + G_n \), \( n = 0 \) to \( b-1 \) are to be positive quantities and hence left hand side of the above expression should be positive. Thus \( P(1) = 1 \) is satisfied iff \( b - \lambda E(X) [E(S) + \pi E(G)] > 0 \) and defining \( \rho \) by

\[
\rho = \frac{\lambda E(X)[E(S) + \pi E(G)]}{b}
\]

we get that \( \rho < 1 \) is the condition to be satisfied for the existence of steady state for the model under consideration.

Equation (6.42) gives the probability generating function of the number of customers in the queue, but involves 'b' unknowns \( c_0, c_1, \ldots, c_{b-1} \). Rouche's theorem of complex variables can be used to find these constants. By Rouche's
theorem, it follows that \( z^b - (1 - \pi)\tilde{S}(\lambda - \lambda X(z)) - \pi\tilde{S}(\lambda - \lambda X(z))\tilde{G}(\lambda - \lambda X(z)) \) has \( b-1 \) zeros inside and one on the unit circle \( |z| = 1 \). Since \( P(z) \) is analytic within and on the unit circle, the numerator of (6.42) must vanish at these points, which gives 'b' equations in 'b' unknowns. We can solve these equations by any suitable numerical technique. Thus, (6.42) gives the probability generating function of the queue size at an arbitrary time.

6.3 Expected Queue Length

The mean queue length \( E(Q) \) is given by

\[
E(Q) = \sum_{n=1}^{\infty} np_n = p'(1)
\]

From (6.42), using L' Hospital's rule and evaluating \( \lim_{z \to 1} \frac{dP(z)}{dz} \), we get

\[
E(Q) = \left[ f_1(X, S, G) \sum_{i=0}^{b-1} (b(b - 1) - i(i - 1))c_i + f_2(X, S, G) \sum_{i=0}^{b-1} (b - i)c_i 
+ f_3(X, S, V) \sum_{i=0}^{a-1} ic_i + f_4(X, S, V) \sum_{i=0}^{a-1} c_i \right]
\]

\[
\left[ 2\lambda E(X)^2(b - S1 - \pi G1)^2 \right]
\]

(6.43)

The functions \( f_1 \) through \( f_4 \) are given by

\[
f_1 = T. [S1 + \pi G1], \quad f_2 = T.[S2 + \pi G2] - T2.[S1 + \pi G1]
\]

\[
f_3 = T.2.b.V1 \text{ and } f_4 = T.[b.V2 + b (b-1) .V1] - T2.b.V1
\]
where

\[
T = X_1 (b - S_1 - \pi G_1), \quad T_2 = X_1 (b(b - 1) - S_2 - \pi(2E(S)E(G) + G_2)) + X_2 (b - S_1 - \pi G_1)
\]

\[
S_1 = \lambda X_1 E(S), \quad S_2 = \lambda X_2 E(S) + \lambda^2 E^2(X) E(S^2),
\]

\[
V_1 = \lambda X_1 E(V), \quad V_2 = \lambda X_2 E(V) + \lambda^2 E^2(X) E(V^2),
\]

\[
G_1 = \lambda X_1 E(G), \quad G_2 = \lambda X_2 E(G) + \lambda^2 E^2(X) E(G^2),
\]

\[
X_1 = E(X) \quad \text{and} \quad X_2 = X^{''} (1)
\]

6.4 Expected Length of Idle Period

Let I be the random variable "idle period". Let U be a random variable such that U = 0, if the server finds at least 'a' customers after first vacation and U = 1, if he finds less than 'a' customers after the first vacation.

Now,

\[
E(I) = E(I / U = 0)P(U = 0) + E(I / U = 1)P(U = 1)
\]

\[
= E(V)P(U = 0) + (E(V) + E(I))P(U = 1)
\]

and since \( P(U = 0) + P(U = 1) = 1 \), solving for E(I), we have

\[
E(I) = E(V) / P(U=0)
\]
Using $\sum_{m=a}^{b} P_{mn}(0) = p_n$ and $G_i = G_i(0)$, from the equation (6.29), we have

$$Q_i(z, 0) = \tilde{V}(\lambda - \lambda X(z)) \sum_{n=0}^{a-1} ((1 - \pi)p_n + G_n)z^n$$

Using the fact that $Q_{ln}(0)$ is the probability that 'n' customers being in the queue after the first vacation, from (6.23), we have

$$\sum_{n=0}^{\infty} Q_{ln}(0)z^n = \tilde{V}(\lambda - \lambda X(z)) \sum_{n=0}^{a-1} ((1 - \pi)p_n + G_n)z^n$$

$$= \left( \sum_{n=0}^{\infty} \alpha_n z^n \right) \left( \sum_{n=0}^{a-1} ((1 - \pi)p_n + G_n)z^n \right)$$

Equating coefficient of $z^n$ ($n = 0, 1, 2, ..., a-1$) on both sides, we get

$$Q_{ln}(0) = \sum_{i=0}^{n} \alpha_i [(1 - \pi)p_{n-i} + G_{n-i}]$$

Therefore,

$$P(U = 0) = 1 - \sum_{n=0}^{a-1} \left( \sum_{i=0}^{n} \alpha_i [(1 - \pi)p_{n-i} + G_{n-i}] \right)$$

(6.44)

where $\alpha_i$ is the probability that 'i' customers arrive during a vacation.

In (6.44), $p_n, q_n$ are unknown constants. We express $P(U=0)$ in terms of the known constants $c_i$ in the following steps.

Taking summation from $j = 1$ to $\infty$ for $Q_j(z, 0)$ given in (6.23) and using

$$p_i = \sum_{m=a}^{b} P_{mi}(0), q_i = \sum_{l=1}^{\infty} Q_{li}(0) \text{ and } G_i = G_i(0), \text{ we get}$$
\[ \sum_{j=1}^{\infty} Q_j(z, 0) = \sum_{n=0}^{\infty} q_n z^n \]

From equations (6.29) and (6.30), we have

\[ \sum_{j=1}^{\infty} Q_j(z, 0) = \tilde{V}(\lambda - \lambda X(z)) \left[ \sum_{n=0}^{\infty} (1 - \pi) \sum_{m=a}^{b} P_{mn}(0) z^n + G_n(0) z^n \right] \]

\[ + \sum_{j=2}^{\infty} \sum_{n=0}^{a-1} Q_{j-1,n}(0) z^n \]

\[ = \tilde{V}(\lambda - \lambda X(z)) \sum_{n=0}^{\infty} [(1 - \pi) p_n + G_n + q_n] z^n \]

\[ = \tilde{V}(\lambda - \lambda X(z)) \sum_{n=0}^{\infty} c_n z^n \]

From the above two equations, we have

\[ \sum_{n=0}^{\infty} q_n z^n = \tilde{V}(\lambda - \lambda X(z)) \sum_{n=0}^{a-1} c_n z^n \]

\[ = \sum_{n=0}^{\infty} \alpha_n z^n \sum_{n=0}^{a-1} c_n z^n \quad (6.45) \]

Equating coefficient of \( z^n \), \( n = 0, 1, 2, \ldots, a-1 \) on both sides of the equation (6.45), we have

\[ q_n = \sum_{i=0}^{n} \alpha_i c_{n-i} \quad (6.46) \]

From the equation (6.44) using \( c_n = (1 - \pi) p_n + q_n + G_n \), we have

\[ P(U = 0) = 1 - \sum_{n=0}^{a-1} \sum_{i=0}^{n} \alpha_i [c_{n-i} - q_{n-i}] \]
which on, substituting for \( q_{n-i} \) from (6.46), gives

\[
P(U = 0) = 1 - \sum_{n=0}^{a-1} \sum_{i=0}^{n} \alpha_i \left[ c_{n-i} - \sum_{j=0}^{n-i} \alpha_j c_{n-i-j} \right]
\]

Therefore, the expected length of the idle period is

\[
E(I) = \frac{E(V)}{1 - \sum_{n=0}^{a-1} \sum_{i=0}^{n} \alpha_i \left[ c_{n-i} - \sum_{j=0}^{n-i} \alpha_j c_{n-i-j} \right]}
\]  

(6.47)

6.5 Expected Length of Busy Period

Let \( B \) be the random variable "busy period". Let \( T \) be the residence time that the server is rendering service or under repair. Therefore, \( T = S \) with probability \( (1 - \pi) \) and \( T = S + G \) with probability \( \pi \). The expected length of residence time is obtained as

\[
E(T) = E(S) + \pi E(G)
\]

We define a random variable \( J \) as, \( J = 0 \), if the server finds less than 'a' customers after the residence time and \( J = 1 \), if the server finds 'a' or more customers after the residence time. Then

\[
E(B) = E(B/J = 0)P(J = 0) + E(B/J = 1)P(J = 1)
\]

\[
= E(T)P(J = 0) + (E(T) + E(B))P(J = 1)
\]

and since \( P(J = 0) + P(J = 1) = 1 \), solving for \( E(B) \), we get
\[ E(B) = E(T) / P(J = 0) \]
\[ = E(T) / \sum_{n=0}^{a-1} [(1 - \pi)p_n + G_n] \]
\[ = E(T) / \sum_{n=0}^{a-1} (c_n - q_n) \]

which, using (6.46), gives

\[ E(B) = \frac{E(T)}{\sum_{n=0}^{a-1} c_n - \sum_{i=0}^{n} \alpha_i c_{n-i}} \quad (6.48) \]

6.6 Cost Model

The total average cost is obtained with the following notations: Let \( c_s \) the start-up cost, \( c_h \) the holding cost per customer, \( c_o \) the operating cost per unit time, \( c_r \) the reward per unit time due to vacation and \( c_r \) the repair cost per unit time.

Since, the length of the cycle is the sum of the idle period and busy period, from the equations (6.47) and (6.48), the expected length of cycle \( E(T_c) \) is

\[ E(T_c) = E(I) + E(B) \]
\[ = \frac{E(V)}{1 - \sum_{n=0}^{a-1} \sum_{i=0}^{n} \alpha_i c_{n-i} - \sum_{j=0}^{n-i} \alpha_j c_{n-i-j}} + \frac{E(T)}{\sum_{n=0}^{a-1} c_n - \sum_{i=0}^{n} \alpha_i c_{n-i}} \]

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The total average cost per unit time is given by

\[
\text{Total average cost} = \text{Start-up cost per cycle} + \text{Repair cost per cycle} + \\
\text{Holding cost of number of customers in the queue} + \\
\text{Operating cost}. \rho - \text{Reward due to vacation per cycle}
\]

\[
= C_s - C_r \left[ \frac{E(V)}{1 - \sum_{n=0}^{a-1} \left( \sum_{i=0}^{n} \alpha_i \left( \sum_{j=0}^{n-i} \alpha_j c_{n-i-j} \right) \right)} + \frac{c_p E(G) E(B)}{E(T)} \right] \frac{1}{E(Tc)} \\
+ c_h E(Q) + c_0 \rho
\]

(6.49)

where \( \rho = \lambda E(X) E(S) / b \)

### 6.7 Numerical Example

A numerical model is analysed with the following assumptions:

1. Service time distribution is \( k \) - Erlang with \( k = 2 \).
2. Batch size distribution of the arrival is geometric with mean 2.
3. Vacation and repair times are exponential with parameter \( \alpha = 10 \) and \( \beta = 8 \), respectively.
4. Service capacity with minimum 'a' = 3 and maximum 'b' = 4.
5. \( \pi = 0.2 \).

\[
\rho = \lambda E(X)[E(S) + \pi E(G)]/b = \frac{\lambda \pi}{b} [k/\mu + \pi/\beta]
\]
Since $k = 2$ and $b = 4$, $z^b - (1 - \pi)\tilde{S}(\lambda - \lambda X(z)) - \pi\tilde{S}(\lambda - \lambda X(z))\tilde{G}(\lambda - \lambda X(z))$ will become a polynomial of degree seven. So, there will be three roots inside, two roots out-side and one on the unit circle.

Using Bairstow's method the zeros of the function $z^b - (1 - \pi)\tilde{S}(\lambda - \lambda X(z)) - \pi\tilde{S}(\lambda - \lambda X(z))\tilde{G}(\lambda - \lambda X(z))$ are obtained. Gauss elimination method is used to solve the simultaneous equations.

Expected length of the queue $E(Q)$, expected length of idle period $E(I)$ and expected length of busy period $E(B)$ are calculated and tabulated as detailed below:

In table 6.1, results are presented for the service rate 1 and arrival rate ranging from 0.5 to 2. For the service rate 1.5 and arrival rate ranging from 0.5 to 3.5 results are given in table 6.2. In table 6.3 the service rate is taken as 2 and arrival rate is ranging from 0.5 to 4.5. Results are presented with service rate 2.5 and arrival rate ranging from 0.5 to 5.5 in table 6.4.

From the tables the following observations are made:

1) As arrival rate $'\lambda'$ increases, the mean queue size increases.

2) When the queue length is more than 3 (minimum customers who can be served in a batch), the expected length of idle period decreases and that of busy period increases.
<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\rho$</th>
<th>E(Q)</th>
<th>E(I)</th>
<th>E(B)</th>
</tr>
</thead>
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<tr>
<td>0.5000</td>
<td>0.2050</td>
<td>1.0613</td>
<td>0.1247</td>
<td>5.0631</td>
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<td>9.4432</td>
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<td><strong>Table 6.1</strong> : Service Rate = 1.0</td>
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<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
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<th>$\rho$</th>
<th>E(Q)</th>
<th>E(I)</th>
<th>E(B)</th>
</tr>
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</tr>
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<tr>
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<td><strong>Table 6.2</strong> : Service Rate = 1.5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$\rho$</td>
<td>$E(Q)$</td>
<td>$E(I)$</td>
<td>$E(B)$</td>
</tr>
<tr>
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<td>--------</td>
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Table 6.3: Service Rate = 2.0

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<th>$\rho$</th>
<th>$E(Q)$</th>
<th>$E(I)$</th>
<th>$E(B)$</th>
</tr>
</thead>
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</tr>
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</tr>
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<td>0.7838</td>
</tr>
<tr>
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<td>1.5894</td>
<td>0.2288</td>
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</tr>
<tr>
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<tr>
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Table 6.4: Service Rate = 2.5