CHAPTER I

Introduction

Graph theory is a fascinating discipline of mathematics which is treasured by the world of science for the simple reason of its lucid proof techniques. Graphs are highly desirable for representing arrangements of discrete objects, where the relationships among the objects are concerned. There are many occasions when we use a group of points joined either by lines or by arrows to illustrate situations of practical interest; the points may represent people or places or atoms, and the arrows or lines may represent the binary relationships. Diagrams of this sort are experienced everywhere under different names, such as circuit diagrams or communication networks in engineering, sociograms in Psychology, organizational structures in Economics etc. Graph theory formally began in 1736 with the publication of Euler's solution [35] to the Königsberg bridge problem. In 1936, De'nesKönig [49] wrote the first book in graph theory, who also suggested the name 'graph' for the binary relationships. Since then the growth of graph theory is tremendous and multidimensional.

With the increasing use of computers in society there has been a need for the study of the topology of networks for effective use. Many optimization problems in the networks can be modeled as graph problems and hence solved using graph theory concepts. For example, the World Wide Web can be modeled as a graph where the web pages
are vertices and the hyperlinks between them are edges in the graph. The applications of graph theory to real world situations emphasize how mathematics is an efficient tool for these studies. Although the idea of graph is very simple, the diagrammatic representation has an intuitive and aesthetic appeal which has attracted all fields of science and engineering. Every graph has several mysterious structures in it. An intensive study of the graph reveals all its secrets which require varied points of view and approach about the graph.

1.1 Motivation for the Work

A graph may be viewed differently according to the field of study. For example an engineer in a mobile company needs to know the minimum number of transmission towers to be build to cover the area and a chemist will earnestly find different warehouses to store volatile chemicals. A common issue that both the engineer and chemist are seeking is a dominating set in their domain of work. Now, instead of a dominating set, why not we think of a related cycle in a graph? A cycle sub structure is quite common in connected graphs other than trees. This enhanced the study of cycle domination. A cycle C of a graph G is dominating if and only if G – C has no edge [42]. A vertex-dominating cycle is a cycle in which every vertex of the graph is adjacent to at least one vertex on the cycle [54]. These are cycles whose vertices dominated other vertices of the graph. Intense research to unearth the richness of dominating cycles is being carried out in various parts of the world. In the process of excavation, graph theorists also identified longest dominating cycles.
If a graph has a Hamilton cycle, then it is also a dominating cycle, perhaps it can be written as the union of two perfect matching if the graph is of even order. When we talk about a perfect matching, by its natural way of definition it covers all vertices of the graph. By adding sufficient edges to a perfect matching, one can construct a connected spanning sub graph containing the perfect matching other than a Hamilton cycle.

Suppose a non Hamiltonian graph has a perfect matching, still can we find a cycle associated with it which is dominating? An answer to this search motivated to define a new kind of dominating cycle called perfect matching dominating cycle. A cycle, in a graph with a perfect matching, is a perfect matching dominating cycle when edges of the perfect matching are in the cycle or some/all of them are pendants to the cycle. Though the idea was imposed initially on graphs with only one perfect matching, a broader definition of such cycles is tried for graphs with at least one perfect matching and it worked well and this thesis evolved.

1.2 Basic Concepts and Definitions

Before stepping into the details of the work, it will be appropriate to have a formal definition of a graph and some basic concepts of graph theory.

**Graph:** A graph G is defined by a non empty set V(G) of elements called vertices, a set E(G) of elements called edges, and a relation of
incidence, which associates with each edge an unordered pair of vertices called its end vertices.

**Simple graph**: A graph without parallel edges and self-loops is called a simple graph.

**Vertex Degree**: The degree of a vertex v, (deg(v)), is the number of edges incident on it, with self-loops counted twice. A vertex with degree one is called a pendant vertex.

**k-regular**: If all vertices of a simple graph G are of degree k, then G is said to be k-regular.

**Edge degree**: The edge degree d(e) of an edge e = uv is defined as the number of its neighbors. i.e. $|N(u) \cup N(v)| - 2$.

**Connected graph**: A graph is said to be connected if there exists a path between every pair of vertices.

**k-connected graph**: If the removal of at least k vertices disconnects the graph then it is said to be k-connected.

**Path graph**: A path graph $P_n$, is a simple graph whose n vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list.
**Cycle graph:** A cycle $C_n$ is a graph with $n$ vertices and $n$ edges whose
vertices can be placed around a circle so that two vertices are adjacent
if and only if they appear consecutively along the circle.

**Complete graph:** A complete graph $K_n$ is a simple graph whose $n$
vertices are pairwise adjacent.

**Bipartite graph:** A graph $G$ is bipartite if its vertex set $V$ can be
partitioned into two disjoint sets $V_1$ and $V_2$ such that every edge of $G$
has one end in $V_1$ and the other end in $V_2$.

**Complete bipartite graph:** A bipartite graph in which every vertex of $V_1$
is adjacent to every vertex of $V_2$ is called a complete bipartite graph
and is denoted as $K_{m,n}$.

**Harary graph:** The Harary graphs, $H_{k,n}$ are the smallest possible
$k$-connected graphs having a maximum of $\left\lceil kn/2 \right\rceil$ edges.

**Hypercube graph:** A hypercube graph $Q_n$, is the simple graph whose
vertices are the $k$-tuples with entries in $\{0, 1\}$ and whose edges are the
pairs of $k$-tuples that differ in exactly one position.

**Tree:** A tree is a connected graph without any cycles.

**Spanning Tree:** A spanning tree of $G$ is a spanning subgraph which is
a tree. In other words it is a tree in $G$ which contains all the vertices of
graph $G$. 

5
**Planar graph**: A graph is planar if it can be drawn on a plane so that no two edges intersect. Otherwise the graph is non-planar.

**Hamilton cycle**: A Hamilton cycle is a cycle of the graph which contains all the vertices of the graph exactly once except the end vertices. A graph with a Hamilton cycle is called as a Hamiltonian graph.

**Dominating set**: In a graph G, a set $S \subseteq V(G)$ is a dominating set if every vertex not in S has a neighbor in S.

**Vertex-dominating cycle**: A cycle $C$ in a graph $G$ is a vertex-dominating cycle if every vertex of $G$ is either in $C$ or adjacent to a vertex of $C$.

**n-sun graph**: A graph with a cycle $C_n$ with one pendant vertex added to each of its vertices is called an n-sun graph.

**Adjacency matrix**: The adjacency matrix $A = (a_{ij})$ of a graph is defined as follows:

$$a_{ij} = \begin{cases} 1 & \text{if the } i^{th} \text{ vertex is adjacent with the } j^{th} \text{ vertex} \\ 0, & \text{otherwise} \end{cases}$$

**Edge contraction**: In a graph $G$, contraction of edge $e = uv$ is the replacement of $u$ and $v$ with a single vertex whose incident edges are the edges other than $e$ that were incident to $u$ or $v$. The resulting graph has one edge less than $G$. 
**Graph minor:** A graph H is a minor of a graph G if H can be obtained from G by contracting edges of G.

**Perfect matching:** A matching in a simple graph G is a collection of nonadjacent edges. A perfect matching M in G is a matching covering all the vertices of G. It is apparent that for a 2n vertex graph, |M| = n.

**Graph product:** Let $G_1$ and $G_2$ be two simple graphs with vertex sets U and V respectively. The Cartesian product $G_1 \square G_2$, the strong product $G_1 \boxtimes G_2$ and the tensor product $G_1 \times G_2$ of $G_1$ and $G_2$ are simple graphs with vertex set $U \times V$ and two vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent if and only if

a) $\{u_1 = u_2$ and $(v_1, v_2) \in G_2\}$ or $\{(u_1, u_2) \in G_1$ and $v_1 = v_2\}$ for $G_1 \square G_2$

b) $\{u_1 = u_2$ and $(v_1, v_2) \in G_2\}$ or $\{(u_1, u_2) \in G_1$ and $v_1 = v_2\}$ or $\{(u_1, u_2) \in G_1 \text{ and } (v_1, v_2) \in G_2\}$ for $G_1 \boxtimes G_2$

c) $\{(u_1, u_2) \in G_1 \text{ and } (v_1, v_2) \in G_2\}$ for $G_1 \times G_2$

**Hamilton decomposition:** A Hamilton decomposition of a graph G is a partitioning of the edge set of G into Hamilton cycles if G is 2d-regular or into Hamilton cycles and a perfect matching if G is (2d+1)-regular.
1.3 Review of Relevant Literature

The question of the development of a theory is important not only to satisfy our curiosity but also because much can be learnt from it for the future. Keeping this in mind, a brief account of the important facts associated with the development of graph theory concepts related to this thesis has been highlighted in this section.

Over the past thirty years, tremendous growth in the areas of computer science and engineering, operations research, molecular chemistry, design and analysis of communication networks, social science, psychology, sociology and so on, has led to a concurrent growth in graph theory due to its various applications to the above said fields.

In literature survey for different subjects of study, the name 'perfect matching' differs. In network applications it is called as a '1-factor'; in chemical molecules it is called as a 'Kekule structure' or a 'dimer covering' and so on. A necessary and sufficient condition for the existence of perfect matching in general graphs is given by Tutte [92] and that for a maximum matching by Berge C [14]. Micali, S. and Vazirani, V.V. [61, 93] has given an $O(\sqrt{|V||E|})$ algorithm for finding maximum matching and hence perfect matching in general graphs. Aldred, Anstee and Locke [2] have studied perfect matchings after vertex deletions.
A Hamilton cycle of a graph $G$ of even order is the union of two perfect matching in $G$. A Hamiltonian graph is not restricted to have only one such cycle; it may contain more than one Hamilton cycle. Moreover, not every connected graph has a Hamilton cycle. Hamiltonian graphs are the basis of this work and hence will be detailed more here. A few theorems give necessary/sufficient conditions on Hamiltonian property of graphs [95]. A result due to Dirac [31] states that, if $G$ is a simple graph with $n \geq 3$ vertices and $\delta \geq \frac{n}{2}$, then $G$ is Hamiltonian. Ore [66] observed that when $u$ and $v$ are nonadjacent vertices in a simple graph $G$ such that $\text{deg}(u) + \text{deg}(v) \geq n$, then $G$ is Hamiltonian if and only if $G + uv$ is Hamiltonian.

Bondy and Chvátal [16] gave a more general form of Ore's statement. A simple graph with $n$ vertices is Hamiltonian if and only if its closure is Hamiltonian. The study of Hamiltonian graphs is ever green in graph theory [20, 37, 45, 55, 62, 78, 100].

The complete graph $K_n$ has a Hamilton decomposition for all $n > 2$ [4]. It is known that any complete graph $K_n$ can be decomposed into $\frac{n-1}{2}$ Hamilton cycles if $n$ is odd and $\frac{n-2}{2}$ Hamilton cycles if $n$ is even using Walecki's construction [6].

The concept of a tree appeared implicitly in the work of Gustav Kirchhoff [47] who employed graph theoretical ideas in the calculation of currents in electrical networks. Later trees were studied by Arthur Cayley [22]. James Joseph Sylvester, George Polya and others used trees in the solution of problems involving the
enumeration of certain chemical molecules [38]. Any connected graph contains a spanning tree. There is a simple and elegant recursive formula for finding the number of spanning trees \( \tau(G) \) in a graph \( G \) and it was first formulated as the matrix tree theorem by Gustav Kirchhoff while obtaining values of current flow in electrical networks.

Graph product is a fascinating constituent of graph theory finding numerous applications in engineering like interconnection network, information theory, automatic library system and so on. Products of graphs were first introduced by Sabidussi [70, 71]. At this point, it might be useful to list some important results in these products [38].

1. Cartesian, strong and tensor products are commutative.
2. The strong product is the union of cartesian and tensor product.
3. \( G_1 \times G_2 \) is connected if, and only if \( G_1 \) and \( G_2 \) are both connected and at least one of \( G_1 \) or \( G_2 \) is non bipartite.
4. \( P_m \times P_n \) is disconnected for every \( m, n > 0 \).
5. \( P_m \times C_n \) is Hamiltonian when \( m = 2 \) and odd \( n \); \( P_2 \times C_n = C_{2n} \).
6. \( P_m \Box P_n \) is the grid graph \( G_{m \times n} \) having vertices corresponding to every pair of integers \( (a, b) \) and \( (a, b) \) is adjacent to \( (a+1, b) \) and \( (a, b+1) \).

One of the fastest-growing areas within graph theory is the study of domination theory. Many people have contributed to the development of domination theory. It was due to the fundamental
efforts of Berge and Ore the concept of domination in graphs was mathematically formulated [13, 65].

Although the mathematical study of dominating sets in graphs began around 1960, it has historical roots dating back to 1862 when De Jaenisch [29] studied the problem of determining the minimum number of queens which are necessary to dominate (cover) an $n \times n$ chessboard which was later studied by Yaglam and Yaglam in 1964 [91]. In 1958 Claude Berge [13] wrote a book on graph theory, in which he defined for the first time the concept of 'domination number' of a graph under the name 'coefficient of external stability'. In 1962 Oystein Ore [65] in his book on graph theory used for the first time the names 'dominating set' and 'domination number'.

Literature survey reveals that Lesniak and Williamson [59] defined dominating cycles and observed that a dominating set $S$ such that the induced subgraph $<S>$ is Hamiltonian is a dominating cycle. In 1977 Cockayne and Hedetniemi [25] published a survey paper on domination related results of their time. They were the first to use the notation $\gamma(G)$ for the domination number of a graph, which subsequently became the accepted notation. Not all graphs have dominating cycles. Bondy and Fan [17], Lu, Liu and Tian [60] derived sufficient conditions for the existence of a dominating cycle. Broersma, Yoshimoto and Zhang [41], Broersma and Veldman [19], Shen and Feng [77] worked on longest dominating cycles. Dominating cycles in bipartite graphs were studied in [7]. Clark,
Colbourn and Erdős [8], Veldman and Fraisse [42] have also contributed to cycle domination. Li, Zang and Wang [62] studied on k-dominating cycles. A vertex-dominating cycle is a generalization of the Hamilton cycle. Thus every Hamilton cycle is a vertex-dominating cycle, but not the converse. An extensive collection of work on domination theory including cycle domination is facilitated by Haynes, Hedetniemi and Slater [41]. The number of papers being published in domination related topics is rapidly increasing.

1.4 Author’s Contribution

The graphs considered for study in this thesis are connected, undirected and of even order 2n. Perfect matching plays a vital role in this entire work; because the study is carried out upon graphs containing a perfect matching.

Chapter II introduces perfect matching sequences, perfect matching dominating cycles and perfect matching minor for simple connected graphs G(V, E) with a perfect matching M. A non repeated finite alternating sequence of edges from M and E - M such that each edge is adjacent to the edge preceding and following it, starting and ending at the same edge of M is called a perfect matching sequence (PM-sequence). The sub graph of G that is formed with the edges of a PM-sequence of length 2n contains exactly one cycle called perfect matching dominating cycle (pmdc). The necessary and sufficient condition for the existence of a perfect matching dominating cycle
(pm\(d\)) is that the perfect matching minor is Hamiltonian. Apart from finding such cycles in general graphs, enumeration of perfect matching dominating cycles is also carried out for some classes of graphs. It is always convenient to represent graphs using matrices for further processing. Hence the matrix of perfect matching and the reduced matrix of the perfect matching are defined. Some graph parameters related to pm\(d\)e are also discussed. These parameters are obtained using the length of the pm\(d\)e and are found for cycle graph, complete graph, complete bipartite graph, Petersen’s graph and wheel graph.

Spanning trees play a very important role in any network since it is a basic structure of the network with minimal number of edges but ensuring connectivity. In line with this, when a non matching edge is removed in the graph of the perfect matching sequence of length 2n it resulted in a spanning tree in which the degree of every vertex is at most 3 and every such tree contains the perfect matching M of the graph. These facts inspired to define spanning trees associated with pm\(d\)es in Chapter III. Perfect matching spanning tree (pmst), edge-Disjoint pmst, perfect matching dominating cycle spanning tree are defined. Enumeration of these spanning trees is carried out for the complete, complete bipartite and Petersen graphs.

One very desirable feature expected out of a graph in the study of combinatorial design theory is edge decomposition with certain structural property. This is taken into account and a new kind of decomposition, called n-sun decomposition is offered in Chapter IV.
An n-sun graph is a cycle with an edge terminating in a vertex of degree one attached to each vertex. Thus the cycle in the n-sun is a \textit{pmde} and hence an n-sun decomposition can be regarded as a type of \textit{pmde} decomposition. To facilitate the task of n-sun decomposition, Walecki’s Hamilton cycle decomposition is utilized. A necessary condition for an n-sun decomposition is also obtained. An n-sun decomposition of even order graphs like complete graph, complete bipartite graph and the k-connected Harary graphs is given.

Chapter V captivates the study of \textit{pmde} in graph products like cartesian, strong and tensor products. Since strong product is the union of Cartesian and tensor, cartesian and tensor products are given more attention. The problem encountered will be twofold. First a perfect matching in the product is found and then existence of \textit{pmde} for that perfect matching is ascertained. It is interesting to observe that not all product graphs have \textit{pmde} for a given choice of perfect matching. In this chapter, some classes of product graphs which are non Hamiltonian are identified to have \textit{pmde}.

In chapter VI, two applications of \textit{pmde}s are identified, because any definition is complete only when it is applied. The rapid advances in the design of digital computers have, of course, had a profound effect on graph theory. The advent of efficient programming languages has made ease the formerly tedious computational aspects of graph theory. Two applications of perfect matching dominating cycles and its allied spanning trees are discussed. The first application finds the number of spanning trees containing a given Kekule
structure in fullerene molecules. The second application is on Bluetooth devices, in which a procedure is given to find spanning tree structure with maximum simultaneous active piconets in the scatternet.

Chapter VII highlights the summary of the thesis and specifies some advantages of the newly contributed pmde and associated spanning trees. This chapter also proposes some future research problems based on the work presented here.