CHAPTER V
Perfect Matching Dominating Cycles in Some Graph Products

The earlier discussions concentrated on finding perfect matching dominating cycles, spanning trees associated with these cycles and decomposing graphs using a class of perfect matching dominating cycles called n-suns. To bring the theory into order, the foregoing work may be generalized to graph products.

Graphs having a particular structure always attract researchers and find applications in many fields [24, 43, 70, 71, 103]. Graph products are gaining enormous importance in computer science. Most of the networks are classified as pure networks, meaning that the entire network is governed by a single set of connectivity rules. Composite or hybrid networks combine two or more pure networks to achieve some advantages like constructing desired size network which cannot be obtained from pure networks. Consider for example the product of the hypercube (8 vertices) and a 4-star (24 vertices) which will have 192 vertices, a number which is not possible to attain from a hypercube or from a star network [9]. Product graphs are preferred in networks because the structure of a product graph can be uniquely specified by its component graphs.

Graph products are binary operations on graphs. There are many different types of graph products like cartesian product, strong
product, tensor product, lexicographic product and so on. Given two graphs $G_1$ and $G_2$ with vertex sets $U$ and $V$ respectively, then the vertex set of any of their product is the cartesian product $U \times V$. The adjacencies between the vertices in cartesian product, strong product and tensor product are defined as below [38].

a) Cartesian product $G_1 \square G_2 : \{(u_1 = u_2 \text{ and } (v_1, v_2) \in G_2) \text{ or } (u_1, u_2) \in G_1 \text{ and } v_1 = v_2\}.

b) Strong product $G_1 \boxtimes G_2 : \{(u_1 = u_2 \text{ and } (v_1, v_2) \in G_2) \text{ or } (u_1, u_2) \in G_1 \text{ and } (v_1, v_2) \in G_2\}.

c) Tensor product $G_1 \times G_2 : \{(u_1, u_2) \in G_1 \text{ and } (v_1, v_2) \in G_2\}.$

More interest and challenge lies in finding \textit{pmdc} in even order non-Hamiltonian graph products. Moreover the strong product is the union of cartesian and tensor product and if either the cartesian or tensor product of $G_1$ and $G_2$ has a \textit{pmdc} then the strong product also has a \textit{pmdc}. Hence the basic concern is to ascertain \textit{pmdc} in cartesian and tensor products. As perfect matching is defined only for even order graphs, in graph products we keep caution about the number of vertices. In this chapter cartesian, strong and tensor product of path, cycle, complete and sun graphs are taken for discussion. The methods of proof used are constructive in nature. The process is twofold - first look for a perfect matching $M$ in the graph product and then secondly analyze whether a \textit{pmdc} exists for $M$ or not.

The notations used for the graph products and its PM-minors shall be made clear before discussion. A vertex in the PM-minor due
to contraction of e = (u, v) ∈ M is denoted as u;v. Let G₁ and G₂ be of
order r and s respectively. Assume that V(G₁) = \{u₀, u₁, ..., uᵣ\},
V(G₂) = \{v₀, v₁, ..., vₛ\}.

Consider the vertex in the product graph for uᵢ = u₂₁ and vⱼ = v₃₂.
In order to distinguish i from j, i and j are separated by a colon, say
w₂₁:₃₂. Then, V(G₁*G₂) = \{w₀:₀, w₀:₁, ..., w₀:ₛ, w₁:₀, w₁:₁, ..., wᵣ:₀, wᵣ:₁, ..., wᵣ:ₛ\}
where * denotes product operation. For simplicity, only the suffix 'i:j'

An example is given in Fig. 5.1 for a PM-minor of P₄ □ P₄ in
which V(P₄) = \{u₀, u₁, u₂, u₃\} as row and \{v₀, v₁, v₂, v₃\} as column.
V(P₄ □ P₄) = \{0:₀, 0:₁, 0:₂, 0:₃, 1:₀, 1:₁, ... , 3:₀, 3:₁, 3:₂, 3:₃\}. Let
M = \{(0:₀, 0:₁), (1:₀, 1:₁), (2:₀, 2:₁), (3:₀, 3:₁), (0:₂, 0:₃), (1:₂, 1:₃),
(2:₂, 2:₃), (3:₂, 3:₃)\}. The PM-minor with respect to M has vertex set
\{0:₀;₀:₁, 1:₀;₁:₁, 2:₀;₂:₁, 3:₀;₃:₁, 0:₂;₀:₃, 1:₂;₁:₃, 2:₂;₂:₃, 3:₂;₃:₃\}. A
semicolon is used to separate labels of the vertex in the product
graphs. This format will be followed throughout the chapter and helps
in finding vertex adjacency in graph products.
The next section deals with the existence of $\text{pmdc}$ in cartesian, strong and tensor products of a few family of graphs like path, cycle, complete and n-sun graphs.

5.1 Perfect Matching Dominating Cycles in Tensor Products

The tensor product is sensitive to connectivity, $G_1 \times G_2$ is connected if and only if both $G_1$ and $G_2$ are connected and at least one of $G_1$ or $G_2$ is not bipartite [10, 11, 38]. If $G_1 \times G_2$ is disconnected it is meaningless in discussing $\text{pmdcs}$ in it. In the following theorems existence of $\text{pmdc}$ in tensor products of cycle graphs, paths, complete graphs and n-suns are studied.

**Theorem 5.1.1:** $C_m \times P_{2n}$ has a $\text{pmdc}$ for all $n > 0$ and $m$ odd.

**Proof:** $C_m \times P_k$ is connected if and only if $m$ is odd and it is non-Hamiltonian for all $m$ and $k$. For the existence of perfect matching, $k$ must be even, say $2n$. The existence of a $\text{pmdc}$ in $C_m \times P_{2n}$ is proved by finding a Hamilton cycle in a PM-minor of it. Let $V(C_m) = \{u_0,
u_1, ..., u_{m-1} \) and \( V(P_{2n}) = \{v_0, v_1, ..., v_{2n-1}\} \) so that \( V(C_m \times P_{2n}) = \{0:0, 0:1, ..., 2n-1, 1:0, ..., m-1:0, m-1:1, ..., m-1:2n-1\} \).

Let \( M = \{(i:2j, i+1:2j+1) / 0 < i < m-1, 0 < j < n-1, i+1 \text{ is taken modulo } m\} \). Consider the following cases to find the PM-minor with respect to \( M \).

**Case (i):** \( n \) is even.

**Claim:** The PM-minor of \( C_m \times P_4 \) is the grid \( G_{m,2} (= C_m \square P_2) \) plus \( m \) copies of \( P_2 \).

Since \( i+1:2j+1 \) is adjacent with \( i+2:2j, i+2:2j+2 \) in \( C_m \times P_4 \), there are three types of adjacencies in the minor after contracting the edges of \( M \). For \( 0 < i < m-1 \), every vertex \( i:2j, i+1:2j+1 \) is adjacent with

(a) \((i+2:2j; i+3:2j+1) \text{ and } (i:2j+2; i+1:2j+3) / 0 < j < n-1\)

(b) \((i+2:2j+2; i+3:2j+3) / 0 < j < n-2 \) where \( i+1, i+2, i+3 \) are taken modulo \( m \).

Adjacency (a) shows that \( C_m \square P_2 \) is a subgraph of \( G_M(C_m \times P_4) \) and (b) gives \( m \) copies of \( P_2 \). Applying the same technique for consecutive pairs \( v_{2i}, v_{2i+1}, 0 < i < n-1 \) of \( P_{2n} \) in \( C_m \times P_{2n} \) we get \( G_M(C_m \times P_{2n}) = (C_m \square P_n) + mP_n \) as depicted in Fig. 5.2. Since \( C_m \square P_n \) is Hamiltonian the PM-minor is Hamiltonian and hence \( C_m \times P_{2n} \) has a pmdc for even \( n \).
Case (ii): n is odd.

The adjacency (b) represents the diagonals in the grid $G_{m,n}$. Hence for odd n, neglect the first row of vertices in $G_{m,n}$ so that the minor has the even grid $G_{m,n-1}$. Construct a Hamilton cycle in $G_{m,n-1}$ and extend it to a Hamilton cycle of $C_m \times P_{2n}$ with the help of a diagonal as shown in the illustration for $G_M(C_m \times P_6)$ in Fig. 5.3 □
Corollary 5.1.2: $C_m \times C_n$ has a pmdc if either $m$ or $n$ is odd.

Proof: $C_m \times C_n$ is connected if and only if $m$ or $n$ is odd. If both $m$ and $n$ are odd then there is an odd number of vertices in the product, which denies perfect matching. Let $m$ be odd and $n=2^r$. Since $C_m \times P_{2r}$ is a subgraph of $C_m \times C_n$, by the above theorem, it has a pmdc $\square$

Theorem 5.1.3: $K_r \times P_{2n}$ has a pmdc for all $r \geq 3$.

Proof: $K_r \times P_{2n}$ is non-Hamiltonian and $C_r \times P_{2n}$ is a subgraph of $K_r \times P_{2n}$. Consider the following cases.

Case (i): When $r$ is odd, by the above theorem, $C_r \times P_{2n}$ has a pmdc and hence the result.

Case (ii): Let $r$ be even, say $r = 2m$. Let $V(K_{2m}) = \{u_0, u_1, \ldots, u_{2m-1}\}$ and $V(P_{2n}) = \{v_0, v_1, \ldots, v_{2n-1}\}$ so that $V(K_{2m} \times P_{2n}) = \{0:0, 0:1, \ldots, 0:2n-1, 1:0, 1:1, \ldots, 1:2n-1, \ldots, 2m-1:0, 2m-1:1, \ldots, 2m-1:2n-1\}$ and choose $M = \{(i:2j, i+1:2j+1)/ 0 \leq i, j < 2m-1, i+1 \text{ is taken modulo } m\}$.

Claim: The perfect matching minor of $K_{2m} \times P_2$ is the multi graph $K_{2m}$.

By the above labeling, the edge set of $K_{2m} \times P_2$ is $\{(i:0, j:1), i \neq j \text{ and } 0 \leq i, j \leq 2m-1\}$. Let $M_1 = \{(i:0, i+1:1)/ 0 \leq i \leq 2m-1, i+1 \text{ is taken modulo } 2m\}$ be a perfect matching of $K_{2m} \times P_2$. Since $K_{2m} \times P_2$ is $(2m-1)$-regular and bipartite, it consists of $2m-1$ copies of perfect matching. In $\overline{GM}(K_{2m} \times P_2)$, every vertex $i:0; i+1:$ is adjacent to
Contracting all edges of $M_i$, gives a multi graph $K_{2m}$. Due to the choice of $M$, there is one edge between $i:0; i+1:1$ and $i+1:0; i+2:1$, $0 \leq i \leq 2m-1$, $i+1, i+2$ taken modulo $2m$ and two parallel edges between every other pair of vertices in the same column. Hence the PM-minor is the complete multi graph $K_{2m}$.

In $G_m(K_{2m} \times P_{2n})$, since $i:2j; i+1:2j+1$ is adjacent with $i:2j+2; i+1:2j+3$, $i+1$ is taken modulo $2n$, $0 \leq j \leq n-2$, the grid $G_{2m,n}$ is a subgraph of the PM-minor. Since even order grid graphs are Hamiltonian, the PM-minor is Hamiltonian and hence $K_{2m} \times P_{2n}$ has a pmdc.

The tensor product $P_2 \times G$ is called the double cover of $G$, that is, if $G$ is already bipartite, its double cover is the disjoint union of two copies of $G$. For convenience, an n-sun graph is denoted by $n_s$.

**Theorem 5.1.4:** $n_s \times P_2$ is a double cover of $n_s$ for even $n$ and it is the $2n$-sun graph for odd $n$.

**Proof:** When $n$ is even $n_s$ become bipartite and hence $n_s \times P_2$ is a double cover of $n_s$. When $n$ is odd, $n_s \times P_2$ is connected. Label the vertices of the cycle in $n_s$, consecutively from 0 to $n-1$ and the pendant vertex to each $i$ as $n+i$, $0 \leq i \leq n-1$. If $V(P_2) = \{0, 1\}$, then $V(n_s \times P_2) = \{0:0, 0:1, 1:0, 1:1, \ldots, n-1:0, n-1:1, n:0, n:1, \ldots 2n-1:0, 2n-1:1\}$ and vertex adjacency is such that (a) $i:0$ is adjacent with $n+i:1$ (b) $i:1$ is adjacent with $n+i:0$, $0 \leq i \leq n-1$, $i+1$ taken modulo $n$ and
the remaining edges are that of the cycle $C_{2n}$ of the $n$-suns - 0:0, 1:1, 2:0, 3:1, ..., n-1:0, 0:1, 1:0, 2:1, ..., n-1:1, 0:0. This cycle of length $2n$ with its $2n$ pendants is the $2n$-sun graph □

An example graph for $3S \times P_2$ is shown in Fig. 5.4.

![Fig. 5.4](image)

The above theorem indicates that for odd $n$, if the pendant edges are chosen as the edges of a perfect matching $M$ then, the PM-minor with respect to $M$ is the cycle graph $C_{2n}$. This motivates the next theorem.

**Theorem 5.1.5**: $n_S \times P_r$ has a pmdc for odd $n$ and even $r$.

**Proof**: $n_S \times P_r$ is non-Hamiltonian for all $n$ and $r$. When $n$ is even $n_S$ becomes bipartite and hence $n_S \times P_r$ is disconnected. A perfect matching does not exist in $n_S \times P_r$ when $r$ is odd. So let $r = 2m$ and $n$ be odd. Label $n_S$ as in Theorem 5.1.4. Let a perfect matching in $n_S \times P_r$ be $M=\{(i:2j, n+i:2j+1) \cup (i:2j+1, n+i:2j), 0 \leq i \leq n-1 \text{ and } 0 \leq j \leq m-1\}$. As only the pendants of $n_S$ are contracted, the cycle structure in $n_S$ is retained as $C_n$. By the labeling scheme of $n_S$, the structure shown in...
Fig. 5.5, for the PM-minor which has $C_n \times P_{2m}$ as a subgraph can be observed.

Every $k^{th}$ vertex in a row has adjacency with $k+2^{nd}$ vertex in the same row; $i:2j; n+i:2j+1$ is adjacent with $i:2j+2; n+i:2j+3$ and $i:2j+1; n+i:2j$ is adjacent with $i:2j+3; n+i:2j+2$. This adjacency helps in the formation of a Hamilton cycle in the PM-minor of $S \times P_{r}$. □
An illustration for $S \times P_6$ is shown in Fig. 5.6.

![Structure of Hamilton cycle in the PM-minor of $S \times P_6$](image)

**Fig. 5.6**

**Corollary 5.1.6:** $n_S \times C_r$ has a pmdc for odd $n$ and even $r$.

**Proof:** Since $n_S \times P_r$ is a subgraph of $n_S \times C_r$, the proof is direct from theorem 5.1.5 □

**Corollary 5.1.7:** $n_S \times m_S$ has a pmdc when $m$ or $n$ is odd.

**Proof:** When both $m$ and $n$ are even, the tensor product is disconnected. Hence either $m$ or $n$ must be odd. As discussed in the previous theorems, here also, contracting the pendants of $n_S$ in $n_S \times m_S$ yields a Hamilton cycle in its PM-minor □

For both $m$ and $n$ odd, a Hamilton cycle in $\overline{G_M(S \times S)}$ and for $n$ or $m$ odd, a Hamilton cycle in $\overline{G_M(S \times 4_S)}$ is given in Fig. 5.7 (labels avoided).
5.2 Perfect Matching Dominating Cycles in Cartesian Products

The next attention is towards cartesian products. It is curious to note that there are a few class of graphs \( G_1 \) and \( G_2 \) whose cartesian product is neither Hamiltonian nor contain a PMDC. For example the book graph, \( B_m = S_{m+1} \square P_2, \ m > 2 \) where \( S_{m+1} \) is the star graph, is neither Hamiltonian nor contain a PMDC. This motivated and suggested the following theorem in cartesian product.

**Theorem 5.2.1:** If \( G_1 \) has a PMDC, then \( G_1 \square P_{2m} \) has a PMDC.

**Proof:** Since \( G_1 \) has a PMDC, \( G_1 \) must be of even order say \( 2n \) and let \( M \) be a perfect matching in it. If the PMDC in \( G_1 \) is a Hamilton cycle \( C_{2n} \) then \( C_{2n} \square P_{2m} \) contains the grid graph \( G_{2m, 2m} \) which is Hamiltonian. On the otherhand, if the PMDC \( C_{2n-k}, \ 1 < k \leq n \), is non Hamiltonian, there are \( k \) pendants to \( C_{2n-k} \). The remaining \( 2n-k \) edges of \( M \) are in the cycle \( C_{2n-k} \). Let the subgraph \( C_{2n-k} \) with its \( k \) pendants
be denoted as G. Draw the cartesian product with $C_{2n-k}$ as the backbone as in Fig. 5.8.

By definition of cartesian product, for every vertex of $P_{2m}$, there is a copy of G in $G_1 \square P_{2m}$. Every copy of G contains the perfect matching $M$ of $G_1$. This can be taken as a perfect matching of $G \square P_{2m}$. Thus $G \square P_{2m}$ contains the grid $G_{2n-k, 2m}$ which is Hamiltonian. The Hamilton cycle of the grid is a pmdc for $G \square P_{2m}$

**Corollary 5.2.2:** $n_5 \square P_{2m}$ has a pmdc.

**Proof:** Since $n_5$ itself is a pmdc, by above theorem $n_5 \square P_{2m}$ has a pmdc

If either the tensor product or cartesian product has a pmdc, then the strong product also have one, since strong product is the union of tensor and cartesian products.
5.3 Perfect Matching Dominating Cycles in Strong Product

The following results are immediate consequences of the fact that strong product is the union of tensor and cartesian products.

(a) \( C_m \boxtimes P_{2n} \) has a \( pmdc \) for all \( n > 0 \) and \( m \) odd.
(b) \( C_m \boxtimes C_n \) has a \( pmdc \) if either \( m \) or \( n \) is odd.
(c) \( K_r \boxtimes P_{2n} \) has a \( pmdc \) for all \( r \geq 3 \).
(d) \( n_s \boxtimes P_r \) has a \( pmdc \) for odd \( n \) and even \( r \).
(e) \( n_s \boxtimes C_r \) has a \( pmdc \) for odd \( n \) and even \( r \).
(f) \( n_s \boxtimes m_s \) has a \( pmdc \) when \( m \) or \( n \) is odd.
(g) If \( G_1 \) has a \( pmdc \), then \( G_1 \boxtimes P_{2m} \) has a \( pmdc \).

Every Hamilton cycle in a graph is a \( pmdc \) of the graph and hence \( pmdcs \) in non-Hamiltonian graph products are challenging and interesting. This chapter has analyzed \( pmdc \) in tensor, cartesian and strong products of some classes of graphs which are Hamiltonian or not.